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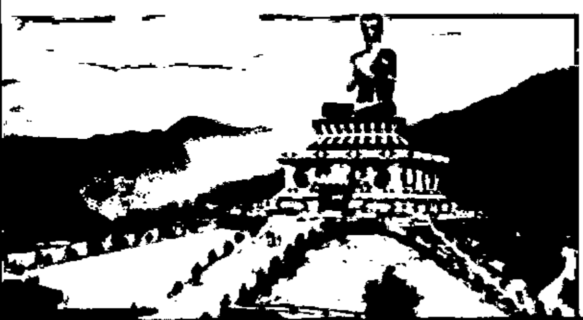


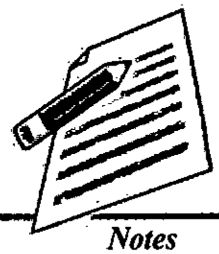
The Pathways To Higher Studies

Mathematics

Class-XII

MATHEMATICS

CLASS 12



Notes

1 SETS

- Understand the concept of set.
- Discuss the types of sets.
- *Describe the finite and infinite sets.*
- Understand the Venn diagram.

Objective of the chapter:

The basic objective of this chapter is to through some light on the initial concepts of sets so that the fundamentals of sets can be learned.

Introduction

Set is a well-defined collection of objects. Examples of sets

1. Players of a cricket team
2. Students of your School
3. Members of your family
4. Vowels in the English alphabet $\{a, e, i, o, u\}$
5. Prime factors of 10, namely $\{2,5\}$
6. Odd natural numbers less than 8, i.e., $\{1, 3, 5, 7\}$
7. Numbers in a Dice $\{1,2,3,4,5,6\}$

Sets in Mathematics

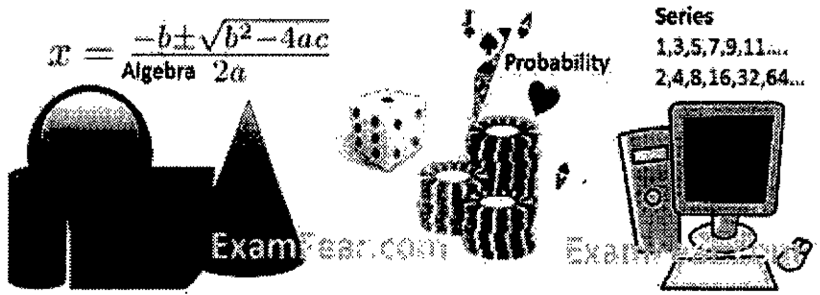
N: *the set of all-natural numbers*

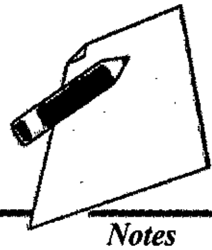
Z: the set of all integers

Q: the set of all rational numbers

R: the set of real numbers

The concept of set serves as a fundamental part of the present-day mathematics. Today this concept is being used in almost every branch of mathematics. Sets are used to define the concepts of relations and functions. The study of geometry, sequences, probability, etc. requires the knowledge of sets.





Notes

The theory of sets was developed by German mathematician Georg Cantor (1845-1918). He first encountered sets while working on "problems on trigonometric series".

Numerical: Let A = Set of all even number. Are 1, 7, 4, 9 member or not of this set.

Solution: Even number set will have numbers 2,4,6,8,10…

Numbers 1, 7 & 9 are not part of this set, while number 4 is part of this set.

Conventions in Set

- o Sets are usually denoted by capital letters A, B, C, X, Y, Z, etc.
- o The elements of a set are represented by small letters a, b, c, d etc.
- o If a is an element of a set A, we say that "a belongs to A" the Greek symbol □ **(epsilon)** is used to denote the phrase 'belongs to'. Thus, we write a □
- o If 'b' is not an element of a set A, we write b □ A and read "b does not belong to A".
- o Objects, elements and members of a set are synonymous terms.

Examples:

If V is set of vowels, a & b are alphabets, then a □ V but b □ V.

P is set of prime factors of 30, then 3 □ P but 15 □ P.

There are two Methods of representing Set

- o Roster or tabular form.
- o Set-builder form.

Roster or Tabular Form

In roster form, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }

E.g., the set of all number in a dice is described in roster form as {1,2,3,4,5,6}.

Points to be noted in roster form:

- o In roster form, the order in which the elements are listed is immaterial. g. The set of all vowels in the English alphabet can be written as {a, e, i, o, u} or {a,u,i,o,e} or {u, e, i, o, a} or {o, e, i, a, u}

- The dots at the end tell us that the list of odd numbers continue indefinitely. E.g.: The set of odd natural numbers is represented by {1, 3, 5, . . .}.
- In roster form, an element is not generally repeated, i.e., all the elements are taken as distinct.

  E.g. The set of letters forming the word 'SCHOOL' is {S, C, H, O, L}.

Set Builder Form

In set-builder form, all the elements of a set possess a single common property which is not possessed by any element outside the set.

- In the set {a, e, i, o, u}, all the elements possess a common property, namely, each of them is a vowel in the English alphabet, and no other letter possess this property.

Denoting this set by V, we write V = {x: x is a vowel in English alphabet}.

- Please note that any other symbol like the letters y, z, etc. could be used.
- The symbol should be followed by a colon ":".
- After the sign of colon, we write the characteristic property possessed by the elements of the set and then enclose the whole description within braces.
- If a set of number doesn't follow any pattern, it can't be written in set builder form.

More Example of set builder form

- A = {x: x is a natural number and $3 < x < 10$} is read as "the set of all x such that x is a natural number and x lies between 3 and 10. Hence, the numbers 4, 5, 6, 7, 8 and 9 are the elements of the set A.

Numerical: Write the set {x: x is a positive integer and $x < 4$} in the roster form.

Solution: Since the set has natural numbers less that 4, Set A = {1, 2, 3}

Numerical: Write the set A = {1, 4, 9, 16, 25, . . .} in set-builder form.

Solution: Number 1, 4, 6, 16, 25.. are squares of natural numbers 1, 2, 3, 4, 5 etc..

Therefore set ={x: x is a square of N}

Empty or null or void set

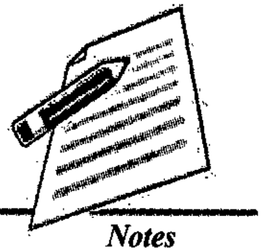
Set which does not contain any element is called the empty set or the null set or the void set. The empty set is denoted by the symbol φ or { }.

Examples of empty sets. Let A = {x: $5 < x < 6$, x is a natural number}. Then A is the empty set, Thus we dente A set by the symbol φ or { }.

Please note that A = {x: $5 < x < 6$, x is a real number} is not empty set, as there are many real number between 5 & 6.

Finite & infinite set

A set which is empty or consists of a definite number of elements is called finite



otherwise, the set is called infinite.

E.g. A = {1, 2, 3, 4, 5} Finite: n (A) =5.

B = {all natural numbers} In-finite: n (B) =infinite

n (S): number of distinct elements

Examples of infinite & finite set:

1. Let W be the set of the days of the week. Then W is finite.
2. Let G be the set of points on a line. Then G is infinite.
3. Let A be set of rats in India, then A is infinite.
4. Let B be set of months in a year, then B is finite.
5. Let P be set of all prime numbers, the P is infinite.

Equal sets

Two sets A and B are said to be equal if they have exactly the same elements and we write A = B.

Otherwise, the sets are said to be unequal and we write A ≠ B.

Examples:

Let A = {1, 2, 3, 4} and B = {3, 1, 4, 2}. Then A = B.

Let C = {1, 2, 3, 4} and D = {1,2,3,5}. Then C ≠ D.

If set A = {1, 2, 3} and B = {2, 2, 1, 3, 3}. Then A=B, since each element of A is in B and vice-versa. That is why we generally do not repeat any element in describing a set.

Sub sets

A set A is said to be a subset of a set B if every element of A is also an element of B.

Consider set A = set of all students in your class, B = set of all students in your School.

We note that every element of A is also an element of B; we say that A is a subset of B.

A is subset of B is expressed in symbols as A □ B. The symbol □ stands for 'is a subset of' or 'is contained in'.

It follows from the above definition that every set A is a subset of itself, i.e. A □ A.

Since the empty set φ has no elements, we agree to say that φ is a subset of every set.

Super Set:

Let A and B be two sets. If A □ B and A ≠ B, B is called superset of A.

1. The set **Q** of rational numbers is a subset of the set **R** of real numbers. We write **Q □ R**
2. Let A = {1, 3, 5} and B = {x: x is an odd natural number less than 6}. Then A □ B and B □ A and hence A = B.
3. Let A = {a, e, i, o, u} and B = {a, b, c, d}. Then A is not a subset of B, also B is not a subset of A.

