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1. ALGEBRA OF MATRICES

STRUCTURE

1.1. INTRODUCTION

In this chapter we shall describe matrices, some special type of arrangement of numbers. The use of matrices helps a lot in mathematical investigations and has become an integral part of mathematics. Contribution of matrices in the fields of business, physics, engineering etc., is indispensable.

1.1.1. Definition of a Matrix

A rectangular arrangement of numbers in a finite number of rows and columns, enclosed in a pair of brackets '[ ]' or '( )' is called a matrix *e.g.,* $A = \begin{bmatrix} 1 & 5 & 6 \\ 3 & 2 & 1 \end{bmatrix}$ is a rectangular arrangement of numbers which has two rows (horizontal lines) '1 5 6', '3 2 1' and three columns (vertical lines) $\begin{matrix} 1' & 5' & 6 \\ 3 & 2 & 1 \end{matrix}$.

Observe that the element 5 lies in Ist row and second column. Similarly, we can tell the position of every entry or element of the matrix A.

1.1.2. Order of a Matrix

If a matrix has 'm' rows and 'n' columns, then the order of the matrix is '$m \times n$' (read as m by n), *e.g.,* the order of the matrix 'A' above is 2×3.

Note. A matrix of order '$m \times n$' has 'mn' elements.

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1.1.3. General Form of a Matrix

An arrangement

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

is a matrix having 'm' rows and 'n' columns. The matrix may be written as $A = [a_{ij}]_{m \times n}$. The representation is called general form of a matrix.

Observe that the element a_{11} lies in the Ist row and Ist column. The element a_{23} lies in IInd row and 3rd column. The element a_{ij} lies in i-th row and j-th column. **The first suffix of every entry indicates the row and IInd suffix indicates the column in which the element lies.**

Remark: A matrix is marely an arrangement and has no numerical value.

1.1.4. Types of Matrices

I. Square Matrix : A matrix $A = [a_{ij}]_{m \times n}$ is called a square matrix if the number of rows and columns in the matrix are equal *i.e.*, if $m = n$.

A square matrix having 'n' rows is called a matrix of order 'n' or 'n' square matrix *e.g.*, the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$ is a square matrix of order 3.

II. Diagonal Elements of a Matrix : An element a_{ij} of a **square** matrix $A = [a_{ij}]_{n \times n}$ is said to be a diagonal element if $i = j$ *i.e.*, the elements a_{11}, a_{22}, ... are diagonal elements.

III. Principal Diagonal : The places along which the diagonal elements lie is called the principal diagonal of a square matrix.

IV. Row Matrix : A matrix having only one row is called a row matrix *e.g.*, the matrix $A = [1\ 3\ 7]_{1 \times 3}$ has one row and three columns is a row matrix.

V. Column Matrix : If a matrix has only one column, then it is called a column matrix *e.g.*, $\begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}_{3 \times 1}$ is a column matrix.

VI. Zero Matrix or Null Matrix : A matrix $A = [a_{ij}]_{m \times n}$ is called a zero matrix or null matrix if $a_{ij} = 0\ \forall\ i$ and j *i.e.*, all the entries of the matrix A are zero *e.g.*,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is a zero matrix.}$$

VII. Diagonal Matrix : A **square matrix** $A = [a_{ij}]_{n \times n}$ is called a diagonal matrix if $a_{ij} = 0\ \forall\ i \neq j$ *i.e.*, all the non-diagonal entries are zero *e.g.*,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix},\ B = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ are diagonal matrices.}$$

VIII. Scalar Matrix : A **diagonal matrix** in which all the diagonal entries are same is called a scalar matrix *e.g.,*

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \text{ is a scalar matrix.}$$

IX. Identity or Unit Matrix : A **diagonal matrix** in which all the diagonal entries are 1, is called an identity or a unit matrix. An identity matrix of order 'n' is denoted by I_n *e.g.,*

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ etc.}$$

X. Upper Triangular Matrix : A **square matrix** $A = [a_{ij}]_n$ is called an upper triangular matrix if $a_{ij} = 0 \ \forall \ i > j$ *i.e.,* all the elements below the principal diagonal are zero *e.g.,*

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix} \text{ is an upper triangular matrix.}$$

XI. Lower Triangular Matrix : A **square matrix** $A = [a_{ij}]_n$ is called a lower triangular matrix if $a_{ij} = 0 \ \forall \ i < j$ *i.e.,* all the elements above the principal diagonal are zero *e.g.,*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 5 & 5 & 2 \end{bmatrix} \text{ is a lower triangular matrix.}$$

XII. Comparable Matrices. Two matrices A and B are said to be **comparable** if they are of the same order *e.g.,*

$$\text{The matrices} \quad A = \begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix}_{2 \times 3} \text{ and } B = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 3 & 6 \end{bmatrix}_{2 \times 3} \text{ are comparable.}$$

XIII. Equal Matrices. Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are **equal** if they are of the same order and their respective entries are equal *i.e.*, if $m = p$ and $n = q$ and $a_{ij} = b_{ij} \ \forall \ i$ and j.

e.g., the two matrices $A = \begin{bmatrix} a & b & 4 \\ c & x & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & 4 \\ 6 & 7 & 6 \end{bmatrix}$ are equal if $a = 0$, $b = 3$, $c = 6$, $x = 7$.

Example 1. *What is the order of the matrix given below ?*

$$A = \begin{bmatrix} 7 & 1 & 9 & -11 \\ 2 & 3 & 8 & 15 \\ -1 & -7 & -12 & 6 \end{bmatrix}$$

Write the elements $a_{12}, a_{21}, a_{24}, a_{31}, a_{34}$ *for the matrix A.*

Sol. The given matrix A has three rows and four columns.

$\therefore$ The order of A is **3 × 4.**

a_{12} = element lying in Ist row and IInd column = **1**

a_{21} = element lying in IInd row and Ist column = **2**

a_{24} = element lying in IInd row and IVth column = **15**
a_{31} = element lying in IIIrd row and Ist column = **– 1**
a_{34} = element lying in IIIrd row and IVth column = **6.**

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Example 2. *Construct a matrix* $A = [a_{ij}]_{2 \times 2}$, *where* $a_{ij} = \dfrac{(i + 2j)^2}{2}$.

Sol. $$A = [a_{ij}]_{2 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ given } a_{ij} = \frac{(i + 2j)^2}{2}$$

so $$a_{11} = \frac{(1 + 2.1)^2}{2} = \frac{9}{2},\ a_{12} = \frac{(1 + 2.2)^2}{2} = \frac{25}{2}$$

$$a_{21} = \frac{(2 + 2.1)^2}{3} = \frac{16}{2} = 8,\ a_{22} = \frac{(2 + 2.2)^2}{2} = \frac{36}{2} = 18$$

so $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{9/2} & \mathbf{25/2} \\ \mathbf{8} & \mathbf{18} \end{bmatrix}.$$

Example 3. *Find x and y if the two matrices* $A = \begin{bmatrix} 2x + 1 & 3y \\ 0 & y^2 - 5y \end{bmatrix}$ *and* $B = \begin{bmatrix} x + 3 & y^2 + 2 \\ 0 & -6 \end{bmatrix}$ *are equal.*

Sol. Since $$\begin{bmatrix} 2x + 1 & 3y \\ 0 & y^2 - 5y \end{bmatrix} = \begin{bmatrix} x + 3 & y^2 + 2 \\ 0 & -6 \end{bmatrix}$$

so there respective entries must be equal.

i.e., $2x + 1 = x + 3$ ...(*i*)

$3y = y^2 + 2$ ...(*ii*)

$y^2 - 5y = -6$ ...(*iii*)

From (*i*), $x = 2$

From (*ii*), $y^2 - 3y + 2 = 0$ ...(*iv*)

and from (*iii*), $y^2 - 5y + 6 = 0$ ...(*v*)

Subtract (*iv*) from (*v*), we get

$-2y + 4 = 0$

⇒ $2y = 4$ ⇒ $\mathbf{y = 2}$

so $\mathbf{x = 2}$ and $\mathbf{y = 2.}$

EXERCISE 1.1

1. What is the order of the matrix A given below :

$$A = \begin{bmatrix} -7 & 8 & 6 & 5 \\ 2 & 7 & 11 & 17 \\ 3 & 9 & -6 & 14 \end{bmatrix}.$$

Write the elements a_{21}, a_{23}, a_{14}, a_{34}, a_{12}.

2. Write the type and order of the following matrices :

(*i*) $\begin{bmatrix} 2 & 3 & 6 \\ 0 & 8 & 7 \\ 1 & 2 & 5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 21 \\ 3 \\ 6 \\ 8 \end{bmatrix}$ (*iii*) $[3 \ \ 4 \ \ 7]$

(*iv*) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (*v*) $\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ (*vi*) $\begin{bmatrix} 8 & 0 \\ 0 & 9 \end{bmatrix}$

(*vii*) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (*viii*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (*ix*) $\begin{bmatrix} 5 & 0 & 0 \\ 6 & 18 & 0 \\ 10 & 0 & 11 \end{bmatrix}$.

3. Construct a 2 × 2 matrix $A = [a_{ij}]$ whose element a_{ij} is given by :

(*i*) $a_{ij} = \dfrac{(i+j)^2}{2}$ (*ii*) $a_{ij} = \dfrac{(i-j)^2}{2}$ (*iii*) $a_{ij} = \dfrac{(i-2j)^2}{2}$.

4. If $\begin{bmatrix} a+b & 2 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$, find the values of a and b.

5. Find the values of a, b, c, d from the following matrix equations :

(*i*) $\begin{bmatrix} 2a+5 & b+7 \\ 3c-8 & 3d+9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$.

Answers

1. 3 × 4 ; 2, 11, 5, 14, 8

2. (*i*) Square matrix, 3 × 3 (*ii*) Column matrix, 4 × 1
(*iii*) Row matrix, 1 × 3 (*iv*) Zero matrix, 2 × 4
(*v*) Scalar matrix, 3 × 3 (*vi*) Diagonal matrix, 2 × 2
(*vii*) Unit matrix, 3 × 3 (*viii*) Unit matrix, 2 × 2
(*ix*) Lower triangular matrix, 3 × 3

3. (*i*) $\begin{bmatrix} 2 & 9/2 \\ 9/2 & 8 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1/2 & 9/2 \\ 0 & 2 \end{bmatrix}$.

4. $a = 2, b = 4$ or $a = 4, b = 2$

5. (*i*) $a = -2, b = -5, c = 3, d = -2$ (*ii*) $a = 1, b = 2, c = 3, d = 4$.

1.2. OPERATIONS ON MATRICES

1.2.1. Scalar Multiplication

Multiplication of a matrix by a scalar.

Let $A = [a_{ij}]_{m \times n}$ be a matrix and let λ be a scalar we define

$$\lambda A = \lambda\, [a_{ij}]_{m \times n} = [\lambda a_{ij}]_{m \times n}$$

i.e., if we multiply a matrix by some constant 'λ' say, then every entry of the matrix is multiplied by λ *e.g.,*

Let $A = \begin{bmatrix} 2 & 5 \\ 2 & 3 \end{bmatrix}$, then $9A = 9\begin{bmatrix} 2 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 \times 2 & 9 \times 5 \\ 9 \times 2 & 9 \times 3 \end{bmatrix} = \begin{bmatrix} 18 & 45 \\ 18 & 27 \end{bmatrix}$.

Note that common is also taken from every entry of the matrix.

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1.2.2. Negative of a Matrix

The negative of a matrix $A = [a_{ij}]_{m \times n}$ is $[-a_{ij}]_{m \times n}$ and is denoted by $-A$ *i.e.,* –ve of a matrix is obtained by multiplying every entry by '–1' or changing the sign of every entry for example :

Let $$A = [a_{ij}]_{2 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

We define $$-A = (-1)A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$$

1.2.3. Addition of Matrices

We can add two matrices only if they are of the same order. Sum of two matrices $A = [a_{ij}]_{m \times n}$, and $B = [b_{ij}]_{m \times n}$ is obtained by adding the respective entries of the two matrices *i.e.,* $A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$.

For example, if $A = \begin{bmatrix} 2 & 3 & 6 \\ 4 & 0 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 6 & 8 \\ 7 & 0 & 8 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 2 & 3 & 6 \\ 4 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 6 & 8 \\ 7 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 2+0 & 3+6 & 6+8 \\ 4+7 & 0+0 & 9+8 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 14 \\ 11 & 0 & 17 \end{bmatrix}.$$

If $C = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, then neither $A + C$ nor $B + C$ is defined.

1.2.4. Properties of Matrix Addition and Scalar Multiplication

(*i*) $\lambda(A + B) = \lambda A + \lambda B$ *i.e.,* scalar multiplication is distributive.

(*ii*) $(\lambda . l)A = \lambda(lA)$.

(*iii*) $1.A = A$.

(*iv*) If A and B are matrices of the same order, then $A + B = B + A$ *i.e.,* matrix addition is commutative.

(*v*) If A, B, C are matrices of the same order, then $(A + B) + C = A + (B + C)$ *i.e.,* matrix addition is associative.

(*vi*) If A is any matrix then $A + O = A = O + A$, where 'O' is a zero matrix of order same as that of A.

Example 1. *If* $A = \begin{bmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{bmatrix}$, *find 7A + 5B.*

Sol. $$7A + 5B = 7\begin{bmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{bmatrix} + 5\begin{bmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} 7\times 3 & 7\times 8 & 7\times 11 \\ 7\times 6 & 7\times -3 & 7\times 8 \end{bmatrix} + \begin{bmatrix} 5\times 1 & 5\times -6 & 5\times 15 \\ 5\times 3 & 5\times 8 & 5\times 17 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 56 & 77 \\ 42 & -21 & 56 \end{bmatrix} + \begin{bmatrix} 5 & -30 & 75 \\ 15 & 40 & 85 \end{bmatrix}$$

$$= \begin{bmatrix} 21+5 & 56-30 & 77+75 \\ 42+15 & -21+40 & 56+85 \end{bmatrix} = \begin{bmatrix} 26 & 26 & 152 \\ 57 & 19 & 141 \end{bmatrix}.$$

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Example 2. *If* $A = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 8 & 11 \\ 6 & 1 \end{bmatrix}$, *then show that :*

(*i*) $A + B = B + A$ (*ii*) $A + (-A) = (-A) + A = O$.

Sol. (*i*) $A + B = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 4+3 & 7+8 \\ 6+7 & 5+2 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 13 & 7 \end{bmatrix}$

$$B + A = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 3+4 & 8+7 \\ 7+6 & 2+5 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 13 & 7 \end{bmatrix}$$

$\therefore$ **A + B = B + A.**

(*ii*) $A = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} \Rightarrow -A = \begin{bmatrix} -4 & -7 \\ -6 & -5 \end{bmatrix}$

$$\therefore \quad A + (-A) = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} + \begin{bmatrix} -4 & -7 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} 4+(-4) & 7+(-7) \\ 6+(-6) & 5+(-5) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$(-A) + A = \begin{bmatrix} -4 & -7 \\ -6 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} -4+4 & -7+7 \\ -6+6 & -5+5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\therefore$ **A + (– A) = (– A) + A = O.**

Example 3. *Find X and Y if* $2X + Y = \begin{bmatrix} 4 & 4 & 7 \\ 7 & 3 & 4 \end{bmatrix}$ *and* $X - 2Y = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

Sol. We have $$2X + Y = \begin{bmatrix} 4 & 4 & 7 \\ 7 & 3 & 4 \end{bmatrix} \quad ...(1)$$

and $$X - 2Y = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad ...(2)$$

(1) × 2 $\Rightarrow$ $$4X + 2Y = 2\begin{bmatrix} 4 & 4 & 7 \\ 7 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 14 \\ 14 & 6 & 8 \end{bmatrix} \quad ...(3)$$

(2) + (3) $\Rightarrow$ $$5X = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 8 & 8 & 14 \\ 14 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 15 & 5 & 10 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{5}\begin{bmatrix} 5 & 10 & 15 \\ 15 & 5 & 10 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{1} & \mathbf{2} \end{bmatrix}.$$

Putting the value of X in (1), we get

$$Y = \begin{bmatrix} 4 & 4 & 7 \\ 7 & 3 & 4 \end{bmatrix} - 2X = \begin{bmatrix} 4 & 4 & 7 \\ 7 & 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 7 \\ 7 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 6 & 2 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix}.$$

EXERCISE 1.2

1. Find the sum of the following matrices :

(*i*) $\begin{bmatrix} -3 & 5 \\ -9 & 11 \end{bmatrix}$ and $\begin{bmatrix} 0 & 7 \\ 18 & -19 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 3 \\ 6 & 8 \end{bmatrix}$ and $\begin{bmatrix} 9 & 3 \\ 6 & 5 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ (*iv*) [2 5] and [7 3].

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2. Does the sum $\begin{bmatrix} 3 & 7 \\ 6 & 6 \\ 8 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 7 \\ 5 & 11 & 8 \end{bmatrix}$ make sense ? If not give the reason.

3. Find A + B if defined, for the following matrices :

(*i*) $A = \begin{bmatrix} 1 & 6 & 8 \\ 5 & 7 & 10 \\ 10 & 12 & 9 \end{bmatrix}, B = \begin{bmatrix} -12 & 5 & 9 \\ 17 & 6 & 5 \\ 4 & 3 & 17 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 7 & -11 & 7 \\ 6 & 9 & 8 \end{bmatrix}, B = \begin{bmatrix} 1 & 6 \\ 8 & 7 \end{bmatrix}$

4. Solve for X : $\begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix} - X = \begin{bmatrix} -3 & 4 \\ 5 & -1 \end{bmatrix}$

5. Verify commutative law of addition for the matrices ;

$$A = \begin{bmatrix} 6 & 8 \\ 3 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}.$$

6. If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$, then find the matrix C such that A + B + C is a zero matrix.

7. For the matrices $A = \begin{bmatrix} 2 & 3 & 6 \\ 1 & 5 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 5 \\ 0 & 8 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} 12 & 3 & 7 \\ 0 & 8 & 7 \end{bmatrix}$. Show that (A + B) + C = A + (B + C).

8. If $A = \begin{bmatrix} 2 & 3 & 6 \\ 1 & 8 & 9 \end{bmatrix}$, find 2A and – 7A.

9. For the matrices $A = \begin{bmatrix} 4 & 3 \\ 6 & 7 \end{bmatrix}, B = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}, C = \begin{bmatrix} 3 & 6 \\ 7 & 8 \end{bmatrix}$. Find (*i*) 2A + 3B (*ii*) A + 5B – 4C.

10. Solve the equation $2\begin{bmatrix} x & z \\ y & u \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$.

11. Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$, find the matrix C such that A + 2C = B.