4. Let A={1,2,3,4} and B={1,2,3,4,5,6}, then A is subset of B, and B is super set of A.
5. Some relation in well-defined sets: N □ Z □ Q □ R

N :{ 1, 2, 3, 4, 5…}	Natural number
Z: (-7, -6, -5, 1, 4, 5, ..}	Integers
Q: {1.2, 1.3, 1.5, 2.2 ...}	Rational Numbers
R: {pie, 1, 3..}	Real Number

Singleton Set

If a set A has only one element, we call it a singleton set. Thus { a } is a singleton set.
E.g. C={x : x □ N+ and x2 = 4} , it has only one element C={2}

Power Set

The collection of all subsets of a set A is called the power set of A.
E.g., consider the set {1, 2}. Let us write down all the subsets of the set {1, 2}.
Subsets of {1,2} are: φ, { 1 }, { 2 } and { 1, 2 }.
The set of all these subsets is called the power set of { 1, 2 }.
In general, if A is a set with n (A) = m, then it can be shown that n [P(A)] = 2m

Universal Set

Universal Set: A set containing all elements of a problem under consideration is called universal set. It is denoted by U.
E.g. while studying the system of numbers, we are interested in the set of natural numbers and its subsets such as the set of all prime numbers, the set of all even numbers, and so forth. This basic set is called the "Universal Set". Here universal set is R.
Numerical: Recommend a universal set for A = {1, 3, 5}, B= {1,2,7}
Solution: Set B= {1,2,3,4,5,6,7,8,} can be a universal set in this case

Open & Closed Intervals

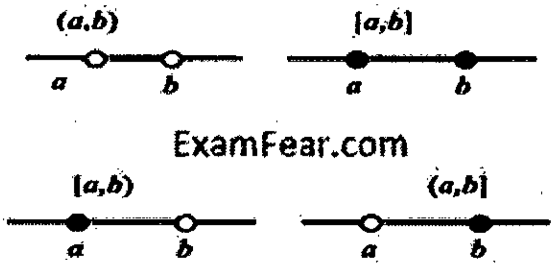
Let a, b □ R and a < b.

Open interval is denoted by (a, b) = {x: a < y < b}. Endpoints element NOT included.

Closed interval is denoted by [a, b] = {x: a ≤ x ≤ b}. Endpoints element included.

We can also have intervals closed at one end and open at the other, i.e.

- [a, b) = {x : a ≤ x < b} is an open interval from a to b, including a but excluding
- (a, b] = { x : a < x ≤ b } is an open interval from a to b including b but excluding



Memory tip:

ONE (Open No endpoint).

Open windows = Light

Closed Window = Dark

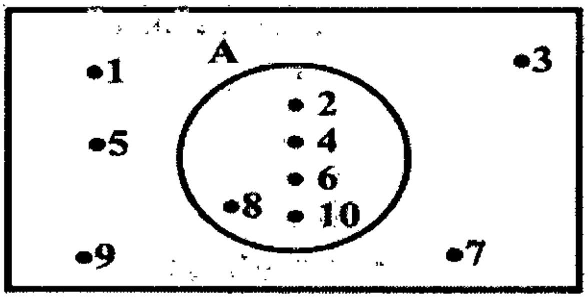
() Looks like O, so it is used for Open interval

[] looks like closed box, so it is used for closed interval

Venn Diagrams

Most of the relationships between sets can be represented by means of diagrams known as Venn diagrams. Venn diagrams are named after the, John Venn. These diagrams consist of rectangles and closed curves usually circles. The universal set is represented usually by a rectangle and its subsets by circles.

In the Venn diagram below: U = {1,2,3, ..., 10} is the universal set of which A = {2,4,6,8,10} is a subset.



Union on Sets

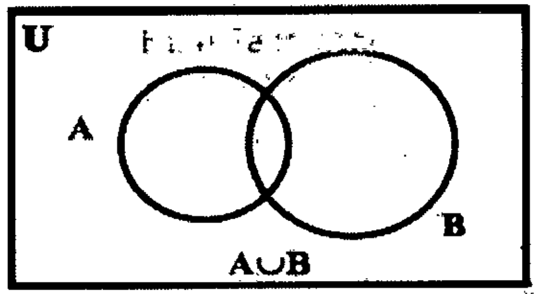
Let A and B be any two sets. The union of A and B is the set which consists of all the elements of A and all the elements of B, the common elements being taken only once. The symbol '□' is used to denote the union. Symbolically, we write A □ B and usually read as 'A union B'.

A □ B = { x : x □A or x □B }

Numerical: Let A = { 2, 4, 6, 8} and B = { 6, 8, 10, 12}. Find A □ B.

Solution We have A □ B = { 2, 4, 6, 8, 10, 12}

Note: Common elements 6 and 8 have been taken only once while writing A □ B. In the Venn diagram below, area in the green represents A □ B



Some Properties of the Operation of Union

(i) A □ B = B □ A (Commutative law)

(ii) (A □ B) □ C = A □ (B □ C) (Associative law)

(iii) A □ φ = A (Law of identity element, φ is the identity of □)

(iv) A □ A = A (Idempotent law)

(v) U □ A = U (Law of U)

Intersection of Sets

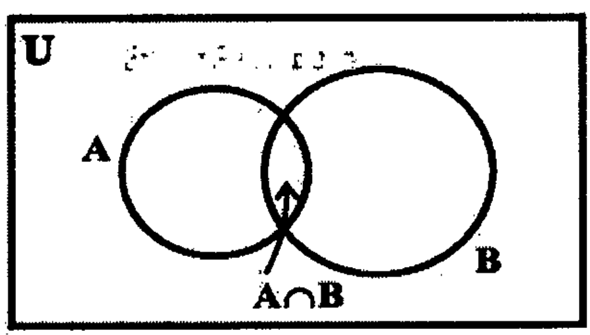
The intersection of sets A and B is the set of all elements which are common to both A and B. The symbol '∩' is used to denote the intersection.

The intersection of two sets A and B is the set of all those elements which belong to both A and B. Symbolically, we write A ∩ B = {x : x □ A and x □ B}.

Numerical: Let A = { 2, 4, 6, 8} and B = { 6, 8, 10, 12}. Find A ∩ B.

Solution We have A ∩B = {6, 8}

In the Venn diagram below, area in the green represents A ∩ B

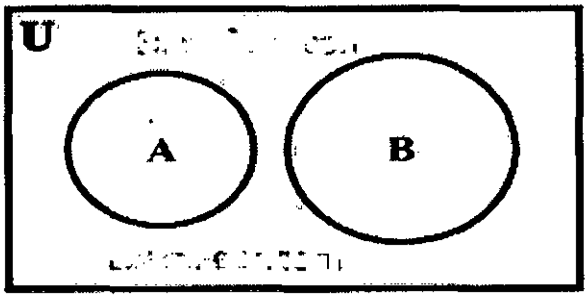


Disjoin sets

If A and B are two sets such that A ∩ B = φ, then A and B are called disjoint sets.

For example, let A = {2, 4, 6, 8 } and B = { 1, 3, 5, 7 }. Then A and B are called disjoint sets, because there are no elements which are common to A and B

The disjoint sets can be represented by means of Venn diagram below. There is no common area shared by A & B, thus A & B are disjoining sets.



Some Properties of Operation of Intersection

(i) $A \cap B = B \cap A$ (Commutative law).

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law).

(iii) $\varphi \cap A = \varphi$, $U \cap A = A$ (Law of φ and U).

(iv) $A \cap A = A$ (Idempotent law)

(v) A ∩ (B □ C) = (A ∩ B) □ (A ∩ C) (Distributive law) i. e., ∩ distributes over □

Difference of Sets

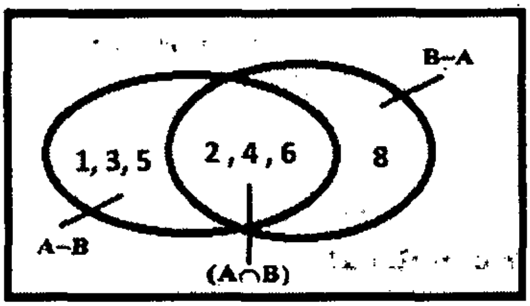
The difference of the sets A and B in this order is the set of elements which belong to A but not to B. Symbolically, we write A – B and read as " A minus B". A – B = { x : x □ A and x □ B }

Example Let A = { 1, 2, 3, 4, 5, 6}, B = { 2, 4, 6, 8 }. Find A – B and B – A

Solution:

A-B= {1,3,5}

B-A ={ 8}



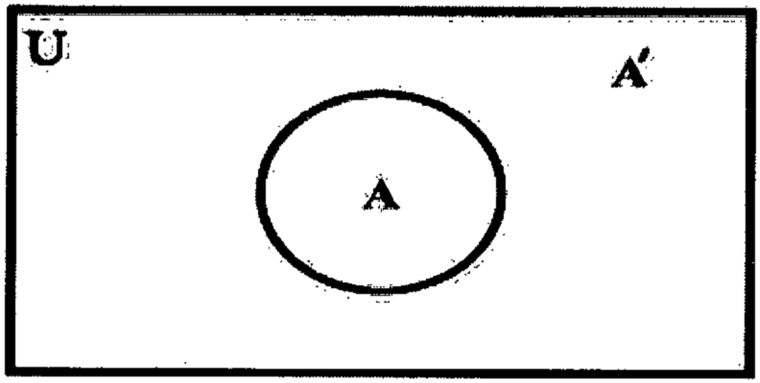
Note that the sets A – B, A ∩ B and B – A are mutually disjoint sets.