12. If $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$, find X and Y.

Answers

1. (*i*) $\begin{bmatrix} -3 & 12 \\ 9 & -7 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 11 & 6 \\ 12 & 13 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (*iv*) [9 8]

2. No, as the order's of the matrices are not same

3. (*i*) $\begin{bmatrix} -11 & 11 & 17 \\ 22 & 13 & 15 \\ 14 & 15 & 26 \end{bmatrix}$ (*ii*) not defined

4. $\begin{bmatrix} 5 & -7 \\ -1 & -1 \end{bmatrix}$

6. $\begin{bmatrix} -3 & 4 & -1 \\ -3 & 0 & -1 \end{bmatrix}$

8. $2A = \begin{bmatrix} 4 & 6 & 12 \\ 2 & 16 & 18 \end{bmatrix}, -7A = \begin{bmatrix} -14 & -21 & -42 \\ -7 & -56 & -63 \end{bmatrix}$

9. (*i*) $\begin{bmatrix} 5 & 21 \\ 24 & 5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -13 & 4 \\ -2 & -40 \end{bmatrix}$

10. $x = 3, y = 6, z = 9, u = 6.$

11. $\begin{bmatrix} \frac{1}{2} & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

12. $X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}, Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

NOTES

1.3. MULTIPLICATION OF MATRICES

The multiplication AB of two matrices A and B is only possible if the number of columns of the premultiplier matrix A equals to the number of rows of the matrix B, the post multiplier.

Similarly, the product BA is defined only if the number of columns of B equals to the number of rows of A.

Let m, n, p be three different numbers.

Let $$A = [a_{ij}]_{m \times n} \quad \text{and} \quad B = [b_{ij}]_{n \times p}$$

the product AB is possible as the number of columns of A are 'n' and equals to the number of rows of B. The order of AB will be $m \times p$. **But the product BA is not defined** as the number of columns of B are p and does not equal to the number of rows of A, which are m in number.

Let $AB = C = [c_{ij}]_{m \times p}$ *i.e.*,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

then
$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + ... + a_{1n}b_{n1}$$
$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + ... + a_{1n}b_{n2}$$
$$\vdots = \vdots \qquad \vdots$$
$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$$

Observe that c_{11} the Ist element of the product 'C' is obtained by adding the multiples of the elements of Ist row with the respective entries of Ist column. The element c_{12} is obtained by adding the multiples of the elements of Ist row with the respective entries of IInd column of the matrix B.

Note that if we multiply by the Ist row of the matrix A, the columns of the matrix B, we get only the elements of Ist row of the product matrix AB. Similarly, when we multiply by IInd row of the matrix A the columns of the matrix B, we get only the elements of IInd row of the product matrix AB and so on.

In general

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \quad i.e.,$$

The element c_{ij} lying in the i-th row and j-th column of the product AB is obtained by adding the multiplies of the elements of i-th row of the matrix A with the j-th column of the matrix B.

For illustration :

Let $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}_{2 \times 3}$$

Since the number of columns of the matrix A are 2 and the number of rows of B are also 2 so the product AB is possible and the order of the matrix AB is 2×3. But BA is not possible.

NOTES

The product AB is

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \downarrow b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}.b_{11} + a_{12}.b_{21} & a_{11}.b_{12} + a_{12}.b_{22} & a_{11}.b_{13} + a_{12}.b_{23} \\ a_{21}.b_{11} + a_{22}.b_{21} & a_{21}.b_{12} + a_{22}.b_{22} & a_{21}.b_{13} + a_{22}.b_{23} \end{bmatrix}.$$

1.3.1. Properties of Matrix Multiplication

1. **Matrix multiplication is not commutative in general**, *i.e.,* for matrices A and B, we need not have AB = BA.

For example, if

(*i*) A and B are 2×3 and 3×4 matrices respectively, then AB is a 2×4 matrix whereas BA is not defined.

(*ii*) A and B are 2×3 and 3×2 matrices respectively, then AB is a 2×2 matrix and BA is a 3×3 matrix.

(*iii*) A and B are 2×2 matrices, then both AB and BA are 2×2 matrices. Even in this case, we may not have AB = BA.

(*iv*) If A is a square matrix, then for a natural no. 'n' we define $A^n = A.A.A...A.$ (n times).

2. **Matrix multiplication is associative** *i.e.*, if A, B and C be matrices of the type $m \times n$, $n \times p$ and $p \times q$ respectively, then **(AB)C = A(BC).**

3. **Matrix multiplication is distributive with respect to addition**

(*a*) If A, B and C are matrices of the type $m \times n$, $n \times p$ and $n \times p$ respectively, then

A(B + C) = AB + AC.

(*b*) If A, B and C are matrices of the type $m \times n$, $m \times n$ and $n \times p$ respectively, then

(A + B)C = AC + BC.

4. **The product of non-zero matrices may be a zero matrix.** For matrices A and B, it may be possible that AB = O and neither A nor B is a zero matrix.

For example, let $A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$.

$$AB = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + 1 \times (-2) & 2 \times 2 + 1 \times (-4) \\ 0 \times 1 + 0 \times (-2) & 0 \times 2 + 0 \times (-4) \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & 4-4 \\ 0-0 & 0-0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

Example 1. *If* $B = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 5 & 3 \\ 2 & 6 & 0 \end{bmatrix}$ *and* $C = \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 6 & 8 \end{bmatrix}$ *and A = BC, find* a_{12}, a_{21}, a_{32} *where* a_{ij} *is the element of A placed at i-th row and j-th column.*

Sol. Since O(B) = 3×3 and O(C) = 3×2 and the number of columns of B = 3 = number of rows of C so BC is defined.

NOTES

a_{12} = (First row of the matrix B) (Second column of the matrix C)

$$= (4 \quad 3 \quad 7)\begin{pmatrix}2\\5\\8\end{pmatrix} = 4 \times 2 + 3 \times 5 + 7 \times 8 = 8 + 15 + 56 = 79$$

a_{21} = (Second row of the matrix B) (Ist column of the matrix C)

$$= 1 \times 3 + 5 \times 1 + 3 \times 6 = 3 + 5 + 18 = 26$$

a_{31} = (Third row of the matrix B) (Ist column of the matrix C)

$$= 2 \times 3 + 6 \times 1 + 0 \times 6 = 6 + 6 = 12.$$

Example 2. *If* $A = \begin{bmatrix}4 & 7\\6 & 2\end{bmatrix}$, $B = \begin{bmatrix}1 & 8\\3 & 7\end{bmatrix}$, *find AB ?*

Sol.

$$AB = \begin{bmatrix}4 & 7\\6 & 2\end{bmatrix}\begin{bmatrix}1 & 8\\3 & 7\end{bmatrix} = \begin{bmatrix}4\times1+7\times3 & 4\times8+7\times7\\6\times1+2\times3 & 6\times8+2\times7\end{bmatrix}$$

$$= \begin{bmatrix}25 & 81\\12 & 62\end{bmatrix}.$$

Example 3. *If* $A = \begin{bmatrix}2 & -1\\3 & 2\end{bmatrix}$, $B = \begin{bmatrix}0 & 4\\-1 & 7\end{bmatrix}$, *find* $3A^2 - 2B$.

Sol.

$$A^2 = \begin{bmatrix}2 & -1\\ \longrightarrow & \\ 3 & 2\end{bmatrix} = \begin{bmatrix}2 & -1\\3\downarrow & 2\end{bmatrix} = \begin{bmatrix}4-3 & -2-2\\6+6 & -3+4\end{bmatrix} = \begin{bmatrix}1 & -4\\12 & 1\end{bmatrix}$$

$$\therefore \quad 3A^2 - 2B = 3\begin{bmatrix}1 & -4\\12 & 1\end{bmatrix} - 2\begin{bmatrix}0 & 4\\-1 & 7\end{bmatrix}$$

$$= \begin{bmatrix}3 & -12\\36 & 3\end{bmatrix} - \begin{bmatrix}0 & 8\\-2 & 14\end{bmatrix} = \begin{bmatrix}\mathbf{3} & \mathbf{-20}\\\mathbf{38} & \mathbf{-11}\end{bmatrix}.$$

Example 4. *If* $A = \begin{bmatrix}2 & 1\\3 & 4\end{bmatrix}$, $B = \begin{bmatrix}1 & 6\\3 & 4\end{bmatrix}$, $C = \begin{bmatrix}4 & 6\\3 & 5\end{bmatrix}$, *verify that (AB)C = A(BC).*

Sol.

$$AB = \begin{bmatrix}2 & 1\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 6\\3 & 4\end{bmatrix} = \begin{bmatrix}2\times1+1\times3 & 2\times6+1\times4\\3\times1+4\times3 & 3\times6+4\times4\end{bmatrix} = \begin{bmatrix}5 & 16\\15 & 34\end{bmatrix}$$

$$\therefore \quad \text{L.H.S.} = (AB)C = \begin{bmatrix}5 & 16\\15 & 34\end{bmatrix}\begin{bmatrix}4 & 6\\3 & 5\end{bmatrix}$$

$$= \begin{bmatrix}5\times4+16\times3 & 5\times6+16\times5\\15\times4+34\times3 & 15\times6+34\times5\end{bmatrix}$$

$$= \begin{bmatrix}20+48 & 30+80\\60+102 & 90+170\end{bmatrix} = \begin{bmatrix}68 & 110\\162 & 260\end{bmatrix}$$

$$BC = \begin{bmatrix}1 & 6\\3 & 4\end{bmatrix}\begin{bmatrix}4 & 6\\3 & 5\end{bmatrix} = \begin{bmatrix}1\times4+6\times3 & 1\times6+6\times5\\3\times4+4\times3 & 3\times6+4\times5\end{bmatrix} = \begin{bmatrix}22 & 36\\24 & 38\end{bmatrix}$$

$$\therefore \quad \text{R.H.S.} = A(BC) = \begin{bmatrix}2 & 1\\3 & 4\end{bmatrix}\begin{bmatrix}22 & 36\\24 & 38\end{bmatrix}$$

$$= \begin{bmatrix}2\times22+1\times24 & 2\times36+1\times38\\3\times22+4\times24 & 3\times36+4\times38\end{bmatrix}$$

$$= \begin{bmatrix}44+24 & 72+38\\66+96 & 108+152\end{bmatrix} = \begin{bmatrix}68 & 110\\162 & 260\end{bmatrix}$$

$\therefore$ **(AB)C = A(BC).**

NOTES

Example 5. *If* $A = [1\ 3\ 4]$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 7 \\ 6 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 1 & 0 \end{bmatrix}$, *verify that* $A(B + C) = AB + AC$.

Sol.
$$B + C = \begin{bmatrix} 2 & 1 \\ 3 & 7 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 8 & 13 \\ 7 & 8 \end{bmatrix}$$

$$\therefore \quad \text{L.H.S.} = A(B + C) = [1 \quad 3 \quad 4] \begin{bmatrix} 5 & 5 \\ 8 & 13 \\ 7 & 8 \end{bmatrix}$$
$$= [5 + 24 + 28 \quad 5 + 39 + 32] = [57\ 76]$$

$$AB = [1 \quad 3 \quad 4] \begin{bmatrix} 2 & 1 \\ 3 & 7 \\ 6 & 8 \end{bmatrix}$$
$$= [1(2) + 3(3) + 4(6)\ 1(1) + 3(7) + 4(8)]$$
$$= [2 + 9 + 24 \quad 1 + 21 + 32] = [35 \quad 54]$$

$$AC = [1 \quad 3 \quad 4] \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 1 & 0 \end{bmatrix}$$
$$= [3 + 15 + 4 \quad 4 + 18 + 0] = [22 \quad 22]$$

$$\therefore \quad \text{R.H.S.} = AB + AC = [35\ 54] + [22\ 22] = [57\ 76].$$

$$\therefore \quad \mathbf{A(B + C) = AB + AC.}$$

Example 6. *Let* $f(x) = x^2 - 5x + 6$. *Find* $f(A)$, *if* $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

Sol. We have
$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}.$$

$f(x) = x^2 - 5x + 6$ implies $f(A) = A^2 - 5A + 6I_3$.

$$A^2 = A.A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1-0 & 1-3+0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}.$$

Now,
$$f(A) = A^2 - 5A + 6I_3$$
$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5 \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 5-10+6 & -1-0+0 & 2-5+0 \\ 9-10+0 & -2-5+6 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2-0+6 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{-3} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{-10} \\ \mathbf{-5} & \mathbf{4} & \mathbf{4} \end{bmatrix}$$

.

Example 7. *Three shopkeepers, Ashok, Ramesh and Ravi, go to a store to buy stationery. Ashok purchases 12 dozen notebooks, 5 dozen pens, 6 dozen pencils. Ramesh purchases 10 dozen notebooks, 6 dozen pens and 7 dozen pencils. Ravi purchases 11* *dozen notebooks 13 dozen pens and 8 dozen pencils. A notebook costs 40 paise, a pen costs Rs. 1.25 and a pencil costs 35 paise. Use matrix multiplication to calculate their individual's bill.*

NOTES

Sol. Let A be the matrix of purchases of Ashok, Ramesh and Ravi.

$$\begin{array}{ccc} & \text{Notebooks} & \text{Pens} & \text{Pencils} \\ & \downarrow & \downarrow & \downarrow \end{array}$$

$$\therefore \quad A = \begin{array}{l}\text{Ashok}\\ \text{Ramesh}\\ \text{Ravi}\end{array} \begin{bmatrix} 12\times12 & 5\times12 & 6\times12 \\ 10\times12 & 6\times12 & 7\times12 \\ 11\times12 & 13\times12 & 8\times12 \end{bmatrix} = \begin{bmatrix} 144 & 60 & 72 \\ 120 & 72 & 84 \\ 132 & 156 & 96 \end{bmatrix}.$$

Let B be the matrix of prices in rupees.

$$\therefore \quad B = \begin{array}{l}\text{Notebook}\\ \text{Pen}\\ \text{Pencil}\end{array} \begin{bmatrix} 0.40 \\ 1.25 \\ 0.35 \end{bmatrix}$$

$\therefore$ The matrix of individual's bill is given by :

$$AB = \begin{bmatrix} 144 & 60 & 72 \\ 120 & 72 & 84 \\ 132 & 156 & 96 \end{bmatrix} \begin{bmatrix} 0.40 \\ 1.25 \\ 0.35 \end{bmatrix}$$

$$= \begin{bmatrix} 144(0.40)+60(1.25)+72(0.35) \\ 120(0.40)+72(1.25)+84(0.35) \\ 132(0.40)+156(1.25)+96(0.35) \end{bmatrix} = \begin{bmatrix} 57.60+75+25.20 \\ 48.00+90+29.40 \\ 52.80+195+33.60 \end{bmatrix} = \begin{bmatrix} 157.80 \\ 167.40 \\ 281.40 \end{bmatrix}$$

$\therefore$ Bill of Ashok = ₹ **157.80**, Bill of Ramesh = ₹ **167.40**, Bill of Ravi = ₹ **281.40.**

Example 8. *A trust fund has ₹ 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 among the two types of bonds if the trust fund must obtain an annual total interest of :*

(*i*) *₹ 1,800* (*ii*) *₹ 2,000* (*iii*) *₹ 1,600.*

Sol. Total amount = ₹ 30,000

Let amount invested in first bond = ₹ x

$\therefore$ Amount invested in second bond = ₹ $(30,000 - x)$

Annual interest on first bond = 5% = $\dfrac{5}{100}$ per rupee

Annual interest on second bond = 7% = $\dfrac{7}{100}$ per rupee

$\therefore$ Matrix of investment = $[x \quad 30,000 - x]$

Matrix of annual interest per rupee = $\begin{bmatrix} 5/100 \\ 7/100 \end{bmatrix}$

$\therefore$ Matrix of total annual interest

$$= [x \quad 30,000 - x] \begin{bmatrix} 5/100 \\ 7/100 \end{bmatrix} = \left[\frac{5x}{100} + \frac{7(30000 - x)}{100}\right]$$

$$= \left[\frac{5x + 210000 - 7x}{100}\right] = \left[\frac{210000 - 2x}{100}\right].$$

$\therefore$ Total annual interest = ₹ $\dfrac{210000 - 2x}{100}$.

(*i*) In this case, total annual interest = ₹ 1800

NOTES

$\Rightarrow \quad \dfrac{210000 - 2x}{100} = 1800 \quad \Rightarrow \quad x = 15000.$

$\therefore$ Investments are ₹ **15000** and ₹ 30000 – ₹ 15000 = ₹ **15000.**

(*ii*) In this case, total annual interest = ₹ 2000.

$\Rightarrow \quad \dfrac{210000 - 2x}{100} = 2000 \quad \Rightarrow \quad x = 5000.$

$\therefore$ Investments are ₹ **5000** and ₹ 30000 – ₹ 5000 = ₹ **25000.**

(*iii*) In this case, total annual interest = ₹ 1600.

$\Rightarrow \quad \dfrac{210000 - 2x}{100} = 1600 \quad \Rightarrow \quad x = 25000$

$\therefore$ Investments are ₹ **25000** and ₹ 30000 – ₹ 25000 = ₹ **5000.**

EXERCISE 1.3

1. Find the order of AB if the orders of A and B are respectively :

(*i*) 2 × 2 and 2 × 3 (*ii*) 4 × 1 and 1 × 3 (*iii*) 4 × 4 and 4 × 1.

2. Find AB where :

(*i*) $A = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} -3 & 1 \\ 2 & 5 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix}, B = \begin{bmatrix} 3 & 9 \\ 1 & 5 \end{bmatrix}$.

3. Compute AB if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 7 & 5 \end{bmatrix}$.

4. If $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, find A^2.

5. If $A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$, then show that $A^2 = A$.

6. Evaluate the following :

(*i*) $[a\ b] \begin{bmatrix} c \\ d \end{bmatrix} + [a\ b\ c\ d] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ (*ii*) $[1\ \ -2\ \ 3] \begin{bmatrix} 7 & -1 & 5 \\ 0 & 2 & 4 \\ -7 & 5 & 0 \end{bmatrix} - [2\ \ -5\ \ 7]$.

7. Given that $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$, show that AB = O.

8. Compute AB – BA, where $A = \begin{bmatrix} 2 & 9 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix}$.

9. If $A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, find A^2 and A^3.

10. Express the following as a single matrix : $\begin{bmatrix} 3 & 2 & 5 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 7 & -8 \\ 5 & 9 \end{bmatrix}$.

11. If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 - xA + yI = O$.

NOTES

12. (*i*) If $A = \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$, find $-A^2 + 6A$. (*ii*) If $A = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$, find $-A^2 + 6A$.

13. Show that $AB \neq BA$, where : $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 1 \\ 2 & 3 & 4 \end{bmatrix}$.

14. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find the value $A^2 - 5A + 6I_3$

15. A factory produces three items A, B and C. Annual sales of products are as given below:

City	*Products*		
	A	B	C
Delhi	5,000	1,000	20,000
Bombay	6,000	10,000	8,000

If the unit sale price of the products are ₹ 2.50, ₹ 1.25 and ₹ 1.50 respectively, find the total revenue in each city, with the help of matrices.

16. There are three families. Family A consists of 2 men, 3 women and 1 child. Family B has 2 men, 1 women and 3 children. Family C has 4 men, 2 women and 6 children. Daily income of men and women are ₹ 20 and ₹ 15.50 respectively and children have no income. Using matrix multiplication, calculate daily income of each family.

Answers

1. (*i*) 2×3 (*ii*) 4×3 (*iii*) 4×1

2. (*i*) $\begin{bmatrix} 2 & 22 \\ 0 & 17 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 10 & 38 \\ 10 & 44 \end{bmatrix}$ **3.** $\begin{bmatrix} 20 & 38 & 26 \\ 47 & 92 & 62 \end{bmatrix}$ **4.** I_3

6. (*i*) $[ac + bd + a^2 + b^2 + c^2 + d^2]$ (*ii*) $[-16\ 15\ -10]$

8. $\begin{bmatrix} 43 & 4 \\ 3 & -43 \end{bmatrix}$ **9.** $\begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 11 & -8 & 0 \\ 8 & -1 & 8 \\ 8 & -4 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 15 & 37 \\ -11 & -1 \end{bmatrix}$ **11.** $x = 9$, $y = 14$

12. (*i*) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ **14.** $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$

15. ₹ 43,750 ; ₹ 39,500 **16.** 86.50, ₹ 55.50, ₹ 111.

1.4. TRANSPOSE OF A MATRIX

Let A be a matrix of order $m \times n$. The $n \times m$ matrix obtained from A by interchanging its rows into columns and columns into rows is called the **transpose** of A and is denoted by A′ or by A^T. For example,

$$\text{let } A = \begin{bmatrix} 2 & 3 & 6 & 8 \\ 5 & -3 & -7 & 4 \\ 9 & 8 & 2 & 1 \end{bmatrix}.$$

The transpose of A is the 4×3 matrix $A' = \begin{bmatrix} 2 & 5 & 9 \\ 3 & -3 & 8 \\ 6 & -7 & 2 \\ 8 & 4 & 1 \end{bmatrix}$.

The rows (respectively columns) of A′ are the columns (respectively rows) of the matrix A.

NOTES

1.4.1. Properties of Transpose of A Matrix

1. (A′)′ = A, where A is any matrix.

2. (A + B)′ = A′ + B′, where A and B are matrices of the same order.

3. (kA)′ = kA′, where A is any matrix and k is any number.

4. (AB)′ = B′A′, where A and B are matrices for which AB is defined.

Remark. Property **4** is known as the reversal law for the transpose of the product.

1.4.2. Symmetric Matrix.

A square matrix $A = [a_{ij}]_{n \times n}$ is a symmetric matrix if A′ = A *i.e.*, $a_{ij} = a_{ji}$ $\forall$ *i* and *j e.g.*,

Let $$A = \begin{bmatrix} 3 & 5 & 9 \\ 5 & 6 & 7 \\ 9 & 7 & 11 \end{bmatrix}.$$

Then $$A' = \begin{bmatrix} 3 & 5 & 9 \\ 5 & 6 & 7 \\ 9 & 7 & 11 \end{bmatrix}' = \begin{bmatrix} 3 & 5 & 9 \\ 5 & 6 & 7 \\ 9 & 7 & 11 \end{bmatrix} = A$$

So A is a symmetric matrix.

1.4.3. Skew Symmetric Matrix

A square $A = [a_{ij}]_{n \times n}$ is a skew symmetric matrix if A′ = – A *i.e.*, $a_{ij} = -a_{ji}$ for every *i* and *j e.g.*, consider

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

We see $$A' = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}' = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} = -A$$

So A is a skew symmetric matrix.

Note. In a skew symmetric matrix, the diagonal entries must be zero.

Example 1. *For the matrices*

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 3 \end{bmatrix}.$$

Verify that (AB)′ = B′A′.

Sol. $$AB = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 6 & 8 \\ 10 & 8 & 10 \end{bmatrix}$$

$$\text{L.H.S.} = (AB)' = \begin{bmatrix} 7 & 6 & 8 \\ 10 & 8 & 10 \end{bmatrix}' = \begin{bmatrix} 7 & 10 \\ 6 & 8 \\ 8 & 10 \end{bmatrix}$$

Also, $B' = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 3 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ -1 & 3 \end{bmatrix}$

$$A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\text{R.H.S.} = B'A' = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 6 & 8 \\ 8 & 10 \end{bmatrix} = \text{L.H.S.}$$

so $(AB)' = B'A'$.

Example 2. *For a square matrix* $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, *prove that ;*

(*i*) *$A + A'$ is a symmetric and* (*ii*) *$A - A'$ is a skew symmetric matrix.*

Sol. (*i*) $A + A' = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$

$$(A + A')' = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix} = A + A'$$

so $A + A'$ is symmetric.

(*ii*) $A - A' = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Now $(A - A')' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -(A - A')$

A shows that $(A - A')$ is skew symmetric.

Example 3. *Find* $\frac{1}{2}(A + A')$ *and* $\frac{1}{2}(A - A')$ *when* $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$. *Also write A as sum of* $\frac{1}{2}(A + A')$ *and* $\frac{1}{2}(A - A')$.

Sol. We have $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

$$\therefore \quad A' = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}' = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

$$\therefore \quad A + A' = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

NOTES

Also, $$A - A' = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2a & 2b \\ -2a & 0 & 2c \\ -2b & -2c & 0 \end{bmatrix}$$

$$\therefore \quad \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & 2a & 2b \\ -2a & 0 & 2c \\ -2b & -2c & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{a} & \mathbf{b} \\ -\mathbf{a} & \mathbf{0} & \mathbf{c} \\ -\mathbf{b} & -\mathbf{c} & \mathbf{0} \end{bmatrix}$$

$$\therefore \quad A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

EXERCISE 1.4

1. Find the transpose of the following matrices :

(*i*) $\begin{bmatrix} 4 & 3 \\ 6 & 9 \end{bmatrix}$ (*ii*) $[1\ 6\ 8\ 9]$

(*iii*) $\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 3 & 8 & 8 & 9 \\ -5 & 6 & 10 & 4 \end{bmatrix}$.

2. Verify that $(A')' = A$ where A is

(*i*) $[3\ 4\ 7]$ (*ii*) $\begin{bmatrix} 12 \\ 1 \\ 3 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 2 & 6 & 8 \\ 5 & 3 & 8 \\ 11 & 9 & 7 \end{bmatrix}$ (*iv*) $\begin{bmatrix} -2 & 3 & 6 & 8 \\ 9 & 5 & 4 & 1 \end{bmatrix}$.

3. Show that the following matrices are symmetric matrices :

(*i*) $\begin{bmatrix} 2 & 3 \\ 3 & 11 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -3 & 5 & -6 \\ 5 & 11 & 15 \\ -6 & 15 & 12 \end{bmatrix}$.

4. Show that the following matrices are skew symmetric :

(*i*) $\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & -7 & -15 \\ 7 & 0 & 6 \\ 15 & -6 & 0 \end{bmatrix}$.

5. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, show that $A + A'$ is a symmetric matrix.

6. Verify that $(kA)' = kA'$ where :

(*i*) $A = \begin{bmatrix} 0 & 3 & 7 \\ 1 & 5 & 6 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 1 & 3 \\ 6 & 8 \\ 11 & 5 \end{bmatrix}$.

7. For matrix $A = \begin{bmatrix} -3 & 6 & 0 \\ 4 & -5 & 8 \\ 0 & -7 & -2 \end{bmatrix}$, find $\frac{1}{2}(A - A')$. (*M.D.U. 2004*)

8. Verify $(A + B)' = A' + B'$ for the matrices $A = \begin{bmatrix} -7 & -8 & 6 \\ 8 & 5 & 9 \\ 4 & 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 9 & 11 \\ 1 & 2 & 3 \\ 6 & 8 & 2 \end{bmatrix}$.

NOTES

9. If $A = \begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$, find $(AB)'$.

10. If $A = \begin{bmatrix} 3 & 6 \\ 1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 6 & 8 \end{bmatrix}$, verify that $(5A + 8B)' = 5A' + 8B'$.

Answers

1. (*i*) $\begin{bmatrix} 4 & 6 \\ 3 & 9 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 \\ 6 \\ 8 \\ 9 \end{bmatrix}$ (*iii*) $[4\ 3\ 7]$ (*iv*) $\begin{bmatrix} 3 & -5 \\ 8 & 6 \\ 8 & 10 \\ 9 & 4 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 15/2 \\ 0 & -15/2 & 0 \end{bmatrix}$ **9.** $\begin{bmatrix} 0 & 1 \\ 15 & -2 \end{bmatrix}$.

NOTES

2. DETERMINANTS

STRUCTURE

A matrix is merely an arrangement and has no numerical value. We in the previous chapter learned some operations (addition, scalar multiplication, multiplication of matrices etc.) on matrices. In this chapter we shall assign a numerical value to a square matrix. We shall describe the way, how to assign values to square matrices ? The value which we associate to a square matrix 'A' is called the determinant of A denoted by det A or Δ or | A |. It is to note that we can assign value only to **square matrices.**

Let $$A = [a_{ij}]_{n \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & & \dots & a_{nn} \end{bmatrix},$$

be a matrix of order n. The determinant associated with the matrix A is denoted by

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & & \dots & a_{nn} \end{vmatrix}.$$

2.1. COMPUTATION OF DETERMINANTS

2.1.1. Determinant of a Square Matrix of Order 1

Let $A = [a_{11}]_{1\times1}$, then $|A| = |a_{11}| = a_{11}$ *i.e.*, the value of the determinant of order 1 is the entry of the determinant itself *e.g.*,
if $A = [-6]$, then $|A| = |-6| = -6$.

NOTES

2.1.2. Determinants of Order 2

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ be a determinant of order 2. The value of Δ is

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Explanation : Multiply the elements of Principal diagonal a_{11} and a_{22} and subtract from it the product of the elements of the diagonal from right to left *i.e.,* $a_{21}a_{12}$. The value of the determinant is $a_{11}a_{22} - a_{21}a_{12}$.

Example : $\Delta = \begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} = 1 \times 2 - 3 \times 5 = 2 - 15 = -13.$

2.1.3. Determinant of Order 3

Determinants may be expanded by any row or column. Let $\Delta = |\ a_{ij}\ |_{3\times3}$ be a determinant of order 3 and we expand it by Ist row.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Explanation. Take element a_{11} of the Ist row, multiply it by the determinant obtained by deleting Ist row and Ist column (*i.e.*, the row and column in which the element a_{11} lies) of the determinant Δ. Also multiply the product by $(-1)^{1+1}$, the power of (-1) which is sum of the suffixes of a_{11}. Thus the constituent corresponding to a_{11} is

$$a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$$

The constituent corresponding to the element a_{12} is

$a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and corresponding to third element of the row a_{13} is

$$a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

The value of the determinant is obtained by adding these constituents.

Example 1. *Evaluate the following determinants*

(*i*) $\begin{vmatrix} 4 & 3 \\ 5 & 12 \end{vmatrix}$ (*ii*) $\begin{vmatrix} -3 & -5 \\ 6 & -7 \end{vmatrix}$ (*iii*) $\begin{vmatrix} 2x+3 & x+2 \\ 2x+1 & x+1 \end{vmatrix}$.

Sol. (*i*) $\Delta = \begin{vmatrix} 4 & 3 \\ 5 & 12 \end{vmatrix} = 4 \times 12 - 5 \times 3 = 48 - 15 = 33.$

(*ii*) $\Delta = \begin{vmatrix} -3 & -5 \\ 6 & -7 \end{vmatrix} = (-3) \times (-7) - 6 \times (-5) = 21 + 30 = 51$

(*iii*) $\Delta = \begin{vmatrix} 2x+3 & x+2 \\ 2x+1 & x+1 \end{vmatrix} = (2x + 3)(x + 1) - (2x + 1)(x + 2)$

$= (2x^2 + 3x + 2x + 3) - (2x^2 + x + 4x + 2)$

$= (2x^2 + 5x + 3) - (2x^2 + 5x + 2) = 3 - 2 = 1.$

NOTES

Example 2. *Solve for x:*

$$\begin{vmatrix} x & 3 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 5 & -4 \\ 3 & 2 \end{vmatrix}.$$

Sol. $\begin{vmatrix} x & 3 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 5 & -4 \\ 3 & 2 \end{vmatrix}$

$\Rightarrow \quad 2x - 15 = 10 + 12$

$\Rightarrow \quad 2x = 22 + 15 = 37$

$\Rightarrow \quad x = \dfrac{37}{2}.$

Example 3. *Find the value of* $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$

Sol. We expand the determinant by Ist row.

$$\Delta = (-1)^{1+1}.1.\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + (-1)^{1+2}.2.\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + (-1)^{1+3}3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1.(45 - 48) - 2.(36 - 42) + 3.(32 - 35)$$

$$= 1.(-3) - 2.(-6) + 3.(-3) = -3 + 12 - 9 = 0.$$

EXERCISE 2.1

1. Evaluate the following determinants :

(*i*) $\begin{vmatrix} 4 & 3 \\ 6 & 9 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 5 & 0 \\ 6 & 7 \end{vmatrix}$ (*iii*) $\begin{vmatrix} -18 & 9 \\ 27 & 15 \end{vmatrix}$ (*iv*) $\begin{vmatrix} 4 & -5 \\ 0 & 6 \end{vmatrix}$.

2. Solve for x :

(*i*) $\begin{vmatrix} 2x & 5 \\ 1 & 3 \end{vmatrix} = 7$ (*ii*) $\begin{vmatrix} 4 & x \\ -3 & 5 \end{vmatrix} = 8.$

3. Evaluate :

(*i*) $\begin{vmatrix} 5 & 2 & 1 \\ 3 & 0 & 2 \\ 8 & 1 & 3 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 2 & 0 & 0 \\ 5 & 9 & 3 \\ 1 & 6 & 7 \end{vmatrix}$ (*iii*) $\begin{vmatrix} 6 & 1 & -3 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{vmatrix}$.

Answers

1. (*i*) 18 (*ii*) 35 (*iii*) – 513 (*iv*) 24

2. (*i*) $x = 2$, (*ii*) $x = -4$,

3. (*i*) 7 (*ii*) 90 (*iii*) 91.

2.2. MINORS AND COFACTORS

2.2.1. Minors

Minor of an element a_{ij} in a square matrix $A = [a_{ij}]_{n \times n}$ is the determinant obtained by deleting the i-th row and j-th column of the determinant of the matrix A. We denote the minor of an element lying at (i, j)th position by M_{ij}.

NOTES

Thus, we see that, minor is obtained by removing the row and column in which the element lies from the determinant. We can talk of the minor of element of a matrix as well as of a determinant.

Example : Let $|A| = \begin{vmatrix} 4 & 3 & 8 \\ 6 & 7 & 5 \\ 3 & 1 & 2 \end{vmatrix}$, then

Minor of a_{11} (= 4) ; $M_{11} = \begin{vmatrix} 7 & 5 \\ 1 & 2 \end{vmatrix} = 14 - 5 = 9$

Minor of a_{12} (= 3) ; $M_{12} = \begin{vmatrix} 6 & 5 \\ 3 & 2 \end{vmatrix} = 12 - 15 = -3$ and so on.

2.2.2. Cofactors

Minors with proper sign are called cofactors. If M_{ij} is the minor of (i, j)th position of a square matrix, then the cofactor A_{ij} is given by

$A_{ij} = (-1)^{i+j} M_{ij}$ *e.g.,* in the example above,

$$A_{11} = (-1)^{1+1} M_{11} = 9.$$

$$A_{12} = (-1)^{1+2} M_{12} = -(-3) = 3.$$

Note. It is clear that the value of $A = |a_{ij}|_{3 \times 3}$ in terms of cofactors is

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

2.2.3. Properties of Determinants

I. If we interchange the rows into columns and columns into rows of a determinant, the value of the determinant remains unchanged *e.g.*,

Let $\Delta = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$

$\therefore$ $\Delta = 1 \times 4 - 2 \times 3 = 4 - 6 = -2$

Now interchange rows into columns and columns into rows, new determinant is

$$\Delta_1 = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

We see that $\Delta_1 = \Delta$.

II. If we interchange any two rows (or columns) of a determinant, the value of the determinant is multiplied by (– 1) *e.g.*,

Let $\Delta = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

and let Δ' be the determinant obtained by interchanging the rows *i.e.,*

$$\Delta' = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

$$\Delta' = 6 - 4 = 2,$$

But $\Delta = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

So $\Delta' = -\Delta$.

NOTES

III. If two rows or columns of a determinant are identical, then the value of the determinant is zero *e.g.,*

Let $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 3 \\ 2 & 0 & 2 \end{vmatrix}$, here 1st and IIIrd column are identical.

Expand it by Ist column

$$\Delta = 1(2-0) - 2(6-6) + 1(0-2) = 2 - 0 - 2 = 0.$$

IV. If we multiply any row (or column) of a determinant by a scalar λ(say) then the value of the new determinant is λ times of the value of the original determinant *e.g.*,

Let $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ 2 & 0 & 1 \end{vmatrix} = 1(1-0) - 2(3-10) + 1(0-2) = 1 + 14 - 2 = 13.$

Let us multiply IInd column of the determinant by 5, and denote the new determinant by Δ', then

$$\Delta' = \begin{vmatrix} 1 & 2\times 5 & 1 \\ 3 & 1\times 5 & 5 \\ 2 & 0\times 5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 10 & 1 \\ 3 & 5 & 5 \\ 2 & 0 & 1 \end{vmatrix} = 1(5-0) - 3(10-0) + 2(50-5)$$

$$= 5 - 30 + 90 = 65$$

So $\Delta' = 65 = 5 \times 13 = 5.\Delta.$

V. If we add a multiple of a row (or column) to any other row (or column), the value of the determinant remains unchanged *e.g.,*

Let $D = \begin{vmatrix} 1 & 5 & 7 \\ 6 & 7 & 2 \\ 1 & 2 & 3 \end{vmatrix}.$

$\therefore$ $\Delta = 1\begin{vmatrix} 7 & 2 \\ 2 & 3 \end{vmatrix} - 5\begin{vmatrix} 6 & 2 \\ 1 & 3 \end{vmatrix} + 7\begin{vmatrix} 6 & 7 \\ 1 & 2 \end{vmatrix}$

$$= (21-4) - 5(18-2) + 7(12-7) = 17 - 80 + 35 = -28.$$

Let Δ' be obtained by operating $R_2 \to R_2 - 3R_1 + 4R_3$.

$\therefore$ $\Delta' = \begin{vmatrix} 1 & 5 & 7 \\ 6-3(1)+4(1) & 7-3(5)+4(2) & 2-3(7)+4(3) \\ 1 & 2 & 3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 5 & 7 \\ 7 & 0 & -7 \\ 1 & 2 & 3 \end{vmatrix} = 1\begin{vmatrix} 0 & -7 \\ 2 & 3 \end{vmatrix} - 5\begin{vmatrix} 7 & -7 \\ 1 & 3 \end{vmatrix} + 7\begin{vmatrix} 7 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (0 + 14) - 5(21 + 7) + 7(14 - 0) = 14 - 140 + 98 = -28.$$

$\therefore$ $\Delta' = \Delta.$

VI. If each element of any row (or column) of a determinant is the sum of two numbers, then the determinant is expressible as the sum of two determinants of the same order.