Complement of a Set

Let U be the universal set and A a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A. Symbolically, we write A′ to denote the complement of A with respect to U. Thus, A′ = {x : x □ U and x □ A }. Obviously A′ = U – A

We note that the complement of a set A can be looked upon, alternatively, as the difference between a universal set U and the set A.

E.g. $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$. Then $A' = \{2,4,6,8,10\}$



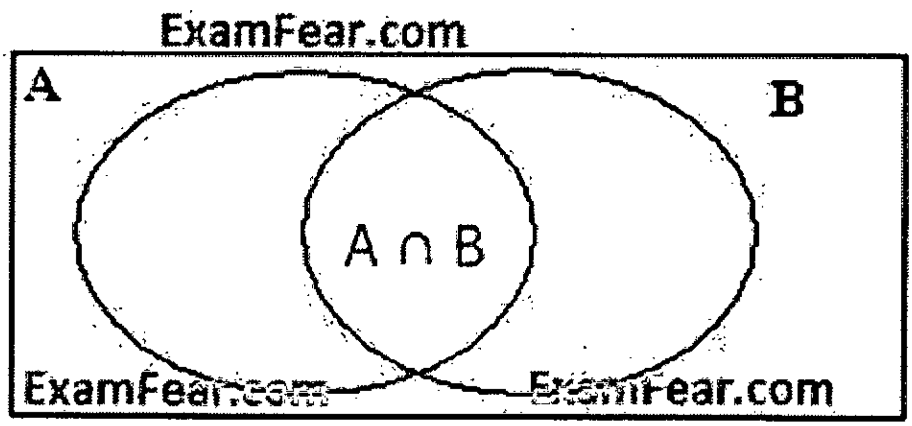
Properties of Complement of Set

1. Complement laws: (i) $A \cup A' = U$ (ii) $A \cap A' = \varphi$
2. De Morgan's law: (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
3. Law of double complementation: $(A')' = A$
4. Laws of empty set and universal set $\varphi' = U$ and $U' = \varphi$.

Number of Elements in a set

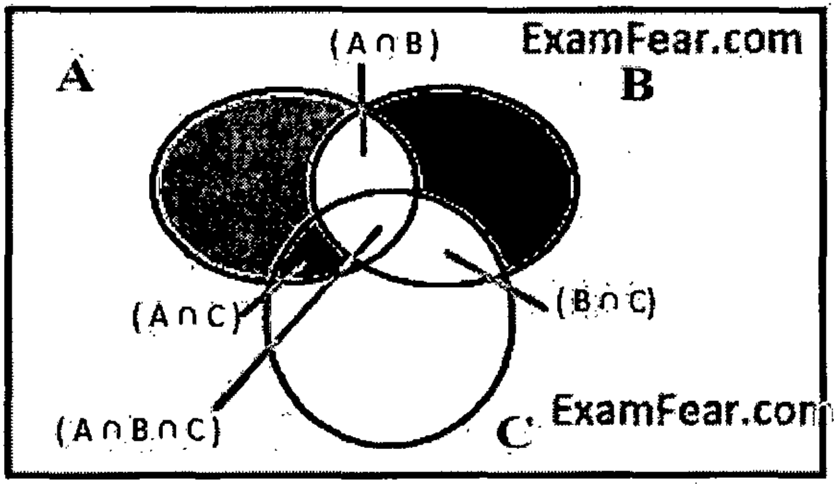
If A, B and C are finite sets, then

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$



Explanation for $n(A \cup B) = n(A) + n(B) - n(A \cap B)$: Since the common elements $A \cap B$ is counted twice with both $n(A)$ & $n(B)$, we subtract it.

- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$



Numerical: In a school there are 20 teachers who teach mathematics or physics. Of these, 12 teach mathematics and 4 teach both physics and mathematics. How many teach physics ?

Solution: Let P denote Physics teachers & M denote Maths Teacher.

$n(M) = 12$

$n(M \cup P) = 20$

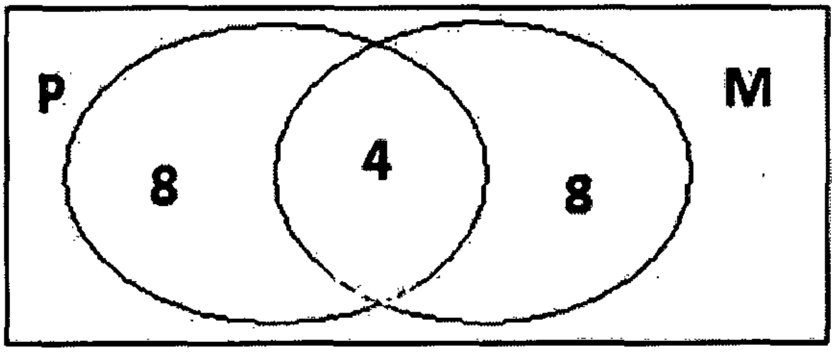
$n(M \cap P) = 4$

Applying formula $\quad n(M \cup P) = n(M) + n(P) - n(M \cap P)$

Or $20 = 12 + n(P) - 4$

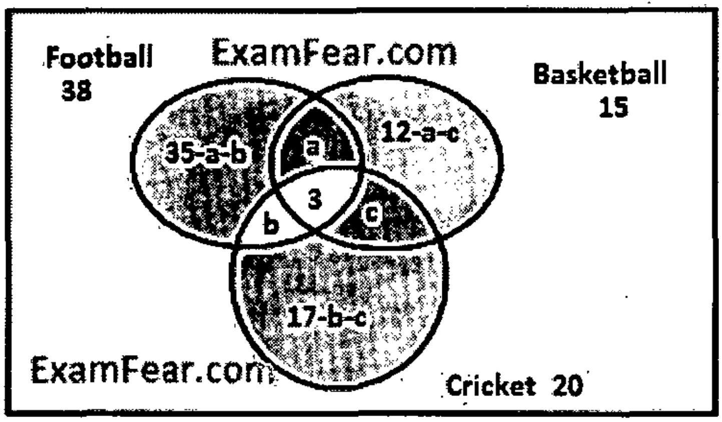
Or $n(P) = 12$

We can also solve this with Venn diagram



Numerical: A college awarded 38 medals in football, 15 in basketball and 20 in cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of the three sports ?

We can represent the data using Venn diagram.



Steps

- o 3 men got all 3 medals, so $n(A \cap B \cap C)$ will be 3. Thus put 3 in the region $n(A \cap B \cap C)$
- o Lets count of orange, blue & pink region be a, b & c. These people received exactly 2 medals.
- o The Purple region will be 35-a-b, since 38 people got medals in football. Total count for football circle is 38. Similarly, grey region will be 12-a-c & Green will be 17-b-c
- o Now it is total that total 58 men received these medals. That is n (A □ B □ C) = 58

  Or $58 = (35-a-b) + a + (12-a-c) + 3 + b + (17-b-c) + c$

  Or $a+b+c = 9$

  Thus 9 people received 2 medals.

  Thus, we can say that 3 men received 3 medals, 9 men received 2 medals & 26 men received 1 medal.

Summary of the Chapter

The concept of set serves as a fundamental part of the present-day mathematics. Today this concept is being used in almost every branch of mathematics. Sets are used to define the concepts of relations and functions. The study of geometry, sequences, probability, etc. requires the knowledge of sets. The theory of sets was developed by German mathematician Georg Cantor (1845-1918). He first encountered sets while working on "problems on trigonometric series". Numerical: Let A = Set of all even number. Are 1, 7, 4, 9 member or not of this set. Numbers 1, 7 & 9 are not part of this set, while number 4 is part of this set Conventions in Sets are usually denoted by capital letters A, B, C, X, Y, Z, etc. The elements of a set are represented by small letters a, b, c, d etc. If a is an element of a set A, we say that "a belongs to A" the Greek symbol □ **(epsilon)** is used to denote the phrase 'belongs to'. Thus, we write a □. If 'b' is not an element of a set A, we write b □ A and read "b does not belong to A". Objects, elements and members of a set are synonymous terms.