Illustration. (*i*) $\Delta = \begin{vmatrix} 2+4 & 6+5 \\ 3 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 3 & 9 \end{vmatrix}$ (Row wise addition)

$$\text{L.H.S.} = \begin{vmatrix} 6 & 11 \\ 3 & 9 \end{vmatrix} = 6(9) - 3(11) = 54 - 33 = 21$$

$$\text{R.H.S.} = \begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 3 & 9 \end{vmatrix} = [2(9) - 3(6)] + [4(9) - 3(5)]$$

$$= (18 - 18) + (36 - 15) = 21$$

$$\therefore \quad \begin{vmatrix} 2+4 & 6+5 \\ 3 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 3 & 9 \end{vmatrix}.$$

NOTES

(*ii*) Similarly, $\begin{vmatrix} 2+4 & 3 \\ 6+5 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 5 & 9 \end{vmatrix}$ (Column wise addition)

$$\text{L.H.S.} = \begin{vmatrix} 6 & 3 \\ 11 & 9 \end{vmatrix} = 6 \times 9 - 11 \times 3 = 54 - 33 = 21$$

R.H.S. = $[2 \times 9 - 6 \times 3] + [4 \times 9 - 5 \times 3] = [18 - 18] + [36 - 15] = 0 + 21 = 21.$

Example 1. *Find all the minors and cofactors of the elements in* $\begin{vmatrix} 4 & 3 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 5 \end{vmatrix}$.

Sol. Let $a_{11} = 4$, $a_{12} = 3$, $a_{13} = 1$

$a_{21} = 1$, $a_{22} = 3$, $a_{23} = 2$

$a_{31} = 2$, $a_{32} = 1$, $a_{33} = 5$.

$$M_{11} = \text{minor of } a_{11} \ (= 4) = \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = 15 - 2 = \mathbf{13}$$

$$M_{12} = \text{minor of } a_{12} \ (= 3) = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = \mathbf{1}$$

$$M_{13} = \text{minor of } a_{13} \ (= 1) = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = \mathbf{-5}$$

$$M_{21} = \text{minor of } a_{21} \ (= 1) = \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 15 - 1 = \mathbf{14}$$

$$M_{22} = \text{minor of } a_{22} \ (= 3) = \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} = 20 - 2 = \mathbf{18}$$

$$M_{23} = \text{minor of } a_{23} \ (= 2) = \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 4 - 6 = \mathbf{-2}$$

$$M_{31} = \text{minor of } a_{31} \ (= 2) = \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 6 - 3 = \mathbf{3}$$

$$M_{32} = \text{minor of } a_{32} \ (= 1) = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 8 - 1 = \mathbf{7}$$

$$M_{33} = \text{minor of } a_{33} \ (= 5) = \begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix} = 12 - 3 = \mathbf{9.}$$

The following are the cofactors :

A_{11} = cofactor of a_{11} (= 4) = $(-1)^{1+1} M_{11} = 1 \times 13 = \mathbf{13}$

A_{12} = cofactor of a_{12} (= 3) = $(-1)^{1+2} M_{12} = -1 \times 1 = \mathbf{-1}$

A_{13} = cofactor of a_{13} (= 1) = $(-1)^{1+3} M_{13} = 1 \times (-5) = \mathbf{-5}$

A_{21} = cofactor of a_{21} (= 1) = $(-1)^{2+1} M_{21} = -1 \times 14 = \mathbf{-14}$

A_{22} = cofactor of a_{22} (= 3) = $(-1)^{2+2} M_{22} = 1 \times 18 = \mathbf{18}$

A_{23} = cofactor of a_{23} (= 2) = $(-1)^{2+3} M_{23} = -1 \times (-2) = \mathbf{2}$

NOTES

A_{31} = cofactor of a_{31} (= 2) = $(-1)^{3+1}$ M_{31} = 1 × 3 = **3**

A_{32} = cofactor of a_{32} (= 1) = $(-1)^{3+2}$ M_{32} = – 1 × 7 = – **7**

A_{33} = cofactor of a_{33} (= 5) = $(-1)^{3+3}$ M_{33} = 1 × 9 = **9.**

Example 2. *Evaluate the following determinants :*

(*i*) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ (*ii*) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$.

Sol. (*i*) $$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Operate $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-bc \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & b(a-c) \end{vmatrix}$$

Taking $(b - a)$ and $(c - a)$ common from R_2 and R_3 respectively, we get

$$\Delta = (b-a)\,(c-a) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix}$$

$$= (b-a)\,(c-a)\,[1 \times (-b) - 1 \times (-c)]$$

($\because$ expanding by C_1)

$$= (b-a)\,(c-a)\,(-b+c) = (a-b)\,(b-c)\,(c-a)$$

(*ii*) $$\Delta = \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$$

Operate $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} 1 & x & x^3 \\ 0 & y-x & y^3-x^3 \\ 0 & z-x & z^3-x^3 \end{vmatrix} = \begin{vmatrix} 1 & x & x^3 \\ 0 & y-x & (y-x)(y^2+yx+x^2) \\ 0 & z-x & (z-x)(z^2+zx+x^2) \end{vmatrix}$$

$$= (y-x)\,(z-x) \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & y^2+yx+x^2 \\ 0 & 1 & z^2+zx+x^2 \end{vmatrix}$$

$$= (y-x)\,(z-x)\,[z^2+zx+x^2-(y^2+yx+x^2)]$$

$$= (y-x)\,(z-x)\,(z^2+zx+x^2-y^2-yx-x^2)$$

$$= (y-x)\,(z-x)\,(z^2+zx-y^2-yx)$$

$$= (y-x)\,(z-x)\,(z^2-y^2+zx-yx)$$

$$= (y-x)\,(z-x)\,[(z-y)\,(z+y)+x(z-y)]$$

$$= (y-x)\,(z-x)\,[(z-y)\,(z+y+x)]$$

$$= (y-x)\,(z-x)\,(z-y)\,(x+y+z)$$

$$= (x-y)\,(y-z)\,(z-x)\,(x+y+z).$$

Example 3. *Show that* $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

NOTES

Sol. Let $\Delta = \begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$, Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \end{vmatrix}.$$

Taking 2 common from C_1, we get

$$\Delta = 2\begin{vmatrix} a+b+c & b+c & c+a \\ a+b+c & c+a & a+b \\ a+b+c & a+b & b+c \end{vmatrix}.$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = 2\begin{vmatrix} a+b+c & -a & -b \\ a+b+c & -b & -c \\ a+b+c & -c & -a \end{vmatrix} = 2(-1)(-1)\begin{vmatrix} a+b+c & a & b \\ a+b+c & b & c \\ a+b+c & c & a \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2 - C_3$, we get

$$\Delta = 2\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix} = -2\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} \qquad (C_1 \leftrightarrow C_2)$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}. \qquad (C_2 \leftrightarrow C_3)$$

Example 3. *Show that*

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$$

Sol. L.H.S. $\Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Operate $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Operate $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & b+c-a & c+a+b \end{vmatrix}$$

$$= 2(a+b+c)\,(a+b+c)^2 \qquad \text{(Expanding by } C_1\text{)}$$

$$= 2(a+b+c)^3 = \text{R.H.S.}$$

NOTES

Example 4. *Show that* $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2.$

Sol. Let $$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

Taking a, b, c common from R_1, R_2, R_3 respectively, we get

$$\therefore \quad \Delta = a.b.c \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} = (abc)(abc) \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

(Taking a, b, c common from C_1, C_2, C_3 respectively)

$$= a^2b^2c^2 \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} \qquad (C_2 \to C_2 + C_1, C_3 \to C_3 + C_1)$$

$$= a^2b^2c^2 \left[-1 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} - 0 + 0 \right] = a^2b^2c^2 \times 4 = \mathbf{4a^2b^2c^2}.$$

Example 5. *Show that* $\begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix} = (a^3 + b^3)^2.$ (M.D.U. 2003)

Sol. Let $$\Delta = \begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix}.$$

Operating $R_1 \to R_1 + R_2 + R_3$, we get $\Delta = \begin{vmatrix} (a+b)^2 & (a+b)^2 & (a+b)^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix}$

$$\therefore \quad \Delta = (a+b)^2 \begin{vmatrix} 1 & 1 & 1 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix}$$

Operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = (a+b)^2 \begin{vmatrix} 1 & 0 & 0 \\ b^2 & a^2 - b^2 & 2ab - a^2 \\ 2ab & b^2 - 2ab & a^2 - b^2 \end{vmatrix}$$

Expanding along R_1, we get $\Delta = (a+b)^2 \left[1. \begin{vmatrix} a^2 - b^2 & 2ab - a^2 \\ b^2 - 2ab & a^2 - b^2 \end{vmatrix} - 0 + 0 \right]$

$$= (a+b)^2 [(a^2 - b^2)^2 - (2ab - a^2)(b^2 - 2ab)]$$
$$= (a+b)^2 [a^4 + b^4 - 2a^2b^2 - 2ab^3 + a^2b^2 + 4a^2b^2 - 2a^3b]$$
$$= (a+b)^2 [a^4 + b^4 + a^2b^2 - 2ab^3 - 2a^3b + 2a^2b^2]$$
$$= (a+b)^2 (a^2 - ab + b^2)^2 = \mathbf{(a^3 + b^3)^2}.$$

NOTES

EXERCISE 2.2

1. Without expanding, show that each of the following determinants is equal to zero :

(i) $\begin{vmatrix} 3 & 8 & 6 \\ 6 & 15 & 12 \\ 7 & 17 & 14 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 10 & 11 \\ 1 & 12 & 13 \\ 1 & 9 & 10 \end{vmatrix}$ (iii) $\begin{vmatrix} 9 & 9 & 18 \\ 1 & -3 & -2 \\ 1 & 9 & 10 \end{vmatrix}$.

2. Without expanding, show that each of the following determinants is equal to zero :

(i) $\begin{vmatrix} 1 & a & abc \\ 1 & b & abc \\ 1 & c & abc \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ (iii) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$.

3. Write the minors and co-factors of the second row of the following denominants and hence evaluate them

(i) $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$.

4. Show that :

(i) $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2 (x+2a)$ (ii) $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$.

5. Show that :

(i) $\begin{vmatrix} x+9 & x & x \\ x & x+9 & x \\ x & x & x+9 \end{vmatrix} = 243(x+3)$ (ii) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$.

6. Show that :

(i) $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

(ii) $\begin{vmatrix} a-b & b-c & c-a \\ b+c & c+a & a+b \\ a & b & c \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$.

7. Show that :

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$ (ii) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$.

8. Show that :

(i) $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

9. Show that :

(i) $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$ (ii) $\begin{vmatrix} 3x+y & 2x & x \\ 4x+3y & 3x & 3x \\ 5x+6y & 4x & 6x \end{vmatrix} = x^3$.

NOTES

10. Solve the equations :

(i) $\begin{vmatrix} x^2 & 0 & 3 \\ x & 1 & -4 \\ 1 & 2 & 0 \end{vmatrix} = 11$ (ii) $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0.$

Answers

3. (i) $M_{21} = 39, M_{22} = 3, M_{23} = -11, A_{21} = -39, A_{22} = 3, A_{23} = 11, \Delta = 231.$

(ii) $M_{21} = (a^2b - bc^2), M_{22} = (ab - bc), M_{23} = c - a, A_{21} = -(a^2b - bc^2), A_{22} = b(a - c),$
$A_{23} = a - c, \Delta = bc^2 - a^2b + ab^2 - b^2c + a^2c - ac^2.$

10. (i) $-\frac{7}{4}$, 1 (ii) 0, $3a$.

2.3. APPLICATIONS OF DETERMINANTS IN SOLVING A SYSTEM OF LINEAR EQUATIONS

2.3.1. Consistency

A system of linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad \text{....}(i)$$

is **consistent** if it has a solution *i.e.*, if we can find x, y, z such that the equations in system (i) are satisfied. Otherwise the system is called **inconsistent** *e.g.*, the system

$x + y = 3$

$x - 2y = 0$ have $x = 2, y = 1$ as a solution so is consistent.

2.3.2. Cramer's Rule (Solution of Equations Using Determinants)

Let
$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

be a system of linear equations in three variables, x, y, z :

Take $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, the determinant of coefficients of the linear equations

The system has a unique solution if and only if $\Delta \neq 0$. If $\Delta = 0$, then the system has either infinitely many solutions or is inconsistent.

Now replace Ist column of the determinant Δ, by the constant terms of the R.H.S. and call the determinant Δ_1 *i.e.*,

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

NOTES

Also write $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ *i.e.*, replace IInd column by d_1, d_2, d_3,

and $$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

Then $$x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta},\ z = \frac{\Delta_3}{\Delta}.$$

Example 1. *Solve the system of equations by Cramer's rule.*

(*i*) $2x + 3y = 5$ (*ii*) $3x + 4y - 7 = 0$

$3x - 2y = 1$ $6x + 8y - 2 = 0.$

Sol. (*i*) The determinant of coefficient of the system is

$$\Delta = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -4 - 9 = -13 \neq 0 \text{ the system is consistent}$$

$$\Delta_1 = \begin{vmatrix} 5 & 3 \\ 1 & -2 \end{vmatrix},$$

the determinant obtained by replacing Ist column with the constant.

$$= -10 - 3 = -13$$

$$\Delta_2 = \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = 2 - 15 = -13.$$

$$\therefore \quad x = \frac{\Delta_1}{\Delta} = \frac{-13}{-13} = 1, \quad y = \frac{\Delta_2}{\Delta} = \frac{-13}{-13} = 1$$

So $x = 1, y = 1$ is a solution.

(*ii*) The equations are

$$3x + 4y = 7 \qquad \ldots(i)$$

$$6x + 8y = 2 \qquad \ldots(ii)$$

$$\Delta = \begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} = 24 - 24 = 0$$

so the system either have no solution or infinitely many solutions. Multiply eq. (*i*) by 2, we get

$$6x + 8y = 14, \quad \text{also from } (ii)$$

$$6x + 8y = 2$$

it shows that $14 = 2$, not possible so the system is inconsistent.

Example 2. *Solve the system of equations by means of determinants*

$$x + 2y + 3z = 6$$

$$2x + 4y + z = 7$$

$$3x + 2y + 9z = 14.$$

Sol. $$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix} = 1(36 - 2) - 2(18 - 3) + 3(4 - 12)$$

$$= 34 - 30 - 24 = -20 \neq 0$$

so the system is consistent and has unique solution.

$$\Delta_1 = \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix} = 6(36-2) - 7(18-6) + 14(2-12)$$

$$= 6 \times 34 - 7 \times 12 + 14(-10)$$

$$= 204 - 84 - 140 = 204 - 224 = -20$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} = 1(63-14) - 2(54-42) + 3(6-21)$$

$$= 49 - 24 - 45 = 49 - 69 = -20$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} = 1(56-14) - 2(28-12) + 3(14-24)$$

$$= 42 - 32 - 30 = -20$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-20}{-20} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-20}{-20} = 1$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-20}{-20} = 1.$$

Example 3. *The sum of three numbers is 20. If we multiply the first number by 2 and add the second number to the result and subtract the third number, we get 23. By adding second and the third numbers to three times the first number, we get 46. Find the numbers by using determinants.*

Sol. Let the numbers be x, y, z.

∴ By the given conditions,

$$x + y + z = 20$$

$$2x + y - z = 23$$

$$3x + y + z = 46.$$

Using **Cramer's rule,** we have

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix} = 1(1+1) - 1(2+3) + 1(2-3)$$

$$= 2 - 5 - 1 = -4 \neq 0.$$

$$\Delta_1 = \begin{vmatrix} 20 & 1 & 1 \\ 23 & 1 & -1 \\ 46 & 1 & 1 \end{vmatrix} = 20(1+1) - 1(23+46) + 1(23-46)$$

$$= 40 - 69 - 23 = -52.$$

$$\Delta_2 = \begin{vmatrix} 1 & 20 & 1 \\ 2 & 23 & -1 \\ 3 & 46 & 1 \end{vmatrix} = 1(23+46) - 20(2+3) + 1(92-69)$$

$$= 69 - 100 + 23 = -8.$$

NOTES

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 20 \\ 2 & 1 & 23 \\ 3 & 1 & 46 \end{vmatrix} = 1(46 - 23) - 1(92 - 69) + 20(2 - 3)$$

$$= 23 - 23 - 20 = -20.$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-52}{-4} = 13,\ y = \frac{\Delta_2}{\Delta} = \frac{-8}{-4} = 2,\ \frac{\Delta_3}{\Delta} = \frac{-20}{-4} = 5.$$

$\therefore$ The numbers are **13, 2, 5.**

EXERCISE 2.3

1. Solve the following simultaneous linear equations using determinants, if consistant.

(*i*) $2x - 5y = 5$, $3x + 2y = -3$ (*ii*) $x - 2y = 4,\ -3x + 5y = -7$.

2. Solve the following linear equations using determinants :

$$2x - 3y + 4z = -9,\ -3x + 4y + 2z = -12,\ 4x - 2y - 3z = -3.$$

3. Solve the system of equations by Cramer's rule :

(*i*) $x + y + z = 9$, $2x + 5y + 7z = 52$ (*M.D.U. 2003*), $2x + y - z = 0$

(*ii*) $x + 3y + 5z - 22 = 0$, $5x - 3y + 2z - 5 = 0$, $9x + 8y - 3z - 16 = 0$

(*iii*) $6x + y - 3z = 5$, $x + 3y - 2z = 5$, $2x + y + 4z = 8$.

4. The equilibrium condition for three related markets are given by the equations :

$$p_1 = \frac{2}{3}p_2 - p_3 + \frac{2}{3};$$

$$p_2 = \frac{17}{6}p_1 + \frac{5}{6}p_3 - \frac{1}{3}$$

and $$p_3 = p_2 - p_1 - 4.$$

Find equilibrium price for each market by using determinants method.

5. The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times the first number, we get 12. Use determinants to find the numbers.

6. To control a certain crop disease it is necessary to use 7 units of chemical A, 10 units of chemical B, and 6 units of chemical C, one barrel of spray P contains 1, 4, 2 units of the chemical, one barrel of spray Q contains 3, 2, 2 units and one barrel of spray R contains 4, 3, 2 units of these chemicals respectively. How much of each type of spray be used to control diseases.

Answers

1. (*i*) $x = -\frac{5}{19}, y = -\frac{21}{19}$ (*ii*) $x = -6, y = -5$

2. $x = -\frac{684}{53}, y = -\frac{321}{53}, z = -\frac{189}{53}$ **3.** (*i*) $x = 1, y = 3, z = 5$

(*ii*) $x = 1, y = 2, z = 3$ (*iii*) $x = 1, y = 2, z = 1$.

4. $p_1 = 8/3, p_2 = 14, p_3 = \frac{22}{3}$. **5.** $x = 3, y = 2, z = 1$

6. P = 3/2, Q = 1/2, R = 1

NOTES

3. ADJOINT AND INVERSE OF MATRICES

STRUCTURE

3.1. ADJOINT AND INVERSE OF MATRICES

3.1.1. Adjoint of a Matrix

Adjoint of **square matrix** is defined as the transpose of the matrix obtained by replacing the elements of the matrix by their respective cofactors. We denote the adjoint of a square matrix 'A' by adj A.

Let $$A = [a_{ij}]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & & & \\ \vdots & \vdots & & & \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix},$$

be a square matrix, then

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & \cdots & A_{2n} \\ \vdots & & & & \\ A_{n1} & \cdots & \cdots & \cdots & A_{nn} \end{bmatrix}$$, where A_{ij}'s are the cofactors of a_{ij}.

Note. Adjoint for non-square matrices is not defined. We state a theorem without proof.

3.1.2. Theorem

For a square matrix A of order '*n*', **A (adj A) = (adj A) A = |A| I_n**, where I_n is the identity matrix of order '*n*'.

(we use the theorem without proof.).

Example 1. *Find the adjoint of the matrix* $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$.

Sol. $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$. We find all cofactors :

$$A_{11} = \text{cofactor of } a_{11} (= 2) = (-1)^{1+1} \mid -1 \mid = -1$$
$$A_{12} = \text{cofactor of } a_{12} (= 1) = (-1)^{1+2} \mid 4 \mid = -4$$
$$A_{21} = \text{cofactor of } a_{21} (= 4) = (-1)^{2+1} \mid 1 \mid = -1$$
$$A_{22} = \text{cofactor of } a_{22} (= -1) = (-1)^{2+2} \mid 2 \mid = 2.$$

$$\therefore \qquad \text{adj } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}' = \begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}' = \begin{bmatrix} \mathbf{-1} & \mathbf{-1} \\ \mathbf{-4} & \mathbf{2} \end{bmatrix}.$$

Example 2. *Find adjoint of A, where* $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$.

Sol. $$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

We know that adj A is the transpose of the matrix obtained by replacing the elements of A by their corresponding cofactors.

$$A_{11} = \text{cofactor of } a_{11} (= 1) = (-1)^{1+1} \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$A_{12} = \text{cofactor of } a_{12} (= -1) = (-1)^{1+2} \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2 + 10) = -12$$

$$A_{13} = \text{cofactor of } a_{13} (= 2) = (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$$

$$A_{21} = \text{cofactor of } a_{21} (= 2) = (-1)^{2+1} \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1 - 0) = 1$$

$$A_{22} = \text{cofactor of } a_{22} (= 3) = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 + 4 = 5$$

$$A_{23} = \text{cofactor of } a_{23} (= 5) = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0 - 2) = 2$$

$$A_{31} = \text{cofactor of } a_{31} (= -2) = (-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11$$

$$A_{32} = \text{cofactor of } a_{32} (= 0) = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5 - 4) = -1$$

$$A_{33} = \text{cofactor of } a_{33} (= 1) = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2 = 5.$$

$$\therefore \qquad \text{adj } A = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}' = \begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{-11} \\ \mathbf{-12} & \mathbf{5} & \mathbf{-1} \\ \mathbf{6} & \mathbf{2} & \mathbf{5} \end{bmatrix}.$$

NOTES

Example 3. *If* $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, *verify that* $A(adj\ A) = (adj\ A)A = |A|\ I_3$.

Sol. We have $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

$$A_{11} = \text{cofactor of } a_{11}(= 2) = (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$$

$$A_{12} = \text{cofactor of } a_{12}(= 1) = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = -(9 - 2) = -7$$

$$A_{13} = \text{cofactor of } a_{13}(= 3) = (-1)^{1+3} \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 6 - 1 = 5$$

$$A_{21} = \text{cofactor of } a_{21}(= 3) = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -(3 - 6) = 3$$

$$A_{22} = \text{cofactor of } a_{22}(= 1) = (-1)^{2+2} \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 6 - 3 = 3$$

$$A_{23} = \text{cofactor of } a_{23}(= 2) = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -(4 - 1) = -3$$

$$A_{31} = \text{cofactor of } a_{31}(= 1) = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 2 - 3 = -1$$

$$A_{32} = \text{cofactor of } a_{32}(= 2) = (-1)^{3+2} \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -(4 - 9) = 5$$

$$A_{33} = \text{cofactor of } a_{33}(= 3) = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 2 - 3 = -1.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -1 & -7 & 5 \\ 3 & 3 & -3 \\ -1 & 5 & -1 \end{bmatrix}' = \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}.$$

$$\therefore \quad A(\text{adj } A) = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2-7+15 & 6+3-9 & -2+5-3 \\ -3-7+10 & 9+3-6 & -3+5-2 \\ -1-14+15 & 3+6-9 & -1+10-3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

$$(\text{adj } A)A = \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

$$= \begin{bmatrix} -2+9-1 & -1+3-2 & -3+6-3 \\ -14+9+5 & -7+3+10 & -21+6+15 \\ 10-9-1 & 5-3-2 & 15-6-3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 2(3-4) - 1(9-2) + 3(6-1)$$

$$= -2 - 7 + 15 = 6.$$

$$\therefore \qquad |A| I_3 = 6\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}6 & 0 & 0\\0 & 6 & 0\\0 & 0 & 6\end{bmatrix}.$$

$$\therefore \qquad \mathbf{A(adj\ A) = (adj\ A)A = |\ A\ |\ I_3.}$$

EXERCISE 3.1

1. Find the adjoint of the following square matrices :

(*i*) $\begin{bmatrix}1 & 0\\5 & 0\end{bmatrix}$ (*ii*) $\begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}$.

2. Find the adjoint of the following square matrices :

(*i*) $\begin{bmatrix}7 & 4\\3 & -6\end{bmatrix}$ (*ii*) $\begin{bmatrix}a & b\\c & d\end{bmatrix}$.

3. Compute the adjoint of the matrix $A = \begin{bmatrix}3 & 4\\5 & 7\end{bmatrix}$ and verify that (adj A) A = | A | I.

4. If $A = \begin{bmatrix}5 & -2\\3 & -2\end{bmatrix}$, verify that A(adj A) = (adj A) A = | A | I_2.

5. If $A = \begin{bmatrix}1 & 2\\3 & 5\end{bmatrix}$ and $B = \begin{bmatrix}2 & 0\\1 & 5\end{bmatrix}$, then verify that adj (AB) = (adj B)(adj A).

6. If $A = \begin{bmatrix}3 & 6\\-5 & 11\end{bmatrix}$ and $B = \begin{bmatrix}-7 & 8\\9 & 11\end{bmatrix}$, then verify that adj (BA) = (adj A)(adj B).

7. If $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, find det (A (adj A)).

8. Find the adjoint of the following square matrices :

(*i*) $\begin{bmatrix}1 & 2 & 3\\0 & 5 & 0\\2 & 4 & 3\end{bmatrix}$ (*ii*) $\begin{bmatrix}1 & -1 & 2\\2 & 3 & 5\\-2 & 0 & 1\end{bmatrix}$.

9. If $A = \begin{bmatrix}-1 & -2 & -2\\2 & 1 & -2\\2 & -2 & 1\end{bmatrix}$, show that adj A = 3A′.

10. For the matrix $A = \begin{bmatrix}1 & -1 & 1\\2 & 3 & 0\\18 & 2 & 10\end{bmatrix}$, show that A(adj A) = O.

11. Find the adjoint of the matrix $\begin{bmatrix}1 & 2 & 3\\2 & 3 & 2\\3 & 3 & 4\end{bmatrix}$. Also verify that

A(adj A) = (adj A)A = | A | I, where A is the given matrix.

12. If $A = \begin{bmatrix}1 & 2 & 3\\0 & 5 & 0\\2 & 4 & 3\end{bmatrix}$, verify that A(adj A) = | A | I_3 = (adj A)A.

Answers

1. (*i*) $\begin{bmatrix}0 & 0\\-5 & 1\end{bmatrix}$ (*ii*) $\begin{bmatrix}4 & 0\\0 & 2\end{bmatrix}$ **2.** (*i*) $\begin{bmatrix}-6 & -4\\-3 & 7\end{bmatrix}$ (*ii*) $\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$

3. $\begin{bmatrix}7 & -4\\-5 & 3\end{bmatrix}$ **7.** $(ad - bc)^2$

NOTES

8. (*i*) $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$

11. $\begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$.

3.2. INVERSE OF A MATRIX

Let A be a square matrix of order n. If there exists a square matrix B of order n such that $AB = BA = I_n$, then B is called the **inverse** of A.

It is clear from the definition that if B is inverse of A, then A is inverse of B. The concept of inverse of matrix is defined only for square matrices.

If inverse of a matrix A exists, then A is called an **invertible matrix.**

3.2.1. Theorem

The inverse of a square matrix, if it exists, is unique.

Proof. Let A be a square matrix of order n such that inverse of A exists.

Let B and C be any two inverses of A.

$\therefore$ By definition, $AB = BA = I_n$ and $AC = CA = I_n$.

We have $B = BI_n = B(AC) = (BA)C = I_nC = C$.

$\therefore$ $B = C$.

$\therefore$ Any two inverses of A are equal matrices.

$\therefore$ The inverse of A is unique.

Notation. The inverse of an invertible matrix A is denoted by A^{-1}.

3.2.2. Theorem

If A and B are invertible matrices of order n, then show that AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. The matrices A and B are invertible.

$\therefore$ A^{-1} and B^{-1} exists and $AA^{-1} = A^{-1}A = I_n$, $BB^{-1} = B^{-1}B = I_n$.

A and B are square matrices of order n, therefore AB is defined.

Also $| AB | = | A | | B | \neq 0$, because A and B are invertible and so $| A | \neq 0$, $| B | \neq 0$.

$\therefore$ AB is invertible, *i.e.*, $(AB)^{-1}$ exists.

Now $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$

and $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I_n)B = B^{-1}B = I_n$.

$\therefore$ We have $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I_n$.

$\therefore$ By definition, $\mathbf{(AB)^{-1} = B^{-1}A^{-1}}$.

3.2.3 Singular and Non-singular Matrices

Definition (**Non-Singular matrix**) : A square matrix 'A' is called a non-singular matrix if and only if $|A| \neq 0$.

If $|A| = 0$, then the matrix A is called **singular**.

e.g., Let $A = \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$, since $|A| = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7 \neq 0$ so A is non-singular.

3.2.4. Theorem

A square matrix is invertible if and only if it is non-singular.

Proof. Let A be a square matrix of order n. Suppose A is non-singular. Then $|A| \neq 0$.

We know $A(\text{adj. } A) = |A| . I_n = (\text{adj } A) . A$

Take 1st part $A.(\text{Adj } A) = |A| . I_n$

Since $|A| \neq 0$ so divide by $|A|$

$$A . \frac{(\text{adj. } A)}{|A|} = I_n \quad ...(i)$$

Take 2nd part

$$|A| I_n = (\text{adj. } A) . A$$

Again divide by $|A|$

$$I_n = \frac{(\text{adj. } A)}{|A|} . A \quad ...(ii)$$

Combine (*i*) and (*ii*);

$$A. \frac{(\text{adj. } A)}{|A|} = I_n = \frac{(\text{adj. } A)}{|A|} . A$$

$$\therefore \quad A^{-1} = \frac{(\text{adj. } A)}{|A|}$$

Conversely, suppose that A is invertible *i.e.,* inverse of A exists. Let B be the inverse of A. Then

$$AB = I = BA \quad \text{(by definition of inverse)}$$

$$\Rightarrow \quad |AB| = |I|$$

$$\Rightarrow \quad |A| . |B| = 1$$

$$\Rightarrow \quad |A| \neq 0$$

$\Rightarrow$ A is non-singular.

WORKING RULES FOR FINDING THE INVERSE OF SQUARE MATRIX A

Step I. *Find the value of* $|A|$.

Step II. *If* $|A| = 0$, *then A does not have its inverse.*

Step III. *If* $|A| \neq 0$, *then A has its inverse* $A^{-1} = \dfrac{adj\ A}{|A|}$.

Step IV. *Find cofactors of all elements of A and compute 'adj A'. Find* A^{-1} *by multiplying 'adj A' with* $\dfrac{1}{|A|}$. *i.e.,* $A^{-1} = \dfrac{1}{|A|}(adj\ A)$

NOTES

Example 1. *Find the inverse of* $\begin{bmatrix} 3 & 6 \\ 7 & 2 \end{bmatrix}$. *Also verify the answer.*

Sol. Let $\quad A = \begin{bmatrix} 3 & 6 \\ 7 & 2 \end{bmatrix}$.

$\therefore \quad |A| = \begin{vmatrix} 3 & 6 \\ 7 & 2 \end{vmatrix} = 6 - 42 = -36 \neq 0.$

$\therefore$ A is non-singular.

$\therefore$ A^{-1} exists and $A^{-1} = \dfrac{\text{adj } A}{|A|}$.

We find adj A :

Cofactors, $\quad A_{11} = (-1)^{1+1} \ | 2 | = 2, \quad A_{12} = (-1)^{1+2} \ | 7 | = -7$

$A_{21} = (-1)^{2+1} \ | 6 | = -6, \quad A_{22} = (-1)^{2+2} \ | 3 | = 3.$

$\therefore \quad \text{adj } A = \begin{bmatrix} 2 & -7 \\ -6 & 3 \end{bmatrix}' = \begin{bmatrix} 2 & -6 \\ -7 & 3 \end{bmatrix}$

$\therefore \quad A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{-36} \begin{bmatrix} 2 & -6 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} \mathbf{-1/18} & \mathbf{1/6} \\ \mathbf{7/36} & \mathbf{-1/12} \end{bmatrix}.$

Verification :

$$AA^{-1} = \begin{bmatrix} 3 & 6 \\ 7 & 2 \end{bmatrix} \times \frac{1}{-36} \begin{bmatrix} 2 & -6 \\ -7 & 3 \end{bmatrix} = -\frac{1}{36} \begin{bmatrix} 6-42 & -18+18 \\ 14-14 & -42+6 \end{bmatrix}$$

$$= -\frac{1}{36} \begin{bmatrix} -36 & 0 \\ 0 & -36 \end{bmatrix} = \frac{-36}{-36} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Similarly, $\quad A^{-1}A = I_2.$

Example 2. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *such that* $ad - bc \neq 0$, *then find the inverse of A.*

Sol. We have $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$\therefore$ $|A| = ad - bc$, which is given to be non-zero.

$\therefore$ A is non-singular and so A is invertible and $A^{-1} = \dfrac{\text{adj } A}{|A|}$.

We find adj A :

Cofactors, $\quad A_{11} = (-1)^2 \ | d | = d, \qquad A_{12} = (-1)^3 \ | c | = -c,$

$A_{21} = (-1)^3 \ | b | = -b, \qquad A_{22} = (-1)^4 \ | a | = a.$

$\therefore \quad \text{adj } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}' = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\therefore \quad A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{\mathbf{1}}{\mathbf{ad - bc}} \begin{bmatrix} \mathbf{d} & \mathbf{-b} \\ \mathbf{-c} & \mathbf{a} \end{bmatrix}.$

Example 3. *Find the inverse of the matrix* $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$.

Sol. Let $\quad A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$.

NOTES

$$\therefore \qquad |A| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix} = 1(8-6) + 1(0+9) + 2(0-6)$$

$$= 2 + 9 - 12 = -1 \neq 0.$$

$\therefore$ A is invertible and $A^{-1} = \dfrac{\text{adj } A}{|A|}$.

We find adj A :

Cofactors,

$$A_{11} = (-1)^2 \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = 8 - 6 = 2 \qquad A_{12} = (-1)^3 \begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -(0+9) = -9$$

$$A_{13} = (-1)^4 \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = 0 - 6 = -6 \qquad A_{21} = (-1)^3 \begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = -(-4+4) = 0$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \qquad A_{23} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -(-2+3) = -1$$

$$A_{31} = (-1)^4 \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = 3 - 4 = -1 \qquad A_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -(-3-0) = 3$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2 + 0 = 2.$$

$$\therefore \qquad \text{adj } A = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-1} \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{-2} & \mathbf{0} & \mathbf{1} \\ \mathbf{9} & \mathbf{2} & \mathbf{-3} \\ \mathbf{6} & \mathbf{1} & \mathbf{-2} \end{bmatrix}.$$

Example 4. *If* $A = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$, *verify that* $(AB)^{-1} = B^{-1}A^{-1}$.

Sol. We have $A = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$.

Inverse of A

$$|A| = \begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix} = 0 - 4 = -4 \text{ ¹ } 0.$$

$\therefore$ A^{-1} exist and equals $\dfrac{\text{adj } A}{|A|}$.

Now $\qquad A_{11} = (-1)^2 |\ 0\ | = 0, \qquad A_{12} = (-1)^3 |\ 4\ | = -4$

$\qquad\qquad A_{21} = (-1)^3 |\ 1\ | = -1, \qquad A_{22} = (-1)^4 |\ 3\ | = 3.$

$$\therefore \qquad \text{adj } A = \begin{bmatrix} 0 & -4 \\ -1 & 3 \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ -4 & 3 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-4} \begin{bmatrix} 0 & -1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1 & -3/4 \end{bmatrix}$$

Inverse of B

$$|B| = \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} = 20 - 0 = 20 \neq 0$$

$\therefore$ B^{-1} exist and equals $\dfrac{\text{adj } B}{|B|}$.

NOTES

Now cofactors $B_{11} = (-1)^2 | 5 | = 5$, $B_{12} = (-1)^3 | 2 | = -2$,

$B_{21} = (-1)^3 | 0 | = 0$, $B_{22} = (-1)^4 | 4 | = 4$.

$$\therefore \quad \text{adj } B = \begin{bmatrix} 5 & -2 \\ 0 & 4 \end{bmatrix}' = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$$

$$\therefore \quad B^{-1} = \frac{\text{adj } B}{|B|} = \frac{1}{20}\begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ -1/10 & 1/5 \end{bmatrix}$$

Inverse of AB

$$AB = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 12+2 & 0+5 \\ 16+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ 16 & 0 \end{bmatrix}.$$

$$\therefore \quad |AB| = \begin{vmatrix} 14 & 5 \\ 16 & 0 \end{vmatrix} = 0 - 80 = -80 \neq 0$$

$\therefore$ $(AB)^{-1}$ exist and equals $\dfrac{\text{adj}(AB)}{|AB|}$.

Now $(AB)_{11} = (-1)^2 | 0 | = 0$, $(AB)_{12} = (-1)^3 | 16 | = -16$

$(AB)_{21} = (-1)^3 | 5 | = -5$, $(AB)_{22} = (-1)^4 | 14 | = 14$.

$$\therefore \quad \text{adj }(AB) = \begin{bmatrix} 0 & -16 \\ -5 & 14 \end{bmatrix}' = \begin{bmatrix} 0 & -5 \\ -16 & 14 \end{bmatrix}$$

$$\therefore \quad (AB)^{-1} = \frac{\text{adj}(AB)}{|AB|} = \frac{1}{-80}\begin{bmatrix} 0 & -5 \\ -16 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 1/16 \\ 1/5 & -7/40 \end{bmatrix}$$

Also
$$B^{-1}A^{-1} = \begin{bmatrix} 1/4 & 0 \\ -1/10 & 1/5 \end{bmatrix}\begin{bmatrix} 0 & 1/4 \\ 1 & -3/4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & \frac{1}{16}+0 \\ 0+\frac{1}{5} & -\frac{1}{40}-\frac{3}{20} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{16} \\ \frac{1}{5} & -\frac{7}{40} \end{bmatrix}$$

$$\therefore \quad \mathbf{(AB)^{-1} = B^{-1}A^{-1}}.$$

Example 5. *If* $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, *then show that* $A^3 - 3A - 2I_3 = O$ *and hence find* A^{-1}.