Review Questions

1. Which of the following collections are sets ?
   (i) The collection of days in a week starting with S.
   (ii) The collection of natural numbers upto fifty.
   (iii) The collection of poems written by Tulsidas.
   (iv) The collection of fat students of your school.
2. Insert the appropriate symbol in blank spaces. If A = {1, 2, 3}.
   (i) 1...........A
   (ii) 4........A.
3. Write each of the following sets in the Roster form :
   (i) A = { x : x z and 5 x 0 □ − ≤ ≤ }.
   (ii) B = { } 2 x : x R and x 1 0 □ − = .
   (iii) C = {x : x is a letter of the word banana}.
   (iv) D = {x : x is a prime number and exact divisor of 60}.
4. Write each of the following sets in the set builder form ?
   (i) A = {2, 4, 6, 8, 10}
   (ii) B = {3, 6, 9,...... ∞ }
   (iii) C = {2, 3, 5, 7}
   (iv) D = {− 2, 2} Are A and B disjoints sets ?
5. Which of the following sets are finite and which are infinite?
   (i) Set of lines which are parallel to a given line.
   (ii) Set of animals on the earth.
   (iii) Set of Natural numbers less than or equal to fifty.
   (iv) Set of points on a circle.
6. Which of the following are null set or singleton?
   (i) A = { x: x R and x is a solution of x 2 0 □ + = 2 } .
   (ii) B = {x : x Z and x is a solution of x 3 0 □ − = }.
   (iii) C = { x : x Z and x is a solution of x 2 0 □ − = 2 }.
   (iv) D = {x: x is a student of your school studying in both the classes XI and XII}
7. In the following check whether A = B or A B≈ .
   (i) A = {a}, B = {x: x is an even prime number}.
   (ii) A = {1, 2, 3, 4}, B = {x: x is a letter of the word guava}.
   (iii) A = {x : x is a solution of 2 x 5x 6 0 − + = },B = {2 , 3}.
8. Insert the appropriate symbol in the blank spaces, given that A= {1, 3, 5, 7, 9}
   (i) φ ....A
   (ii) {2, 3, 9}..........A
   (iii) 3............A
   (iv) 10...................A

9. Given that A = {a, b}, how many elements P (A) has?
10. Let A = { φ , {1} ,{2}, {1,2}}. Which of the following is true or false?
    (i) {1, 2} □ A
    (ii) φ□A.
11. Which of the following statements are true or false?
    (i) Set of all boys, is contained in the set of all students of your school.
    (ii) Set of all boy students of your school, is contained in the set of all students of your school.
    (iii) Set of all rectangles, is contained in the set of all quadrilaterals.
    (iv) Set of all circles having centre at origin is contained in the set of all ellipses having centre at origin.
12. If A = { 1, 2, 3, 4, 5 } , B= {5, 6, 7} find
    (i) A−B
    (ii) B−A.
13. Let N be the universal set and A, B, C, D be its subsets given by A = {x : x is a even natural number}, B = {x : x□N and x is a multiple of 3} C = {x : x□N and x ≥5}, D = {x : x□N and x ≤ 10}
    Find complements of A, B, C and D respectively.
14. Write the following sets in the interval form.
    (a) {x R:-8 x 3} □ < <
    (b) {x R: 3 2x 7} □ ≤ □
15. Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {1, 2, 3, 4} and B = {2, 4, 6, 8}, then verify the following
    (i) (A′)′ = A
    (ii) (B′)′ = B
    (iii) A ∩ A′ = φ
    (iv) (A □ B)′ = A′ ∩ B′

2 RELATIONS

- Understand the concept of Relations.
- Discuss the properties of relations.
- Describe the empty and extreme relation.
- Understand the types of relations.

Objective of the chapter:

The basic objective of this chapter is to through some light on the initial concepts of relations so that the fundamentals of relations can be learned.

Introduction

We approach the concept of relations in different aspects using real life sense, Cryptography and Geometry through Cartesian products of sets.

In our day-to-day life very often, we come across questions like, "*How is he related to you?*" Some probable answers are,

i. He is my father.

ii. He is my teacher.

iii. He is not related to me.

From this we see that the word relation connects a person with another person. Extending this idea, in mathematics we consider relations as one which connects mathematical objects. Examples,

i. A number m is related to a number n if m divides n in N.

ii. A real number x is related to a real number y if $x \leq y$.

iii. A point p is related to a line L if p lies on L.

iv. A student X is related to a school S if X is a student of S.

Illustration 1.1 (Cryptography) for centuries, people have used ciphers or codes, to keep confidential information secure. Effective ciphers are essential to the military, to financial institutions and to computer programmers. The study of the techniques used in creating coding and decoding these ciphers is called cryptography.

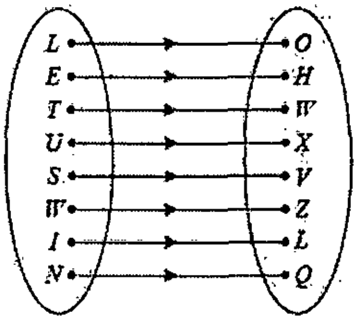


Figure 1.4

One of the earliest methods of coding a message was a simple substitution. For example, each letter in a message might be replaced by the letter that appears three places later in the alphabet.

Using this coding scheme, "LET US WIN" becomes "OHW XVZ LQ". This scheme was used by Julius Caesar and is called the Caesars cipher. To decode, replace each letter by the letter three places before it. This concept is used often in Mental Ability Tests. The above can be represented as an arrow diagram as given in Figure 1.4.

This can be viewed as the set of ordered pairs

$\{(L, O), (E, H), (T, W), (U, X), (S, V), (W, Z), (I, L), (N, Q)\}$

Which is a subset of the Cartesian product $C \times D$ where $C = \{L, E, T, U, S, W, I, N\}$ and $D = \{O, H, W, X, V, Z, L, Q\}$.

Illustration 1.2 (Geometry) Consider the following three equations

(i) 2x − y = 0

(ii) x2 − y = 0

(iii) x − y2 = 0

(i) 2x − y = 0

The equation $2x - y = 0$ represents a straight line. Clearly the points, (1, 2), (3, 6) lie on it whereas (1, 1), (3, 5), (4, 5) are not lying on the straight line. The analytical relation between x and y is given by $y = 2x$. Here the values of y depend on the values of x. To denote this dependence, we write $y = f(x)$. The set of all points that lie on the straight line is given as $\{(x, 2x) : x \square \mathrm{R}\}$. Clearly this is a subset of R × R. (See Figure 1.5.)

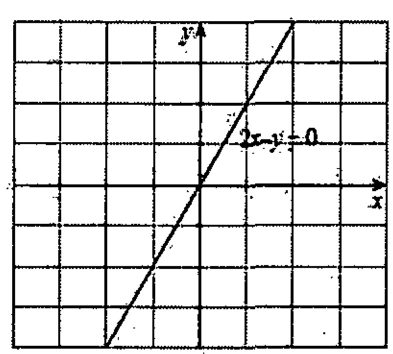


Figure 1.5

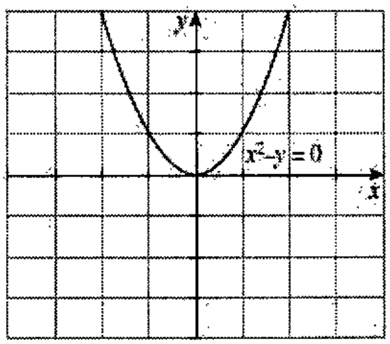


Figure 1.6

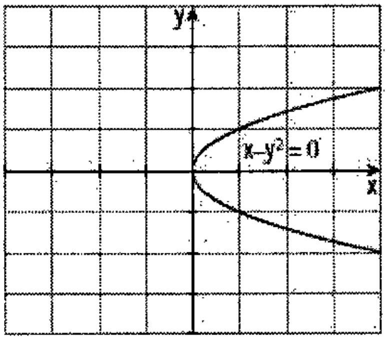


Figure 1.7

(ii) x2 − y = 0.

As we discussed earlier, the relation between *x* and *y* is *y* = *x*2. The set of all points on the curve is *{(x, x2) : x □ R}* (See Figure 1.6). Again this is a subset of the Cartesian product RxR.