Sol. We have $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1(-1) + 1(1) = 1 + 1 = 2 \neq 0$

so A^{-1} exists

Now
$$A^2 = A.A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

and
$$A^3 = A^2 . A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}$$

$$\therefore \quad A^3 - 3A - 2I = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} - 3\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - 2$$

$$= \begin{bmatrix} 2-0-2 & 3-3-0 & 3-3-0 \\ 3-3-0 & 2-0-2 & 3-3-0 \\ 3-3-0 & 3-3-0 & 2-0-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

$\therefore \quad A^3 - 3A - 2I = O \quad \ldots(1)$

To find inverse,

Since $|A| \neq 0$ so A^{-1} exists

Now $\quad A^3 - 3A - 2I = 0$

$\therefore \quad A^{-1}(A^3 - 3A - 2I) = A^{-1}0 = 0$

$\Rightarrow \quad A^{-1}A^3 - 3A^{-1}A - 2A^{-1}I = 0$

$\Rightarrow \quad A^2 - 3I - 2A^{-1} = 0$

$\Rightarrow \quad 2A^{-1} = A^2 - 3I$

$$\Rightarrow \quad A^{-1} = \frac{1}{2}(A^2 - 3I) = \frac{1}{2}\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

EXERCISE 3.2

1. For what value of k, the matrix $\begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix}$ has no inverse ?
2. Find the inverse of the following matrices :

   (*i*) $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$.
3. Find the sum of $\begin{bmatrix} 2 & -3 \\ 5 & -7 \end{bmatrix}$ and its multiplicative inverse.
4. Find the inverse of $A = \begin{bmatrix} 3 & 5 \\ 7 & -11 \end{bmatrix}$ and verify that $AA^{-1} = A^{-1}A = I_2$.
5. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^{-1} = \frac{1}{19}A$.
6. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, find $(AB)^{-1}$.
7. If $A^{-1} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}$, find $(AB)^{-1}$.
8. Verify that $(AB)^{-1} = B^{-1}A^{-1}$, where :

   (*i*) $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$

   (*ii*) $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$.
9. If $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$.

NOTES

10. Find the inverse of the following matrices :

(*i*) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$.

11. Find the inverse of the following matrices and verify your result :

(*i*) $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 3 \\ 8 & 5 & 10 \end{bmatrix}$.

12. If $A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 5 & 9 \\ 6 & 4 & 7 \end{bmatrix}$ and AB = BA = I, find B.

13. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, find A^{-1} and show that $A^{-1} = A^2$.

14. (*i*) If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, show that $A^2 - 5A - 14I = O$ and hence find A^{-1}.

(*ii*) If $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$, show that $A^2 + 4A - 42\,I = O$ and hence find A^{-1}.

Answers

1. $\frac{3}{2}$

2. (*i*) $\begin{bmatrix} 1/17 & -5/17 \\ 3/17 & 2/17 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 5/22 & 3/22 \\ -2/11 & 1/11 \end{bmatrix}$ (*iii*) $\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$

3. $\begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$ **4.** $\begin{bmatrix} 11/68 & 5/68 \\ 7/68 & -3/68 \end{bmatrix}$

6. $\begin{bmatrix} -47 & 39/2 \\ 41 & -17 \end{bmatrix}$ **7.** $\begin{bmatrix} -29/6 & 13/6 \\ -4/3 & 2/3 \end{bmatrix}$

10. (*i*) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (*ii*) $\frac{1}{5}\begin{bmatrix} -2 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$ (*iii*) $\frac{1}{4}\begin{bmatrix} -8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$

11. (*i*) $\begin{bmatrix} 4 & 3 & -3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (*iii*) $\frac{1}{5}\begin{bmatrix} 5 & -10 & 0 \\ 14 & -18 & -3 \\ -11 & 13 & 2 \end{bmatrix}$

12. $-\frac{1}{55}\begin{bmatrix} -1 & 8 & -10 \\ 47 & 9 & -25 \\ -26 & -12 & 15 \end{bmatrix}$ **13.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

14. (*i*) $-\frac{1}{14}\begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$ (*ii*) $\frac{1}{42}\begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$

3.3. SOLUTION OF SYSTEM OF LINEAR EQUATIONS BY MATRIX METHOD

In this section, we shall use inverse of matrices in solving systems of linear equations.

Let us consider the system

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

of three linear equations in three variables x, y, z.

This system can be expressed in the form of matrices as

$$\begin{bmatrix} a_1x+b_1y+c_1z \\ a_2x+b_2y+c_2z \\ a_3x+b_3y+c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

$$\Rightarrow \quad \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow \quad AX = B, \quad \text{where } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

If $|A| \neq 0$, then A^{-1} exists.

Multiplying $AX = B$ by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B.$$

$$\Rightarrow \quad (A^{-1}A)X = A^{-1}B \quad \Rightarrow \quad IX = A^{-1}B \quad \Rightarrow \quad X = A^{-1}B.$$

$\therefore$ The system has **unique** solution given by $\mathbf{X = A^{-1}B}$.

Remark 1. $|A| \neq 0$ is the necessary and sufficient condition for the above system of linear equations to have *unique solution*.

Remark 2. The above method of solving three equations in three variables is general and is applicable to systems containing n (≥ 2) linear equations in n variables.

WORKING RULES

Step I. *Express the given system in the standard form :*

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3.$$

Write $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Find the value of the determinant $|A|$.

Step II. *If* $|A| \neq 0$, *then the given system has unique solution. If* $|A| = 0$, *then the system has either no solution or infinitly many solutions.*

Step III. *In case* $|A| \neq 0$ *evaluate* A^{-1}.

Step IV. *Find x, y, z by using the equation* $X = A^{-1}B$.

NOTES

Example 1. *Solve the equations :* $x + 2y = 4,\ 2x + 5y = 9.$

Sol. The given equations are :

$$x + 2y = 4$$
$$2x + 5y = 9.$$

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 4 \\ 9 \end{bmatrix}.$

∴ The system is AX = B.

$$|\ A\ | = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1 \neq 0.$$

∴ The system has unique solution, $X = A^{-1}B$.

For $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, we have $A_{11} = 5,\quad A_{12} = -2,\quad A_{21} = -2,\quad A_{22} = 1$

∴ $$\text{adj } A = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}' = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

∴ $$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

The solution is $X = A^{-1}B$.

∴ $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 20 - 18 \\ -8 + 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

∴ $\mathbf{x = 2,\ y = 1.}$

Example 2. *Solve the equations :* $2x + y + z = 1,\ x - 2y - z = 1.5,\ 3y - 5z = 9.$

Sol. The given equations are :

$$2x + y + z = 1$$
$$x - 2y - z = 1.5$$
$$0x + 3y - 5z = 9.$$

Let $$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1.5 \\ 9 \end{bmatrix}.$$

∴ The given system is AX = B.

Now $$|\ A\ | = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{vmatrix}$$

$$= 2(10 + 3) - 1(-5 - 0) + 1(3 - 0) = 26 + 5 + 3 = 34 \neq 0.$$

∴ The system has unique solution, $X = A^{-1}B$.

For A, we have

$$A_{11} = \begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix} = 13,\quad A_{12} = -\begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} = 5,\quad A_{13} = \begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} = 3,$$

$$A_{21} = -\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} = 8,\quad A_{22} = \begin{vmatrix} 2 & 1 \\ 0 & -5 \end{vmatrix} = -10,\quad A_{23} = -\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = -6,$$

$$A_{31} = \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = 1,\quad A_{32} = -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 3,\quad A_{33} = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -5.$$

$$\therefore \qquad \text{adj A} = \begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}' = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{\text{adj A}}{|A|} = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$\therefore$ The unique solution $X = A^{-1} B$ is ;

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1.5 \\ 9 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13+12+9 \\ 5-15+27 \\ 3-9-45 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ -1.5 \end{bmatrix}$$

$\therefore \qquad$ $x = \mathbf{1}$, $y = \mathbf{0.5}$, $z = \mathbf{-1.5}$.

Example 3. *Gaurav purchases 3 pens, 2 bags and 1 instrument box and pays ₹ 41. From the same shop, Dheeraj purchases 2 pens, 1 bag and 2 instrument boxes and pays ₹ 29, while Ankur purchases 2 pens, 2 bags and 2 instrument boxes and pays ₹ 44. Translate the problem into a system of equations. Solve the system of equations by matrix method and hence find the cost of 1 pen, 1 bag and 1 instrument box.*

Sol. Let price of 1 pen = ₹ x, price of 1 bag = ₹ y

and price of 1 instrument box = ₹ z.

$\therefore$ By the given conditions,

$$3x + 2y + z = 41$$
$$2x + y + 2z = 29$$
$$2x + 2y + 2z = 44.$$

Let $$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 41 \\ 29 \\ 44 \end{bmatrix}. \ \backslash \quad AX = B.$$

Now $$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 3(2-4) - 2(4-4) + 1(4-2) = -6 - 0 + 2 = -4 \neq 0.$$

$\therefore$ The system AX = B has unique solution.

For A, $A_{11} = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$, $\quad A_{12} = -\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0$, $\quad A_{13} = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2$,

$A_{21} = -\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = -2$, $\quad A_{22} = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = 4$, $\quad A_{23} = -\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = -2$

$A_{31} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, $\quad A_{32} = -\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = -4$, $\quad A_{33} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -1$

$$\therefore \qquad \text{adj A} = \begin{bmatrix} -2 & 0 & 2 \\ -2 & 4 & -2 \\ 3 & -4 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 3 \\ 0 & 4 & -4 \\ 2 & -2 & -1 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{\text{adj A}}{|A|} = \frac{1}{-4}\begin{bmatrix} -2 & -2 & 3 \\ 0 & 4 & -4 \\ 2 & -2 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 2 & -3 \\ 0 & -4 & 4 \\ -2 & 2 & 1 \end{bmatrix}$$

∴ The unique solutions $X = A^{-1} B$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 2 & -3 \\ 0 & -4 & 4 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 41 \\ 29 \\ 44 \end{bmatrix} = \begin{bmatrix} \dfrac{82+58-132}{4} \\ 15 \\ \dfrac{-82+58+44}{4} \end{bmatrix} = \begin{bmatrix} \dfrac{8}{4} \\ 15 \\ \dfrac{20}{4} \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ 5 \end{bmatrix}$$

∴ $x = 2, y = 15, z = 5$

∴ Price of one pen = ₹ **2**

Price of one bag = ₹ **15**

Price of one instrument box = ₹ **5.**

EXERCISE 3.3

Solve the following systems of linear equations using matrix method (Q. No. 1–6) :

1. (*i*) $x + 2y = 1$, $3x + y = 4$ (*ii*) $5x + 7y = -2$, $4x + 6y = -3$ (*iii*) $3x - 2y = 6$, $5x + 3y = 1$

2. (*i*) $5x + 2y = 4$, $7x + 3y = 5$ (*ii*) $3x - 4y = 5$, $4x + 2y = 3$ (*iii*) $5x - 7y = 2$, $7x - 5y = 3$

3. (*i*) $5x + 2y = 3$, $3x + 2y = 5$ (*ii*) $3x + 4y = -1$, $2x + 5y = 4$ (*iii*) $3x + 4y = 5$, $x - y = -3$

4. (*i*) $2x + 3y + 3z = 5$, $x - 2y + z = -4$, $3x - y - 2z = 3$ (*ii*) $x + 2y - 3z = 6$, $3x + 2y - 2z = 3$, $2x - y + z = 2$ (*iii*) $x + y - z = 1$, $3x + y - 2z = 3$, $x - y - z = -1$

5. (*i*) $2x + 3y + 4z = 8$, $3x + y - z = -2$, $4x - y - 5z = -9$ (*ii*) $u - 2v + w = 1$, $2u + v + w = 1$, $u + v - 2w = -2$ (*iii*) $x + y + z = 1$, $x - 2y + 3z = 2$, $x - 3y + 5z = 3$

6. (*i*) $3x + 4y + 7z = 14$, $2x - y + 3z = 4$, $x + 2y - 3z = 0$ (*ii*) $-x + 2y + 5z = 2$, $2x - 3y + z = 15$, $-x + y + z = -3$ (*iii*) $4x + 2y + 3z = 2$, $x + y + z = 1$, $3x + y - 2z = 5$

7. A salesman has the following record of sales during the past three months for three items A, B and C which have the different rates of commission :

Months	*Sale of Units* A	B	C	*Total Commission (in ₹)*
January	90	100	20	800
February	130	50	40	900
March	60	100	30	850

Find out the rates of commission on items A, B and C, by matrix method.

8. Find A^{-1}, by adj. method where $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$

Hence, solve the system of linear equations :

$x + 2y - 3z = -4$, $2x + 3y + 2z = 2$, and $3x - 3y - 4z = 11$.

9. Solve the following set of simultaneous equations by matrix inverse method :

$$2x + 4y - z = 9$$
$$3x + y + 2z = 7$$
$$x + 3y - 3z = 4$$

Answers

1. (*i*) $x = 7/5, y = -1/5$ (*ii*) $x = 9/2, y = -7/2$

(*iii*) $x = 20/19, y = -27/19$

2. (*i*) $x = 2, y = -3$ (*ii*) $x = 1, y = -1/2$

(*iii*) $x = 11/24, y = 1/24$

3. (*i*) $x = -1, y = 4$ (*ii*) $x = -3, y = 2$ (*iii*) $x = -1, y = 2$

4. (*i*) $x = 1, y = 2, z = -1$ (*ii*) $x = 1, y = -5, z = -5$ (*iii*) $x = 2, y = 1, z = 2$

5. (*i*) $x = 1, y = -2, z = 3$ (*ii*) $x = 0, y = 0, z = 1$

(*iii*) $x = 1/2, y = 0, z = 1/2$

6. (*i*) $x = 1, y = 1, z = 1$ (*ii*) $x = 2, y = -3, z = 2$

(*iii*) $x = \frac{1}{2}, y = \frac{3}{2}, z = -1$

7. Rs. 2, Rs. 4, Rs. 11 **8.** $x = 3, y = -2, z = 1$ **9.** $x = 1, y = 2, z = 1.$

3.4. INVERSE OF A MATRIX BY ELEMENTARY OPERATIONS

3.4.1. Elementary Row Operation

There are three types of elementary row operations :

(*i*) **The interchange of any two rows.**

The interchange of ith and jth rows is an elementary row operation to be denoted by $R_i \leftrightarrow R_j$.

(*ii*) **The multiplication of the elements of a row by a non-zero number.**

If the elements of ith row of a matrix are multiplied by non-zero number λ, then this elementary row operation is denoted by $R_i \to \lambda R_i$.

(*iii*) **The addition of multiple of the elements of one row to the corresponding elements of another row.**

If λ times the elements of jth row are added to the corresponding elements of the ith row, then this elementary row operation is denoted by $R_i \to R_i + \lambda R_j$.

Illustration :

Let $$A = \begin{bmatrix} 3 & 3 & 4 \\ 8 & 2 & 1 \end{bmatrix}$$

The matrix B obtained by applying $R_2 \to R_2 + 2R_1$ is

$$B = \begin{bmatrix} 3 & 3 & 4 \\ 14 & 8 & 9 \end{bmatrix}$$

The matrix after applying elementary operation $R_1 \leftrightarrow R_2$ is

$$\begin{bmatrix} 8 & 2 & 1 \\ 3 & 3 & 4 \end{bmatrix}$$

and the matrix obtained by applying $R_1 \to 2R_1$ is $\begin{bmatrix} 6 & 6 & 8 \\ 8 & 2 & 1 \end{bmatrix}$

3.4.2. Theorem

An elementary row operation on the product of two matrices is equivalent to the same elementary row operation on the pre-factor.

Note. Proof of this theorem is beyond the scope of this book.

NOTES

3.4.3. Theorem

If A is an invertible square matrix such that BA = I, then prove that B is the inverse of A.

Proof. Let C be the inverse of A.

$\therefore$ AC = CA = I. Also BA = I.

We have B = BI = B(AC) = (BA)C = IC = C. $\therefore$ B = C.

$\therefore$ B is the inverse of A.

3.4.4. Method to Find Inverse of a Square Matrix by Using Elementary Row Operations

Let A be a non-singular square matrix.

$\therefore$ A = IA, where I is the identity matrix of the same order as A.

By **Theorem 3.4.2**, an elementary row operation on A on the left side of 'A = IA' is equivalent to the same elementary row operation on the pre-factor (= I) on the right side of A = IA. By applying elementary row operations, the matrix A on the left side is reduced to I and the same elementary row operations, in the same sequence are applied on the identity matrix on the right side.

Let I on the right side be changed to B, when A on the left side is changed to I

$\therefore$ We have I = BA.

By **Theorem 3.4.5**, B is the required inverse of the matrix A.

Remark. (*i*) If the matrix A is singular then we cannot reduce A to I by applying elementary row operations.

(*ii*) To find inverse by elementary row operetions, Ist make the matrix an upper triangular matrix.

Example 1. *Find the inverse of the matrix* $\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, *using elementary row operations*.

Sol. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$.

$$| A | = \begin{vmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{vmatrix} = 1(-1-8) + 2(0+8) + 3(0-2)$$

$$= -9 + 16 - 6 = 1 \neq 0.$$

$\therefore$ A^{-1} exists.

Write A = IA.

$$\therefore \quad \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

To find inverse, first we make L.H.S. matrix an upper triangular matrix. Since the top entry of the first column of A is already 1, to make other entries of the Ist column zero we add suitable multiplies of first row

Operating $R_3 \to R_3 + 2R_1$, we get

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ 0 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

Operating $R_2 \to (-1)R_2$, we get

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -4 \\ 0 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

Suitable multiplies of second row are added to the other rows to make other elements of second column zero.

Operating $R_1 \to R_1 + 2R_2$ and $R_3 \to R_3 + 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix} A$$

Operating $R_3 \to (-1)R_3$, we get

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ -2 & 2 & -1 \end{bmatrix} A$$

Suitable multiplies of third row are added to the other rows to make other elements of third column zero.

Operating $R_1 \to R_1 + 5R_3$ and $R_2 \to R_2 + 4R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix} A.$$

$\therefore$ $I = BA$, where $B = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} \mathbf{-9} & \mathbf{8} & \mathbf{-5} \\ \mathbf{-8} & \mathbf{7} & \mathbf{-4} \\ \mathbf{-2} & \mathbf{2} & \mathbf{-1} \end{bmatrix}$.

Example 2. *Find the inverse of* $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ *using elementary row operations.*

Sol. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 2(3-4) - 1(9-2) + 3(6-1) = -2 - 7 + 15 = 6 \neq 0.$$

$\therefore$ A^{-1} exists.

Write $A = IA$.

NOTES

$$\therefore \quad \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operating $R_1 \leftrightarrow R_3$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Operating $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A$$

Operating $R_2 \leftrightarrow R_3$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} A$$

Operating $R_2 \to \left(-\frac{1}{3}\right) R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1/3 & 0 & 2/3 \\ 0 & 1 & -3 \end{bmatrix} A$$

Operating $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 + 5R_2$, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 \\ -5/3 & 1 & 1/3 \end{bmatrix} A$$

Operating $R_3 \to \left(-\frac{1}{2}\right) R_3$, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 \\ 5/6 & -1/2 & -1/6 \end{pmatrix} A$$

Operating $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/6 & 1/2 & -1/6 \\ -7/6 & 1/2 & 5/6 \\ 5/6 & -1/2 & -1/6 \end{bmatrix} A.$$

$$\therefore \quad I = BA, \text{ where } B = \begin{bmatrix} -1/6 & 1/2 & -1/6 \\ -7/6 & 1/2 & 5/6 \\ 5/6 & -1/2 & -1/6 \end{bmatrix}.$$

$$\therefore \quad \mathbf{A^{-1} = \begin{bmatrix} -1/6 & 1/2 & -1/6 \\ -7/6 & 1/2 & 5/6 \\ 5/6 & -1/2 & -1/6 \end{bmatrix}.}$$

EXERCISE 3.4

Find the inverses of the following matrices by using elementary row operations :

1. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ **9.** $\begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$.

Answers

1. $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$ **4.** $\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$ **6.** $\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1/3 & 0 \\ 3 & 2/3 & -1 \end{bmatrix}$

8. $\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ **9.** $\frac{1}{19}\begin{bmatrix} 1 & -6 & 1 \\ -5 & -8 & 14 \\ 3 & 1 & 3 \end{bmatrix}$ **10.** $\frac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -1 & 1/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}$ **12.** $-\frac{1}{3}\begin{bmatrix} -4 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$.

3.5. ELEMENTARY COLUMN OPERATIONS

An elementary operation involving columns of a matrix is called an **elementary column operation :**

There are three types of elementary column operations :

(I) The interchange of any two columns.

The interchange of ith and jth columns is an elementary column operation to be denoted by $C_i \leftrightarrow C_j$.

(II) The multiplication of the elements of a column by a non-zero number.

If the elements of ith column of a matrix are multiplied by non-zero number λ, then this elementary column operation is denoted by $C_i \leftrightarrow \lambda C_i$.

(III) The addition of multiple of the elements of one column to the corresponding elements of another column.

If λ times the elements of jth column are added to the corresponding elements of the ith column, then this elementary column operation is denoted by $C_i \leftrightarrow C_i + \lambda C_j$.

NOTES

Illustration :

Let $$A = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 0 & 8 \\ 3 & 9 & 18 \end{bmatrix}.$$

If B, C and D are matrices obtained from A after applying elementary column operations $C_2 \leftrightarrow C_3$, $C_1 \leftrightarrow 6C_1$ and $C_2 \leftrightarrow C_2 + (-3)C_1$ respectively, then we have :

$$B = \begin{bmatrix} 1 & 6 & 5 \\ 2 & 8 & 0 \\ 3 & 18 & 9 \end{bmatrix}, \ C = \begin{bmatrix} 6\times 1 & 5 & 6 \\ 6\times 2 & 0 & 8 \\ 6\times 3 & 9 & 18 \end{bmatrix} = \begin{bmatrix} 6 & 5 & 6 \\ 12 & 0 & 8 \\ 18 & 9 & 18 \end{bmatrix}$$

and $$D = \begin{bmatrix} 1 & 6+(-3)\times 1 & 6 \\ 2 & 8+(-3)\times 2 & 8 \\ 3 & 18+(-3)\times 3 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 2 & 8 \\ 3 & 9 & 18 \end{bmatrix}.$$

3.5.1. Theorem

An elementary column operation on the product of two matrices is equivalent to the same elementary column operation on the post-matrix.

Note. Proof of this theorem is beyond the scope of this book.

3.5.2. Theorem

If A is an invertible square matrix such that AB = I, then prove that B is the inverse of A.

Proof. Let C be the inverse of A.

∴ AC = CA = I. Also AB = I.

We have B = IB = (CA)B = C(AB) = CI = C ∴ B = C

∴ B is the inverse of A.

3.5.3. Method of Finding Inverse of a Square Matrix by Using Elementary Column Operations

Let A be a non-singular square matrix.

∴ A = AI,

where I is the identity matrix of the same order as A.

By **Theorem 3.5.4,** an elementary column operation on A on the left side of 'A = AI' is equivalent to the same elementary column operation on the post-factor (= I) on the right side of A = AI. By applying elementary column operations, the matrix A on the left side is reduced to I and the same elementary column operations in the same sequence are applied on the identity matrix on the right side.

Let I on the right side be changed to B, when A on the left side is changed to I.

∴ We have I = AB.

By **Theorem 3.5.2,** B is the required inverse of the matrix A.

Remark. (*i*) If the matrix A is singular then we cannot reduce A to I by applying elementary column operations.

(*ii*) To find inverse by elementary column operations, Ist we reduce the matrix to lower triangular matrix.

Example 1. *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & 5 \\ 3 & 8 \end{bmatrix}$ *by using elementary column operations.*

Sol. $$A = \begin{bmatrix} 1 & 5 \\ 3 & 8 \end{bmatrix}.$$

$$|A| = \begin{vmatrix} 1 & 5 \\ 3 & 8 \end{vmatrix} = 8 - 15 = -7 \neq 0. \quad \therefore A^{-1} \text{ exists.}$$

Write $A = AI$.

$$\therefore \quad \begin{bmatrix} 1 & 5 \\ 3 & 8 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We reduce the matrix to lower triangular matrix. First make all the entries of Ist row 0 except Ist entry.

Apply $C_2 \to R_2 - 5C_1$

$$\begin{bmatrix} 1 & 0 \\ 3 & -7 \end{bmatrix} = A \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$$

Operating $C_2 \to -\frac{1}{7} C_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 5/7 \\ 0 & -1/7 \end{bmatrix}$$

Now suitable multiple of second column is added to the first column to make the other element of second row zero. Operating $C_1 \to C_1 - 3C_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} -8/7 & 5/7 \\ 3/7 & -1/7 \end{bmatrix}$$

$$\therefore \quad I = AB, \quad \text{where } B = \begin{bmatrix} -8/7 & 5/7 \\ 3/7 & -1/7 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \mathbf{-8/7} & \mathbf{5/7} \\ \mathbf{3/7} & \mathbf{-1/7} \end{bmatrix}.$$

Example 2. *Find the inverse of* $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ *by using elementary column operations.*

Sol. Let $$A = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}.$$

$$|A| = \begin{vmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{vmatrix}$$

$$= 4(0 - 4) - 3(3 - 4) + 3(4 - 0) = -16 + 3 + 12 = -1 \neq 0$$

$\therefore$ A^{-1} exists.

We have $A = AI$

$$\therefore \quad \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

NOTES

Operating $C_1 \to C_1 - C_2$, we get

$$\begin{bmatrix} 1 & 3 & 3 \\ -1 & 0 & -1 \\ 0 & -4 & -3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 3C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 2 \\ 0 & -4 & -3 \end{bmatrix} = A \begin{bmatrix} 1 & -3 & -3 \\ -1 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \to C_2 - C_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ 0 & -1 & -3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & -3 \\ -1 & 1 & 3 \\ 0 & -1 & 1 \end{bmatrix}$$

Operating $C_1 \to C_1 + C_2$, $C_3 \to C_3 - 2C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

Operating $C_3 \to (-1)C_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ -1 & -1 & -3 \end{bmatrix}$$

Operating $C_1 \to C_1 + C_3$, $C_2 \to C_2 + C_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$$

$\therefore$ $\quad I = AB$, where $B = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$

$\therefore$ $\quad A^{-1} = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{3} \\ \mathbf{-1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{-4} & \mathbf{-4} & \mathbf{-3} \end{bmatrix}$.

EXERCISE 3.5

Find the inverses of the following matrices by using elementary column operations :

1. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ **3.** $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

5. $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$ **6.** $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ **12.** $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$.

Answers

1. $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ **4.** $\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 7 & -10 \\ 2 & 3 \end{bmatrix}$ **6.** $\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1/3 & 0 \\ 3 & 2/3 & -1 \end{bmatrix}$

8. $\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ **9.** $\frac{1}{19}\begin{bmatrix} 1 & -6 & 1 \\ -5 & -8 & 14 \\ 3 & 1 & 3 \end{bmatrix}$ **10.** $\frac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -1 & 1/5 \\ 0 & -1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}$ **12.** $-\frac{1}{3}\begin{bmatrix} -4 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$.

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4. BASIC MATHEMATICS AND FINANCE

STRUCTURE

4.1. SIMPLE AND COMPOUND INTEREST

In any money transaction there is a lender, who gives money, and a borrower, who receives money. The amount of loan borrowed is called the Principal. The borrower pays a certain amount for the use of this money. This called Interest. Interest is always calculated on the principal borrowed.

The sum of the principal and the interest is called the Amount.

Interest is of two kinds

Simple and Compound

If the interest is calculated only, on a certain sum borrowed it is called simple interest.

In compound interest, the borrower and the lender fill up a unit of time *e.g.*, yearly, half-yearly, quarterly or monthly to settle the previous account. The interest due at the end of the first unit of time is added to the principal and the amount so obtained becomes the principal for the second unit of time. Similarly, the amount after the second unit of time becomes the principal for third unit of time and so on. The interest due at the end of intermediate units of time is not paid to the lender rather it gets added to the principal and the amount thus obtained at the end of the last unit of time and the original principal is called the compound interest.

4.2. IMPORTANT RELATIONS

I. Since rate of interest is usually given as rate percent, we write

$$I = \frac{\text{Profit}}{100}$$

$$\text{Amount (A)} = \text{Principal (P)} + \text{Interest (I)}$$

Where P = Principal

r = r% per annum

t = time period

II. When interest is compounded annually,

$$A = P\left[1 + \frac{r}{100}\right]^t$$

III. When interest is compounded half-yearly

$$A = P\left[1 + \frac{r/2}{100}\right]^{2t} = P\left[1 + \frac{r}{100}\right]^{2t}$$

IV. When interest is compounded quarterly

$$A = P\left[1 + \frac{r/4}{100}\right]^{4t} = P\left[1 + \frac{r}{400}\right]^{4t}$$

V. When interest is compounded monthly

$$A = P\left[1 + \frac{r/12}{100}\right]^{12t} = P\left[1 + \frac{r}{1200}\right]^{12t}$$

VI. When interest is r_1 % for the first year, r_2 % for second year and r_3 % for third year.

$$A = P\left[1 + \frac{r_1}{100}\right]\left[1 + \frac{r_2}{100}\right]\left[1 + \frac{r_3}{100}\right]$$

VII. When time is fraction of years, for example $2\frac{1}{3}$ years, then

$$A = P\left[1 + \frac{r}{100}\right]^2 \times \left[1 + \frac{r/3}{100}\right]$$

VIII. Present worth (P) of ₹ A due it years hence is given by

$$P = \frac{A}{\left(1 + \frac{r}{100}\right)^t}$$

IX. CI = A – P

X. When C.I. is compounded annually, the ratio of S.I to C.I at the same rate percent per annum and for the same period is given by

$$\frac{\text{S.I}}{\text{C.I}} = \frac{rt}{100\left[\left(1 + \frac{r}{100}\right)^t - 1\right]}$$

General formula is

$$\text{Amount} = \text{Principal}\left[1 + \frac{\text{rate}}{100}\right]^{\text{time period}}$$

NOTES

SOLVED EXAMPLES

Example 1. *Find the interest on ₹ 1460 at 10% from 5th Feb, 2020 to 25th April 2020.*

Sol. $P = ₹\ 1460$

$r = 10\%$

2008 is a leap year

$\therefore \quad t = (24 + 31 + 25) \text{ days} = 80 \text{ days} = \dfrac{80}{365} \text{ year}$

$$I = \frac{Prt}{100} = \frac{1460 \times 10 \times 80}{36500} = ₹\ 32$$

Example 2. *Ram lent ₹ 1200 to Shyam for 5 years and ₹ 1500 to Mohan for 2 years received altogether ₹ 900 as interest find the rate per annum.*

Sol. $I = I_1 + I_2$

$$I = \frac{P_1 r t_1}{100} + \frac{P_1 r t_2}{100}$$

$$I = \frac{r}{100}(P_1 t_1 + P_2 t_2)$$

Here $I = ₹\ 900,\ P_1 = ₹\ 1200,\ t_1 = 5 \text{ years},\ P_2 = \ 1500$

$$t_2 = \frac{100I}{P_1 t_1 + P_2 t_2}$$

$$r = \frac{100 \times 900}{(1200 \times 5) + (1500 \times 2)} = \frac{90,000}{9,000}$$

$r = 10\%$

Example 3. *Find the annual installments that will discharge a debt of ₹ 12900 due in 4 years at 5% per annum simple interest.*

Sol. Let each equal amount installments be ₹ x.

First installments is paid after one year and hence will remain with the lender for the remaining = 3 years. Similarly, second installments will remain with the lender for 2 years, third installments for 1 year and the final fourth installments remain ₹ x as such

$$A = A_1 + A_2 + A_3 + A_4$$

$$A = P\left(\frac{100 + rt}{100}\right)$$

$$A = x\left[\frac{100 + 5 \times 3}{100} + \frac{100 + 5 \times 2}{100} + \frac{100 + 5 \times 1}{100} + \frac{100 + 5 \times 0}{100}\right]$$

$$12900 = x\left[\frac{115 + 110 + 105 + 100}{100}\right] = \frac{430}{100}x$$

$$x = \frac{12900 \times 100}{430}$$

$x = ₹\ 3000$

NOTES

Example 4. *₹ 1500 is invested at the rate of 10% simple interest and interest is added to the principal after every 5 years. In how many years will it amount to ₹ 2500?*

Sol. First we will find the simple interest for years

$$\text{S.I} = \frac{1500 \times 5 \times 10}{100} = ₹\ 750$$

∴ Now Principal after 5 years = 1500 + 750 = 2250

Final amount = ₹ 2250

S.I. = 2500 – 2250 = ₹ 250

$$t = \frac{250 \times 100}{2250 \times 10} = \frac{10}{1} \text{ years}$$

$$\text{Total time} = 5 + \frac{10}{9} \text{ years}$$

$$= \frac{55}{9} \text{ years} = 6\frac{1}{9} \text{ years}$$

EXERCISE 4.1

1. A sum of money at simple interest amounts it ₹ 2240 in 2 years and ₹ 2600 in 5 years find the sum.
2. ₹ 7914 is divided into three parts in such a way that the first part at 3% per annum after 8 years, the second part at 4% per annum after 5 years and third part at 6% per annum after 2 years give equal amounts find each part.
3. Find the annual installments that will discharge a debt of ₹ 5600 due in 5 years at 4% per annum simple interest.
4. Ravi invests two equal amounts in two books giving 10% and 12% rate of interest respectively. At the end of year, the interest earned is ₹ 1650. Find the sum invested in each book.
5. The difference of interest on a certain sum at 4% per annum for 3 years and at 5% per annum for 2 years in ₹ 100. Find the sum.
6. Ram earns ₹ 1320 in 3 years from his investment of ₹ 5000 at a certain rate of simple interest and ₹ 4000 at 2% higher. Find the rates of interest.
7. A certain sum of money amount to ₹ 4720 in 3 years at 6% per annum simple interest. In how many years will it amounts to ₹ 5680 at the same rate of interest?

Answers

1. ₹ 2000

2. ₹ 2520 at 3% for 8 years, ₹ 2604 at 4% for 5 years, ₹ 2790 at 6% for 2 years

3. ₹ 1000 **4.** ₹ 7500 **5.** ₹ 5000 **6.** 4%

7. 7 years

SOLVED EXAMPLES

Example 1. *Find the present worth of ₹ 9261 due 3 years at 5% per annum compounded yearly.*

Sol.
$$A = P\left(1 + \frac{r}{100}\right)^t$$

NOTES

$$P = \frac{A}{\left(1 + \frac{r}{100}\right)^t}$$

A = ₹ 9261

r = 5% per annum and t = 3 years

$$P = \frac{9261}{\left(1 + \frac{5}{100}\right)^3} = \frac{9261}{\frac{9261}{8000}} = ₹\ 8000$$

Example 2. *The simple interest on a certain sum for 2 years is ₹ 50 and the compound interest is ₹ 55. Find the rate of interest per annum and the sum.*

Sol. The difference between C.I and S.I for 2 years period is 5 because C.I also includes interest for the second year on the first year's interest.

C.I – S.I = ₹ (55 – 50) = ₹ 5

$$\text{First year's S.I} = \frac{₹\ 50}{2} = ₹\ 25$$

So, ₹ 5 is in the interest on 25 for 1 year

$$l = \frac{Ptr}{100}$$

$$r = \frac{100\ I}{Pt}$$

I = ₹ 5

P = ₹ 25

t = 1 year

$$r = \frac{100 \times 5}{25 \times 1}$$

r = 20% per annum

Now, to find the principal sum we use the S.I given for 2 years.

$$P = \frac{100\ I}{rt}$$

I = ₹ 50

r = 20% per annum

t = 2 yrs

$$P = \frac{100 \times 50}{20 \times 2}$$

P = ₹ 125

Example 3. *A man borrows ₹ 20,000 and agrees to pay both the interest and the principal in 4 equal annual installments. If interest is calculated at 5% annually, find the annual installments.*

Sol. Let the four equal installments be ₹ x.