(iii) x − y2 = 0

As above, the relation between x and y is $y^2 = x$ or $y = \pm\sqrt{x}$, $x \geq 0$. The equation can also be re-written as $y = +\sqrt{x}$ and $y = -\sqrt{x}$. The set of all points on the curve is the union of the sets $\{(x, \sqrt{x})\}$ and $\{(x, -\sqrt{x})\}$, where x is a non-negative real number, are the subsets of the Cartesian product $\mathbb{R} \times \mathbb{R}$. (See Figure 1.7).

From the above examples we intuitively understand what a relation is. But in mathematics, we have to give a rigorous definition for each and every technical term we are using. Now let us start defining the term "relation" mathematically.

Definition of Relation

Let *A = {p, q, r, s, t, u}* be a set of students and let *B = {X, Y, Z, W}* be a set of schools. Let us consider the following "relation".

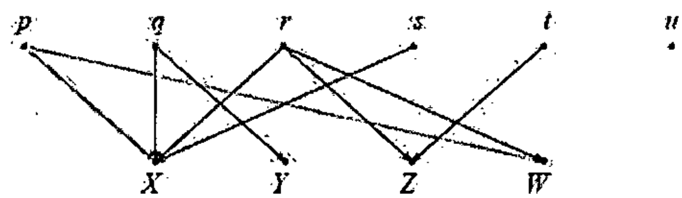
A student *a* □ *A* is related to a school *S* □ *B* if "*a*" is studying or studied in the school *S*.

Let us assume that *p* studied in *X* and now studying in *W*, *q* studied in *X* and now studying in *Y*, *r* studied in *X* and *W*, and now studying in *Z*, *s* has been studying in *X* from the beginning, *t* studied in *Z* and now studying in no school, and *u* never studied in any of these four schools.

Though the relations are given explicitly, it is not possible to give a relation always in this way. So let us try some other representations for expressing the same relation:

(i) $\begin{matrix} p & p & q & q & r & r & r & s & t \\ X & W & X & Y & X & Z & W & X & Z \end{matrix}$

(ii)



(iii) $\{(p, X), (p, W), (q, X), (q, Y), (r, X), (r, Z), (r, W), (s, X), (t, Z)\}$

(iv) $pRX, pRW, qRX, qRY, rRX, rRZ, rRW, sRX, tRZ.$

Among these four representations of the relation, the third one seems to be more convenient and comfortable to deal with a relation in terms of sets.

The set given in the third representation is a subset of the Cartesian product $A \times B$. In Illustrations 1.1 and 1.2 also, we arrived at subsets of a Cartesian product.

Definition 1.2

Let A and B be any two non-empty sets. A *relation* R from A to B is defined as a subset of the Cartesian product of A and B. Symbolically $R \subseteq A \times B$.

A relation from A to B is different from a relation from B to A.

The set *{a □ A : (a, b) □ R for some b □ B}* is called the ***domain*** of the relation.

The set *{b □ B : (a, b) □ R for some a □ A}* is called the ***range*** of the relation.

Thus the domain of the relation R is the set of all first coordinates of the ordered pairs and the range of the relation R is the set of all second coordinates of the ordered pairs.'

Illustration 1.3 Consider the diagram in Figure 1.8. Here the alphabets are mapped onto the natural numbers. A simple cipher is to assign a natural number to each alphabet. Here a is represented by 1, b is represented by 2, . . . , z is represented by 26. This correspondence can be written as the set of ordered pairs *{(a, 1), (b, 2), . . . , (z, 26)}*. This set of ordered pairs is a relation. The domain of the relation is *{a, b, . . . , z}* and the range is *{1, 2, . . . , 26}*.

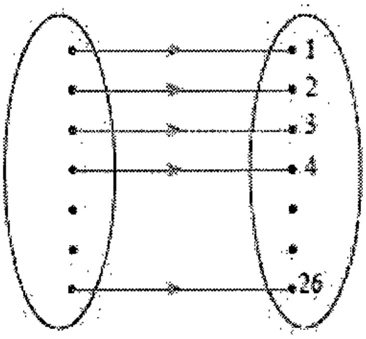


Figure 1.8

Now we recall that the relations discussed in Illustrations 1.1 and 1.2 also end up with subsets of the Cartesian product of two sets. So the term *relation* used in all discussions we had so far, fits with the mathematical term *relation* defined in Definition 1.2.

The domain of the relation discussed in Illustration 1.1 is the set *{L, E, T, U, S, W, I, N}* and the range is *{O, H, W, X, V, Z, L, Q}*. In Illustration 1.2, the domain and range of the relation discussed for the equation $2x - y = 0$ are R and R (See Figure 1.9); for the equation $x2 - y = 0$, the domain is R and the range is $[0, \infty)$ (See Figure 1.10); and in the case of the third equation $x - y2 = 0$, the domain is $[0, \infty)$ and the range is R (See Figures 1.11 and 1.12).

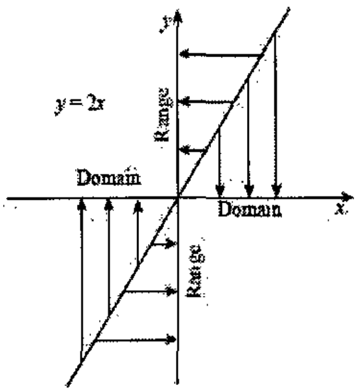


Figure 1.9

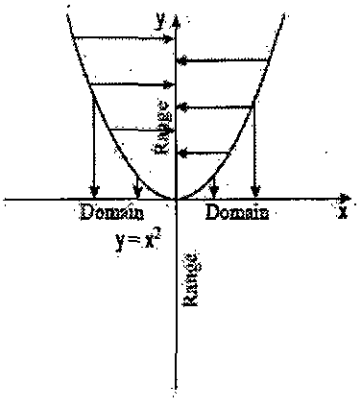


Figure 1.10

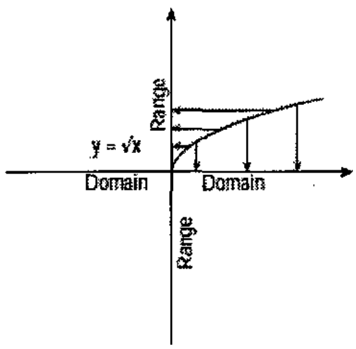


Figure 1.11

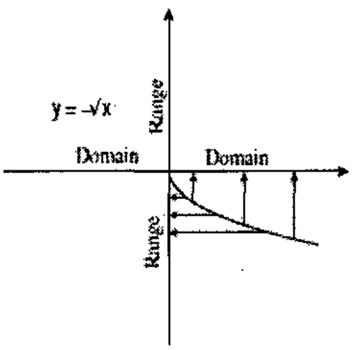


Figure 1.12

CLASS–12

Mathematics

Notes

Note that, the domain of a relation is a subset of the first set in the Cartesian product and the range is a subset of second set. Usually, we call the second set as ***co-domain*** of the relation. Thus, the range of a relation is the collection of all elements in the co-domain which are related to some element in the domain. Let us note that the range of a relation is a subset of the co-domain.

For any set A, □ and $A \times A$ are subsets of $A \times A$. These two relations are called ***extreme relations***.

The former relation is an ***empty relation*** and the latter is a ***universal relation.***

We will discuss more about domain, co-domain and the range in the next section namely, "Functions".

If R is a relation from A to B and if (x, y) □ R, then sometimes we write xRy (read this as "x is related to y") and if (x, y) □/ R, then sometimes we write xRy (read this as "x is not related to y").

Though the general definition of a relation is defined from one set to another set, relations defined on a set are of more interest in mathematical point of view. That is, relations in which the domain and the co-domain are the same area of more interest. So let us concentrate on relations defined on a set.

1. Type of Relations

Consider the following examples:

I. Let $S = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 3), (2, 3)\}$ on S.

ii. Let $S = \{1, 2, 3, 10\}$ and define "m is related to n, if m divides n".

iii. Let C be the set of all circles in a plane and define "a circle C is related to a circle C, if the radius of C is equal to the radius of C".

iv. In the set S of all people define "a is related to b, if a is a brother of b".

v. Let S be the set of all people. Define the relation on S by the rule "mother of".

In the second example, as every number divides itself, "a is related a for all a □ S"; the same is true in the third relation also. In the first example "a is related a for all a □ S" is not true as 2 is not related to 2.

It is easy to see that the property "if a is related to b, then b is related to a" is true in the third but not in the second.

It is easy to see that the property "if a is related to b and b is related to c, then a is related to c" is true in the second and third relations but not in the fifth.

Definition 1.3

Let S be any non-empty set. Let R be a relation on S. Then

- R is said to be *reflexive* if a is related to a for all $a \in S$.
- R is said to be *symmetric* if a is related to b implies that b is related to a.
- R is said to be *transitive* if "a is related to b and b is related to c" implies that a is related to c.

These three relations are called *basic relations*.

These properties, together with some more properties are very much studied in mathematical structures. Let us define them now.

Let us rewrite the definitions of these basic relations in a different form:

Let S be any non-empty set. Let R be a relation on S. Then R is

- reflexive if "(a, a) □ R for all a □ S".
- symmetric if "(a, b) □ R □ (b, a) □ R".
- transitive if "(a, b), (b, c) □ R □ (a, c) □ R".

> **Definition 1.4**
>
> Let S be any set. A relation on S is said to be an *equivalence relation* if it is reflexive, symmetric and transitive.

Let us consider the following two relations.

- In the set $S1$ of all people, define a relation $R1$ by the rule: "a is related to b, if a is a brother of b".
- In the set $S2$ of all males, define a relation $R2$ by the rule: "a is related to b, if a is a brother of b".

The rules that define the relations on $S1$ and $S2$ are the same. But the sets are not same. $R1$ is not a symmetric relation on $S1$ whereas $R2$ is a symmetric relation on $S2$. This shows that not only the rule defining the relation is important, the set on which the relation is defined, is also important. So whenever one considers a relation, both the relation as well as the set on which the relation is defined have to be given explicitly. Note that the relation $\{(1, 1), (2, 2), (3, 3), (1, 2)\}$ is reflexive if it is defined on the set $\{1, 2, 3\}$; it is not reflexive if it is defined on the set $\{1, 2, 3, 4\}$.

Illustration 1.4

1. Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 1), (2, 2), (3, 3), (1, 3), (4, 4), (1, 2), (3, 1)\}$. As $(1, 1)$, $(2, 2)$, $(3, 3)$ and $(4, 4)$ are all in R, it is reflexive. Also for each pair (a, b) □ R the pair (b, a) is also in R. So R is symmetric. As $(2, 1)$, $(1, 3)$ □ R and $(2, 3)$ □/ R, we see that R is not transitive. Thus R is not an equivalence relation.
2. Let P denote the set of all straight lines in a plane. Let R be the relation defined on P as Rm if is parallel to m.

   This relation is reflexive, symmetric and transitive. Thus it is an equivalence relation.
3. Let A be the set consisting of children and elders of a family. Let R be the relation defined by aRb if a is a sister of b.

   This relation is to be looked into carefully. A woman is not a sister of herself. So it is not reflexive. It is not symmetric also. Clearly it is not transitive. So it is not an equivalence relation. (If we consider the same relation on a set consisting only of females, then it becomes symmetric; even in this case it is not transitive).

4. On the set of natural numbers let R be the relation defined by xRy if $x + 2y = 21$. It is better to write the relation explicitly. The relation R is the set
   $\{(1, 10), (3, 9), (5, 8), (7, 7), (9, 6), (11, 5), (13, 4), (15, 3), (17, 2), (19, 1)\}$.
   As $(1, 1)$ □/ R it is not reflexive; as $(1, 10)$ □ R and $(10, 1)$ □/ R it is not symmetric.
   As $(3, 9)$ □ R, $(9, 6)$ □ R but $(3, 6)$ □/ R, the relation is not transitive.
5. Let $X = \{1, 2, 3, 4\}$ and R = □, where □ is the empty set.
   As $(1, 1)$ □/ R it is not reflexive. As we cannot find a pair (x, y) in R such that (y, x) □/ R, the relation is not 'not symmetric'; so it is symmetric. Similarly it is transitive.
6. The universal relation is always an equivalence relation.
7. An empty relation can be considered as symmetric and transitive.
8. If a relation contains a single element, then the relation is transitive.

   Let us discuss some more special relations now.

Example 1.10 Check the relation $R = \{(1,1), (2,2), (3,3), \ldots, (n,n)\}$ defined on the set $S = \{1, 2, 3, \ldots, n\}$ for the three basic relations.

Solution:
As $(a, a) \in R$ for all $a \in S$, R is reflexive.
There is no pair (a, b) in R such that $(b, a) \notin R$. In other words, for every pair $(a, b) \in R$, (b, a) is also in R. Thus R is symmetric.
We cannot find two pairs (a, b) and (b, c) in R, such that $(a, c) \notin R$. Thus the statement "R is not transitive" is not true; therefore, the statement "R is transitive" is true; hence R is transitive.
Since R is reflexive, symmetric and transitive, this relation is an equivalence relation.

From the very beginning we have denoted all the relations by the same letter R. It is not necessary to do so. We may use the Greek letter ρ (Read as rho) to denote relations. Equivalence relations are mostly denoted by "□".

If a relation is not of required type, then by inserting or deleting some pairs we can make it of the required type. We do this in the following problem.

Example 1.11 Let $S = \{1, 2, 3\}$ and $\rho = \{(1,1), (1,2), (2,2), (1,3), (3,1)\}$.

(i) Is ρ reflexive? If not, state the reason and write the minimum set of ordered pairs to be included to ρ so as to make it reflexive.

(ii) Is ρ symmetric? If not, state the reason, write minimum number of ordered pairs to be included to ρ so as to make it symmetric and write minimum number of ordered pairs to be deleted from ρ so as to make it symmetric.

(iii) Is ρ transitive? If not, state the reason, write minimum number of ordered pairs to be included to ρ so as to make it transitive and write minimum number of ordered pairs to be deleted from ρ so as to make it transitive.

(iv) Is ρ an equivalence relation? If not, write the minimum ordered pairs to be included to ρ so as to make it an equivalence relation.

Solution:

(i) ρ is not reflexive because $(3, 3)$ is not in ρ. As $(1, 1)$ and $(2, 2)$ are in ρ, it is enough to include the pair $(3, 3)$ to ρ so as to make it reflexive.

(ii) ρ is not symmetric because $(1, 2)$ is in ρ, but $(2, 1)$ is not in ρ. It is enough to include the pair $(2, 1)$ to ρ so as to make it symmetric.
It is enough to remove the pair $(1, 2)$ from ρ so as to make it symmetric

(iii) ρ is not transitive because $(3,1)$ and $(1,3)$ are in ρ, but $(3,3)$ is not in ρ. To make it transitive we have to include $(3,3)$ in ρ. Even after including $(3,3)$, the relation is not transitive because $(3,1)$ and $(1,2)$ are in ρ, but $(3,2)$ is not in ρ. To make it transitive we have to include $(3,2)$ also in ρ. Now it becomes transitive. So $(3,3)$ and $(3,2)$ are to be included so as to make ρ transitive.

But if we remove $(3,1)$ from ρ, then it becomes transitive.

(iv) We have seen that

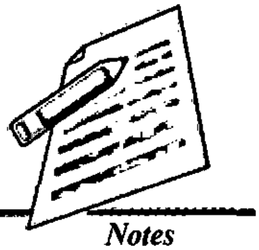
- to make ρ reflexive, we have to include $(3,3)$;
- to make ρ symmetric, we have to include $(2,1)$;
- and to make ρ transitive, we have to include $(3,3)$ and $(3,2)$.

To make ρ as an equivalence relation we have to include all these pairs. So after including the pairs the relation becomes $\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(3,2)\}$

But this relation is not symmetric because $(3,2)$ is in the relation and $(2,3)$ is not in the relation. So we have to include $(2,3)$ also. Now the new relation becomes

$$\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(3,2),(2,3)\}$$

It can be seen that this relation is reflexive, symmetric and transitive, and hence it is an equivalence relation. Thus we have to include $(3,3)$, $(2,1)$, $(3,2)$ and $(2,3)$ to ρ so as to make it an equivalence relation.



Notes

Now let us learn how to create relations having certain properties through the following example.

Example 1.12 Let $A = \{0,1,2,3\}$. Construct relations on A of the following types:

(i) not reflexive, not symmetric, not transitive.
(ii) not reflexive, not symmetric, transitive.
(iii) not reflexive, symmetric, not transitive.
(iv) not reflexive, symmetric, transitive.
(v) reflexive, not symmetric, not transitive.
(vi) reflexive, not symmetric, transitive.
(vii) reflexive, symmetric, not transitive.
(viii) reflexive, symmetric, transitive.

Solution:

(i) Let us use the pair $(1,2)$ to make the relation "not symmetric" and consider the relation $\{(1,2)\}$. It is transitive. If we include $(2,3)$ and not include $(1,3)$, then the relation is not transitive. So the relation $\{(1,2),(2,3)\}$ is not reflexive, not symmetric and not transitive. Similarly we can construct more examples.

(ii) Just now we have seen that the relation $\{(1,2)\}$ is transitive, not reflexive and not symmetric.

(iii) *Let us start with the pair $(1,2)$. Since we need symmetricity, we have to include the pair $(2,1)$.* At this stage as $(1,1)$, $(2,2)$ are not here, the relation is not transitive. Thus $\{(1,2),(2,1)\}$ is not reflexive; it is symmetric; and it is not transitive.