Present worth of the first installments

$$= \frac{x}{\left(1 + \frac{5}{100}\right)} = \frac{x}{1 + \frac{1}{20}} = \frac{x}{\frac{21}{20}} = \frac{20}{10}x$$

Similarly, present worth of the second installments

$$= \frac{x}{\left(1+\frac{5}{100}\right)^2} = \left(\frac{20}{10}\right)^2 x$$

Present worth of third installments

$$= \frac{x}{\left(1+\frac{5}{100}\right)^3} = \left(\frac{20}{10}\right)^3 x$$

Sum of the present worth of all four installments

$$= \frac{20}{21}x + \left(\frac{20}{21}\right)^2 x + \left(\frac{20}{21}\right)^3 x + \left(\frac{20}{21}\right)^4 x$$

$$= \left(\frac{20}{21}\right)x\left[1+\frac{20}{21}+\left(\frac{20}{21}\right)^2+\left(\frac{20}{21}\right)^3\right]$$

$$= \frac{20}{21}x\left[9261+\frac{8820+8400+8000}{9261}\right]$$

$$= x\left(\frac{20}{21}\right)\left(\frac{34481}{9261}\right) = \frac{689620}{194481}x$$

but $\qquad \frac{689620}{194481}x = 20000$

$$x = 20000 \times \frac{194481}{689620}$$

$$x = ₹\ 5640$$

Example 4. *A man borrows ₹ 1000 and repays the loan by yearly installments of ₹ 100, the first installments being paid one year after the loan. After how many years will he be out of debt interest being reckoned throughout at 4 percent per annum.*

Sol. Suppose n is required number of years

The values of installments are

$$\frac{100}{1.04}, \frac{100}{(1.04)^2}, ... \frac{100}{(1.04)^n} \qquad \left[\because \quad \frac{A}{\left(1+\frac{r}{100}\right)^n} = P\right]$$

Hence $\qquad 1000 = \frac{100}{104} + ... + \frac{100}{(1.04)^n}$

$$= \frac{100}{1.04}\left\{1+\frac{1}{1.04}+.....+\frac{100}{(1.04)^{n-1}}\right\}$$

$$= \frac{100}{1.04}\frac{1-(1.04)^{-n}}{1-(1.04)^{-1}}$$

Thus, $\qquad 4 = 1 - (1.04)^{-4}$

i.e., $\qquad (1.04)^n = \frac{5}{3}$

NOTES

Therefore, $h = \frac{\log 5 - \log 3}{\log 1.04} = \frac{.6990 - .4771}{.0170}$

$= 13.05$ approve

Thus by slightly increasing the last payment, the debt would be discharged in 13 years.

EXERCISE 4.2

1. A man invests ₹ 2500 and get interest at 4% per annum during the first year and 5% during the second year. How much total amount does he get at the end of second year?
2. The compound interest on a sum of money at 4% per annum for 2 years is ₹ 204 what would be the simple interest on this sum at the same rate and for the same period?
3. The compound interest on a certain sum money for 2 years at 10% per annum is ₹ 420. Find the simple interest on the same sum at the same rate and for the same period.
4. The difference between simple and compound interest at the same rate for ₹ 5000 for 2 years is ₹ 72. Find the ratio of interest.
5. ₹ 5115 is to be divided between Ram and Shyam who are respectively 18 years and 21 years old. They invest their shares in bonds which given them 20% per annum interest, compounded yearly. Both get equal amount when they attain the age of 25 years. Find the shares of each.
6. An amount of ₹ 3640 borrowed at 20% per annum, compounded annually, is to be repaid in 3 equal installments. Find the amount of each installments.
7. A owes B ₹ 33275 in 3 years B owes A ₹ 43923 in 4 years. The rate of interest is 10% per annum, compounded yearly. They now decide to settle their account by a ready money payment. How much need to be paid and it whom?
8. A sum of money was lent at compound interest for 2 years at 20% per annum compounded yearly. If the interest is compounded half yearly ₹ 723 are received more, find the sum.

Answers

1. ₹ 2730 **2.** ₹ 200 **3.** ₹ 400
4. 12% per annum **5.** Ram's share = ₹ 1875 Shyam's shares = ₹ 3240 **6.** ₹ 1720
7. B has to pay A ₹ 50000 **8.** ₹ 30,000

4.3. CONTINUOUS COMPOUNDING OF INTEREST

In practical situations, it is observed that as the frequency of compounding increases, the amount also increases. When the frequency of compounding increases indefinitely, then the interest is said to be **compounded continuously**. In such cases, at any instant of time, the investment increases in proportion of its current value.

To illustrate, let us consider that principal = ₹ 100 and rate of interest = 10% p.a.

If the interest is compounded annually, then

$$A = 100\left(1 + \frac{10}{100}\right)^1 = ₹\ 110$$

If the interest is compounded half-yearly, then

$$A = 100\left(1 + \frac{\frac{10}{2}}{100}\right)^2 = ₹\ 110.25$$

If the interest is compounded quarterly, then

$$A = 100\left(1 + \frac{\frac{10}{4}}{100}\right)^4 = ₹\ 110.38$$

If the interest is compounded monthly, then

$$A = 100\left(1 + \frac{\frac{10}{12}}{100}\right)^{12} = ₹\ 110.53$$

Thus we see that the amount increases as we increase the frequency of compounding.

Let P be the principal and r% be the rate of interest p.a. If the frequency of compounding is k, then amount after n years

$$= P\left(1 + \frac{\frac{r}{k}}{100}\right)^{kn}$$

Let the interest be compounded continuously and let A be the amount after n years

$$A = \lim_{k\to\infty} P\left(1 + \frac{\frac{r}{k}}{100}\right)^{kn} = P \lim_{k\to\infty}\left(1 + \frac{r}{100k}\right)^{\left(\frac{100k}{r}\right)\left(\frac{rn}{100}\right)}$$

Now when $k \to \infty$, $\dfrac{100k}{r}$ also $\to \infty$

$$A = P\left[\text{Lim}\left(1 + \frac{r}{100k}\right)^{\frac{100k}{r}}\right]^{\frac{rn}{100}}$$

$$= Pe^{\frac{rn}{100}} \qquad \left[\because \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m = e\right]$$

Thus if the rate of interest is r% p.a. and the interest is compounded continuously then after n years, the amount of the principal is given by

$$A = Pe^{\frac{rn}{100}}$$

SOLVED EXAMPLES

Example 1. *Let ₹ 5000 be invested at 12% p.a. Find the amount after 3 years if the interest is compounded continuously (take e = 2.71828)*

Sol. Here, Principal (P) = ₹ 5000

Rate (r) = 12%, n = 3

NOTES

It is a case of continuous compounding

$\therefore$ Amount after 3 years = $Pe^{\frac{rn}{100}}$

i.e., $A = 5000\,(2.71828)^{\frac{12\times 3}{100}}$

or $A = 5000\ (2.71828)0.36$

or $\log A = [\log 5000 + 0.36\ (\log 2.71828)]$

or $\log A = [3.6990 + 0.36\ (0.4343)] = (3.8555)$

$\therefore$ $A = \text{antilog}\ (3.8555) = 7169$

Hence the amount after 3 years is ₹ 7169.

Example 2. *How long will it take for a principal to be three times of itself, if money is worth 10% p.a. compounded continuously?*

Sol. Let the principal = P, $\therefore$ Amount = 3P

Rate of interest (r) = 10% p.a.

Let the time period = n years

According to the given condition of the problem,

$$3P = Pe^{\frac{10n}{100}} \qquad \left[\because A = Pe^{\frac{rn}{100}}\right]$$

or $$3 = e^{\frac{10n}{100}} \Rightarrow 3 = e^{(0.1)n}$$

$\therefore$ $$\log 3 = \log^{\,e(0.1)n} = (0.1)n \log e$$

$\therefore$ $$n = \frac{\log 3}{(0.1)\log e} = \frac{0.4771}{(0.1)\log(2.71828)}$$

$$= \frac{0.4771}{(.1)\times(0.4343)} = \frac{0.4771}{0.04343}$$

$$= 10.99 = 11 \text{ years (nearly)}$$

Hence it will take 11 years (nearly) for the principal to be three times

EXERCISE 4.3

1. If ₹ 5700 is invested at 11% p.a., find the amount after 3 years if the interest is compounded continuously.
2. If ₹ 5000 is invested at 8% p.a., find the amount after 2 years if the interest is compounded continuously.
3. How long will it take for a principal to double if money is worth 9% p.a. compounded continuously?
4. How long will it take for a principal to double if money is worth 7% per annum compounded continuously. Give your answers to nearest years.

Answers

1. ₹ 7929 **2.** ₹ 5868 **3.** 8 years

4. 10 years (nearly)

4.4. PROBLEMS ON EFFECTIVE RATE OF INTEREST, DEPRECIATION AND POPULATION

NOTES

4.4.1. Effective Rate of Interest

When interest is compounded more often than once per year, the given annual rate is called the *nominal annual rate* or *nominal rate*. The rate of interest actually earned in one year is called the effective *annual rate* or the *effective rate*.

Thus the effective rate of interest is defined as the rate which when compounded annually, gives the same amount of interest as a nominal rate compounded several times each year.

Note 1. If the interest is calculated only at the end of an year, both the effective and nominal rate are equal.

Note 2. Case of continuous compounding:

Let r% be the rate of interest and interest is compounded continuously, then amount of principal after one year = $Pe^{\frac{r}{100}}$.

SOLVED EXAMPLES

Example 1. *Find the effective rate of interest of 10% p.a. payable half-yearly.*

Sol. Let the principal = ₹ 100

Here time = 1 year = 2 half years, rate = 10% p.a. = 5% per half year

Using the formula, $A = P\left(1+\frac{r}{100}\right)^t$, we have

$$A = 100\times\left(1+\frac{5}{100}\right)^2 = 100\times\frac{21}{20}\times\frac{21}{20} = ₹\ \frac{441}{4}$$

$$\therefore \qquad \text{C.I.} = A - P = ₹\ \frac{441}{4} - 100 = ₹\ \frac{41}{4} = ₹\ 10\frac{1}{4}$$

Hence the effective rate of interest = $10\frac{1}{4}$%

Example 2. *Find the effective rate of interest 6% p.a. compounded continuously.*

Sol. Let the principal be ₹ 100

Here rate of interest (r) = 6%

After one year,

$$\text{Amount} = 10e^{\frac{6}{100}} = 10e^{.06} \qquad \left[\because\ A = Pe^{\frac{rn}{100}}, \text{here } n = 1\right]$$

$$\therefore \qquad \log A = [\log 100 + (0.06)\log e]$$

$$= [\log 100 + (0.06)(\log 2.7183)]$$

$$= [2 + (0.06)(0.4343)] = [2 + 0.0261] = [2.0261]$$

$$\therefore \qquad A = \text{antilog}\ (2.0261) = ₹\ 106.2$$

∴ Interest after 1 year = ₹ 106.2 – ₹ 100 = ₹ 6.2

Hence the effective rate of interest = 6.2%

NOTES

Example 3. *Which is a better investment, 8% p.a. compounded half yearly or 7.5% compounded continuously.*

Solution. Let the principal be ₹ 100 in both types of investments.

First investment:

Here rate = 8% p.a. = 4% half yearly

$\therefore$ $$A = 100\left(1+\frac{4}{100}\right)^2 = 100(1.04)2 = ₹\ 108.16$$

$\therefore$ Interest on 100 after 1 year = ₹ 108.16 – ₹ 100 = ₹ 8.16

$\therefore$ Effective rate of interest = 8.16% half yearly.

Second investment:

Here rate = 7.5 p.a. compounded continuously

$\therefore$ After one year, $$A = 100e^{\frac{7.5}{100}} \qquad \left[\because\ A = Pe^{\frac{rn}{100}}, \text{here } n = 1\right]$$

$$= 100\ e0.075$$

$\therefore$ $$\log A = [\log 100 + (0.075) \log (2.7183)]$$

$$= [2 + (0.075)\ (0.4343)] = [2 + 0.326] = [2.0326]$$

$\therefore$ $$A = \text{antilog } (2.0326) = 107.7$$

$\therefore$ Interest on ₹ 100 after 1 year = ₹ 107.7 – ₹ 100 = ₹ 7.7

$\therefore$ Effective rate of interest = 7.7%

Hence the first investment is better.

4.4.2. Problems on Depreciation

It is generally observed that the value of all articles decrease with the passage of time. This decrease in value is called depreciation. The depreciated value can be calculated by using the formula.

$$A = P\left(1-\frac{r}{100}\right)^t$$

where A is the depreciated value; P, the present value and r the rate of depreciation.

Example 5. *The value of a machinery depreciates by 5% annually. If its present value in ₹ 210000, find its value after 4 years.*

Solution. Let A be the depreciated value of the machinery after 4 years

Present value of machinery, P = ₹ 210000

Rate of depreciation = 5%, t = 4 years

Using the formula, $A = P\left(1-\frac{r}{100}\right)^t$, we have

$$A = 210000\left(1-\frac{5}{100}\right)^5 = 210000\times\left(\frac{95}{100}\right)^4 = 210000\ (0.95)^4$$

Taking log on both sides, we have

$$\log A = \log 210000 + 4 \log 0.95$$

$$= 5.3222 + (4.\bar{1}.9777)$$

$= 5.3222 + 4(-1 + 0.9777)$

$= 5.3222 - 4 + 3.9108 = 5.2330$

$\therefore \quad A = \text{antilog}\ (5.2330) = 171000$

Hence the value of machinery after 4 years is ₹ 171000

Example 5. *A property decreases in value every year at the rate of $6\frac{1}{4}\%$ of its value at the beginning of the year. If its value at the end of 3 years was ₹ 21093.95, find its value at the beginning of the first year.*

Solution. Let A = value at the end of 3 years = ₹ 21093.95

P = value at the beginning of the first year

$r = 6\frac{1}{4}\% = \frac{25}{4}\%$, $t = 3$ years

Using the formula, $A = P\left(1 - \frac{r}{100}\right)^t$, we have

$$21093.5 = P\left(1 - \frac{\frac{25}{4}}{100}\right)^3 = P\left(1 - \frac{25}{400}\right)^3 = P\left(\frac{15}{16}\right)^3$$

$$P = \frac{21093.95 \times 16 \times 16 \times 16}{15 \times 15 \times 15} = 25600.24$$

Hence the value of property at the beginning of the first year was ₹ 25600.24.

Example 6. *A machine depreciates at the rate of 10% of its value at the beginning of an year. The machine was purchased for ₹ 10000 and the scrap value realized when sold was ₹ 3855. Find how many years the machine was used for?*

Solution. Let P = value at the beginning of the first year = ₹ 10000

A = value at the end of t years = ₹ 3855

Rate of depreciation = 10%

Using the formula $A = P\left(1 - \frac{r}{100}\right)^t$, we have

$$3855 = 10000\left(1 - \frac{10}{100}\right)^t \Rightarrow 3855 = 10000(0.9)^t$$

Taking logarithms on both sides, we have

$$\log 3855 = \log 10000 + t \log 0.9$$

i.e., $3.586 = 4.00 + t \log \frac{9}{10}$

or $3.586 - 4 = t(\log 9 - \log 10)$

or $-0.414 = t(0.9542 - 1)$

or $-0.414 = t \times (-0.0458)$

$$t = \frac{0.414}{0.0458} = 9.04 \text{ years nearly}$$

NOTES

4.4.3. Problems on Population

The formula for finding the increase in population is also the same as that of compound interest. If r is the rate of increase per 100, then population after t year is given by

$$A = P\left(1+\frac{r}{100}\right)^t$$

Note 1. If the increase is r per 1000, then

$$A = P\left(1+\frac{r}{1000}\right)^t$$

2. If the rate of increase is different for different years viz., r_1, r_2, r_3, ... r_n then population after t years is given by

$$A = \left(1+\frac{r_1}{100}\right)\left(1+\frac{r_2}{100}\right)+.....+\left(1+\frac{r_n}{100}\right)$$

Example 7. *The population of a town is 140000. If it increases by 5% annually, what will be the population of the town after 5 years?*

Solution. Let A be the population after 5 years

Here present population, P = 140000, r = 5% annually, t = 5 years

$$\therefore \quad A = 140000\left(1+\frac{5}{100}\right)^5 = 140000\ (1.05)^5$$

Taking log on both sides, we have

$$\log A = \log 140000 + 5 \log 1.05 = 5.1461 + 5(0.0212)$$
$$= 5.1461 + 0.1060 = 5.2521$$

or $$A = \text{antilog } (5.2521) = 178600$$

Hence population of town after 5 years will be **178600**.

Example 8. *If the population of a town decreases 6.25% annually and the present population is 20, 480,000; find its population after three years.*

Solution. Let A be the population after 3 years

Here present population = 20,480,000

$$\text{Decrease} = 6.25\% = \frac{625}{100} = \frac{25}{4}$$

$$\therefore \quad A = 20480000\left(1-\frac{25}{4\times100}\right)^3 = 20480000\left(1-\frac{1}{16}\right)^3$$

$$= 20480000\times\frac{15}{16}\times\frac{15}{16}\times\frac{15}{16} = \mathbf{16875000.}$$

Example 9. *The bacteria in a culture increases by 5% in the first hour, decreases by 5% in the second hour and again increases by 5% in the third hour. If the count of the bacteria at the end of the third hour is 8.379 × 10^8, find the original count of bacteria in the sample.*

Solution. Let P = original count of bacteria in the sample

A = count of bacterial at the end of 3rd hour = 8.379 × 10^8

Now $$A = P\left(1+\frac{r_1}{100}\right)\left(1+\frac{r_2}{100}\right)\left(1+\frac{r_3}{100}\right)$$

$\therefore \quad 8.379 \times 10^8 = P\left(1+\frac{5}{100}\right)\left(1-\frac{5}{100}\right)\left(1+\frac{5}{100}\right)$

$[\because \ r_1 = 5\%, r_2 = -5\%, r_3 = 5\%]$

or $\quad 8.379 \times 10^8 = P\left(\frac{21}{20}\right)\left(\frac{19}{20}\right)\left(\frac{21}{20}\right)$

$\therefore \quad P = \frac{8.379 \times 10^8 \times 20 \times 20 \times 20}{21 \times 19 \times 21} = \frac{8379 \times 10^5 \times 8000}{8379}$

$= 10^5 \times 8000 = 8 \times 10^8$

Hence the original count of bacteria in the sample is 8×10^8.

EXERCISE 4.4

1. Find the effective rate of interest corresponding to the nominal rate of 10% per annum, if it is converted to
   (*i*) half yearly (*ii*) quarterly
2. (*i*)Find the effective rate equivalent to the nominal rate of $4\frac{1}{2}\%$ per year compounded quarterly.
   (*ii*) Find the effective rate of interest of 10% p.a. compounded monthly.
3. (*i*) Find the effective rate of interest of 9% p.a. compounded continuously.
   (*ii*) Find the effective rate of interest of 7.5% p.a. compounded continuously.
4. Which is a better investment, 12% compounded quarterly or 12.2% compounded continuously.
5. A machine is depreciated in such a way that the value of the machine at the end of any year is 90% of its value at the beginning of the year. The cost of the machine was ₹ 12000 and its was sold eventually as waste metal for ₹ 200. Find the number of years during which the machine was in use.
6. The population of a village is 5000. Find the population at the end of 3 years if the population increases every year by 10% of what it is at the beginning of the year.
7. The population of a developing countries increases every year by 2.3% of the population at the beginning of that year. In what time will the population double itself?
   $\left[\textbf{Hint.}\ 2P = P\left(1+\frac{2.3}{100}\right)^n \Rightarrow n = 31 \text{ years}\right]$
8. If the annual growth rate of a population is 50 per thousand and the present population is 600 millions, what will be the population in 25 years time?
9. A car factory increased its production of cars from 80000 in 2000 to 92610 in 2003. Find the annual rate of growth of production of cars.

Answers

1. (*i*) 10.25% (*ii*) 10.38% **2.** (*i*) 4.58 (*ii*) 10.47%
3. (*i*) 9.4% (*ii*) 7.7% **4.** 2nd
5. 39 years (approx) **6.** 6655 **7.** 31 years
8. 2033 millions **9.** 5%