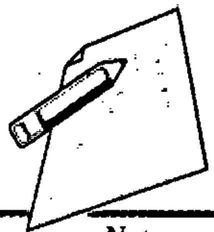
(iv) If we include the pairs $(1,1)$ and $(2,2)$ to the relation discussed in (iii), it will become transitive. Thus $\{(1,2),(2,1),(1,1),(2,2)\}$ is not reflexive; it is symmetric and it is transitive.

(v) For a relation on $\{0,1,2,3\}$ to be reflexive, it must have the pairs $(0,0)$, $(1,1)$, $(2,2)$, $(3,3)$. Fortunately, it becomes symmetric and transitive. Therefore, as in (i) if we insert $(1,2)$ and $(2,3)$ we get the required one. Thus $\{(0,0),(1,1),(2,2),(3,3),(1,2),(2,3)\}$ is reflexive; it is not symmetric and it is not transitive.

(vi) Proceeding like this we get the relation $\{(0,0),(1,1),(2,2),(3,3),(1,2)\}$ that is reflexive, transitive and not symmetric.

(vii) As above we get the relation $\{(0,0),(1,1),(2,2),(3,3),(1,2),(2,3),(2,1),(3,2)\}$ that is reflexive, symmetric and not transitive.

(viii) We have the relation $\{(0,0),(1,1),(2,2),(3,3)\}$ which is reflexive, symmetric and transitive.



Example 1.13 In the set $\mathbf{Z}$ of integers, define mRn if $m - n$ is a multiple of 12. Prove that R is an equivalence relation.

Solution:

As $m - m = 0$ and $0 = 0 \times 12$, hence mRm proving that R is reflexive.

Let mRn. Then $m - n = 12k$ for some integer k; thus $n - m = 12(-k)$ and hence nRm. This shows that R is symmetric.

Let mRn and nRp: then $m - n = 12k$ and $n - p = 12\ell$ for some integers k and ℓ. So $m - p = 12(k + \ell)$ and hence mRp. This shows that R is transitive.

Thus R is an equivalence relation.

Theorem 1.1: The number of relations from a set containing m elements to a set containing n elements is $2mn$. In particular the number of relations on a set containing n elements is 2^{n^2}.

Proof. Let A and B be sets containing m and n elements respectively. Then $A \times B$ contains mn elements and $A \times B$ has 2^{mn} subsets. Since every subset of $A \times B$ is a relation from A to B, there are 2^{mn} relations from a set containing m elements to a set containing n elements.

Taking $A = B$, we see that the number of relations on a set containing n elements is 2^{n^2}.

Definition 1.5

If R is a relation from A to B, then the relation R^{-1} defined from B to A by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

is called the *inverse* of the relation R.

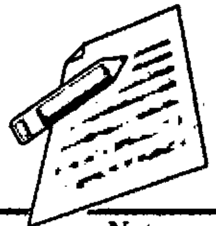
For example, if $R = \{(1, a), (2, b), (2, c), (3, a)\}$, then

$R-1 = \{(a, 1), (b, 2), (c, 2), (a, 3)\}$.

It is easy to see that the domain of R becomes the range of $R-1$ and the range of R becomes the domain of $R-1$.

Summary of the Chapter

We approach the concept of relations in different aspects using real life sense, Cryptography and Geometry through Cartesian products of sets. In our day-to-day life very often, we come across questions like, "*How is he related to you?*". Some probable answers are, i. He is my father. ii. He is my teacher. iii. He is not related to me. From this we see that the word relation connects a person with another person. Extending this idea, in mathematics we consider relations as one which connects mathematical objects. Examples, i. A number m is related to a number n if m divides n in N. ii. A real number x is related to a real number y if $x \leq y$. iii. A point p is related to a line L if p lies on L. iv. A student X is related to a school S if X is a student of S. Though the general definition of a relation is defined from one set to another set, relations defined on a set are of more interest in mathematical point of view. That is, relations in which the domain and the co-domain are the same are of more interest. So let us concentrate on relations defined on a set.



Notes

Review Questions

Exercise - 1.2

1. Discuss the following relations for reflexivity, symmetricity and transitivity:
   (i) The relation R defined on the set of all positive integers by "mRn if m divides n".
   (ii) Let P denote the set of all straight lines in a plane. The relation R defined by "ℓRm if ℓ is perpendicular to m".
   (iii) Let A be the set consisting of all the members of a family. The relation R defined by "aRb if a is not a sister of b".
   (iv) Let A be the set consisting of all the female members of a family. The relation R defined by "aRb if a is not a sister of b".
   (v) On the set of natural numbers the relation R defined by "xRy if $x + 2y = 1$".
2. Let $X = \{a, b, c, d\}$ and $R = \{(a, a), (b, b), (a, c)\}$. Write down the minimum number of ordered pairs to be included to R to make it
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence
3. Let $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (a, c)\}$. Write down the minimum number of ordered pairs to be included to R to make it
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence
4. Let P be the set of all triangles in a plane and R be the relation defined on P as aRb if a is similar to b. Prove that R is an equivalence relation.
5. On the set of natural numbers let R be the relation defined by aRb if $2a + 3b = 30$. Write down the relation by listing all the pairs. Check whether it is
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence
6. Prove that the relation "friendship" is not an equivalence relation on the set of all people in Chennai.
7. On the set of natural numbers let R be the relation defined by aRb if $a + b \leq 6$. Write down the relation by listing all the pairs. Check whether it is
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence
8. Let $A = \{a, b, c\}$. What is the equivalence relation of smallest cardinality on A? What is the equivalence relation of largest cardinality on A?
9. In the set Z of integers, define mRn if $m - n$ is divisible by 7. Prove that R is an equivalence relation.

3 FUNCTIONS

- Understand the concept of Functions.
- Discuss the properties of Functions.
- Describe the domain and co-domain of functions.
- Understand the types of Functions.
- Discuss the ways of representing functions

Objective of the chapter:

The basic objective of this chapter is to through some light on the initial concepts of functions so that the fundamentals of functions can be learned.

Introduction

Suppose that a particle is moving in the space. We assume the physical particle as a point. As time varies, the particle changes its position. Mathematically at any time the point occupies a position in the three-dimensional space R3. Let us assume that the time varies from 0 to 1. So the movement or *functioning* of the particle decides the position of the particle at any given time t between 0 and 1. In other words, for each t □ [0, 1], the functioning of the particle gives a point in R3. Let us denote the position of the particle at time t as $f(t)$.

Let us see another simple example. We know that the equation $2x-y=0$ describes a straight line. Here whenever x assumes a value, y assumes some value accordingly. The movement or *functioning* of y is decided by that of x. Let us denote y by $f(x)$. We may see many situations like this in nature. In the study of natural phenomena, we find that it is necessary to consider the variation of one quantity depending on the variation of another.

The relation of the time and the position of the particle, the relation of a point in the x-axis to a point in the y-axis and many more such relations are studied for a very long period in the name *function*. Before Cantor, the term function is defined as a rule which associates a variable with another variable. After the development of the concept of sets, a function is defined as a rule that associates for every element in a set A, a unique element in a set B. However the terms *rule* and *associate* are not properly defined mathematical terminologies. In modern mathematics every term we use has to be defined properly. So, a definition for function is given using relations.

Suppose that we want to discuss a test written by a set of students. We shall see this as a relation. Let A be the set of students appeared for an examination and let $B = \{0, 1, 2, 3, \ldots 100\}$ be the set of possible marks. We define a relation R as follows: A student a is related to a mark b if a got b marks in the test.

We observe the following from this example:

- Every student got a mark. In other words, for every a □ A, there is an element b □ B such that (a, b) □ R.
- A student cannot get two different marks in any test. In other words, for every a □ A, there is definitely only one b □ B such that (a, b) □ R. This can be restated in a different way: If (a, b), (a, c) □ R then $b = c$.

Relations having the above two properties form a very important class of relations, called functions.

Let us now have a rigorous definition of a function through relations.

Definition 1.6

Let A and B be two sets. A relation f from A to B, a subset of $A \times B$, is called a **function** from A to B if it satisfies the following:

(i) for all $a \in A$, there is an element $b \in B$ such that $(a, b) \in f$.

(ii) if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.

That is, a function is a relation in which each element in the domain is mapped to exactly one element in the range.

A is called the ***domain*** of f and B is called the ***co-domain*** of f. If (a, b) is in f, then we write $f(a) = b$; the element b is called the ***image*** of a and the element a is called a ***pre-image*** of b and $f(a)$ is known as the value of f at a. The set $\{b : (a, b)$ □ f for some a □ $A\}$ is called the ***range*** of the function. If B is a subset of R, then we say that the function is a ***real-valued function.***

Two functions f and g are said to be ***equal functions*** if their domains are same and $f(a) = g(a)$ for all a in the domain.

If f is a function with domain A and co-domain B, we write $f: A \rightarrow B$ (Read this as f is from A to B or f be a function from A to B). We also say that f maps A into B. If $f(a) = b$, then we say f maps a to b or a is mapped onto b by f, and so on.

The range of a function is the collection of all elements in the co-domain which have pre-images. Clearly the range of a function is a subset of the co-domain. Further the first condition says that every element in the domain must have an image; this is the reason for defining the domain of a relation R from a set A to a set B as the set of all elements of A having images and not as A. The second condition says that an element in the domain cannot have two or more images.

Naturally one may have the following doubts:

- In the definition, why we use the definite article "the" for image of a and the indefinite article "a" for pre-image of b?

- We have a condition stating that every element in the domain must have an image; is there any condition like "every element in the co-domain must have a pre-image"? If not, why?
- We have a condition stating that an element in the domain cannot have two or more images; is there any condition like "an element in the co-domain cannot have two or more pre-images"? If not, why?

As an element in the domain has exactly one image and an element in the co-domain can have more than one pre-image according to the definition, we use the definite article "the" for image of a and the indefinite article "a" for pre-image of b. There are no conditions as asked in the other two questions; the reason behind it can be understood from the problem of students' mark we considered above.

We observe that every function is a relation but a relation need not be a function.

Let $f = \{(a, 1), (b, 2), (c, 2), (d, 4)\}$.

Is f a function? This is a function from the set $\{a, b, c, d\}$ to $\{1, 2, 4\}$. This is not a function from $\{a, b, c, d, e\}$ to $\{1, 2, 3, 4\}$ because e has no image. This is not a function from $\{a, b, c, d\}$ to $\{1, 2, 3, 5\}$ because the image of d is not in the co-domain; f is not a subset of $\{a, b, c, d\} \times \{1, 2, 3, 5\}$. So whenever we consider a function the domain and the co-domain must be stated explicitly.

The relation discussed in Illustration 1.1 is a function with domain $\{L, E, T, U, S, W, I, N\}$ and co-domain $\{O, H, W, X, V, Z, L, Q\}$. The relation discussed in Illustration 1.3 is again a function with domain $\{a, b, \ldots, z\}$ and the co-domain $\{1, 2, 3, \ldots, 26\}$.

In Illustration 1.2, we discussed three relations, namely

(i) $y = 2x$

(ii) $y = x2$

(iii) $y2 = x$.

Clearly (i) and (ii) are functions whereas (iii) is not a function, if the domain and the co-domain are R. In (iii) for the same x, we have two y values which contradict the definition of the function. But if we split into two relations, that is, $y = \sqrt{x}$ and $y = -\sqrt{x}$ then both become functions with same domain non-negative real numbers and the co-domains $[0, \infty)$ and $(-\infty, 0]$ respectively.

1. Ways of Representing Functions

(a) Tabular Representation of a Function

When the elements of the domain are listed like $x1, x2, x3 \ldots xn$, we can use this tabular form. Here, the values of the arguments $x1, x2, x3 \ldots xn$ and the corresponding values of the function $y1, y2, y3 \ldots yn$ are written out in a definite order.

x	x_1	x_2	$\ldots$	x_n
y	y_1	y_2	$\ldots$	y_n

(b) Graphical Representation of a Function

When the domain and the co-domain are subsets of R, many functions can be represented using a graph with x-axis representing the domain and y-axis representing the co-domain in the (x, y)-plane.

We note that the first and second figures in Illustration 1.2 represent the functions $f(x) = 2x$ and $f(x) = x2$ respectively. Usually the variable x is treated as independent variable and y as a dependent variable. The variable x is called the ***argument*** and $f(x)$ is called the value.

(c) Analytical Representation of a Function

If the functional relation $y = f(x)$ is such that f denotes an analytical expression, we say that the function y of x is represented or defined analytically. Some examples of analytical expressions are

$$x^3 + 5, \ \frac{\sin x + \cos x}{x^2 + 1}, \ \log x + 5\sqrt{x}.$$

That is, a series of symbols denoting certain mathematical operations that are performed in a definite sequence on numbers, letters which designate constants or variable quantities.

Examples of functions defined analytically are

$$\text{(i) } y = \frac{x-1}{x+1} \quad \text{(ii) } y = \sqrt{9 - x^2} \quad \text{(iii) } y = \sin x + \cos x \quad \text{(iv) } A = \pi r^2.$$

One of the usages of writing functions analytically is finding domains naturally. That is, the set of values of x for which the analytical expressions on the right-hand side has a definite value is the natural domain of definition of a function represented analytically.

Thus, the natural domain of the function,

$$\begin{array}{llllllll} \text{(i)} & y = x^3 + 3 & \text{is} & (-\infty, \infty) & \text{(ii)} & y = x^4 - 2 & \text{is} & (-\infty, \infty) \\ \text{(iii)} & y = \frac{x-1}{x+1} & \text{is} & \mathbb{R} - \{-1\} & \text{(iv)} & y = +\sqrt{4 - x^2} & \text{is} & -2 \leq x \leq 2. \end{array}$$

Now recall the domain of the functions (i) $y = 2x$, (ii) $y = x^2$, (iii) $y = +\sqrt{x}$, (iv) $y = -\sqrt{x}$ which are analytical in nature described earlier.

Sometimes we may come across piece-wise defined functions. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & \text{if } -\infty < x \leq -2 \\ 2x & \text{if } -2 < x \leq 3 \\ x^2 & \text{if } 3 < x \leq \infty. \end{cases}$$

Depending upon the value of x, we have to select the formula to be used to find the value of f at any point x. To find the value of f at any real number, first we have to find to which interval x belongs to; then using the corresponding formula we can find the value of f at that point. To find $f(6)$ we know $3 \leq 6 \leq \infty$ (or 6 □ $[3, \infty)$); so we use the formula $f(x) = x2$ and find $f(6) = 36$. Similarly $f(-1) = -2$, $f(-5) = 0$ and so on.

If the function is defined from R or a subset of R then we can draw the graph of the function. For example, if $f : [0, 4] \rightarrow R$ is defined by $f(x) = x/2 + 1$, then we can plot the points $(x, x/2 + 1)$ for all □ $[0, 4]$. Then we will get a straight line segment joining (0, 1) and (4, 3). (See Figure 1.13)

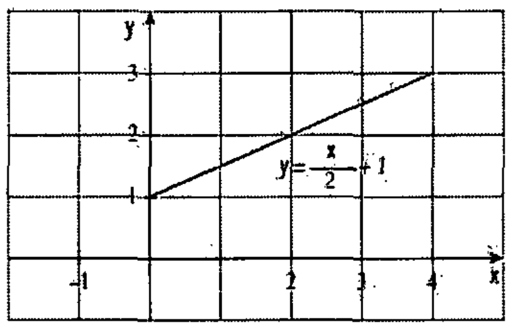


Figure 1.13

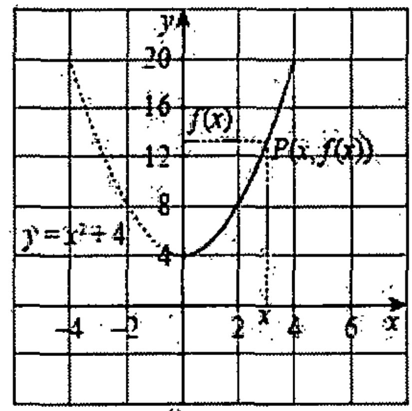


Figure 1.14

Consider another function $f(x) = x2 + 4$, $x \geq 0$. The function will be given by its graph. (See Figure 1.14)

Let x be a point in the domain. Let us draw a vertical line through the point x. Let it meet the curve at P. The point at which the horizontal line drawn through P meets the y-axis is $f(x)$. Similarly using horizontal lines through a point y in the co-domain, we can find the pre-images of y.

Can we say that any curve drawn on the plane be considered as a function from a subset of R to R? No, we cannot. There is a simple test to find this.

Vertical Line Test

As we noted earlier, the vertical line through any point x in the domain meets the curve at some point, then the y-coordinate of the point is $f(x)$. If the vertical line through a point x in the domain meets the curve at more than one point, we will get more than one value for $f(x)$ for one x. This is not allowed in a function. Further, if the vertical line through a point x in the domain does not meet the curve, then there will be no image for x; this is also not possible in a function. So we can say,

"If the vertical line through a point x in the domain meets the curve at more than one point or does not meet the curve, then the curve will not represent a function".

The curve indicated in Figure 1.15 does not represent a function from [0, 4] to R because a vertical line meets the curve at more than one point (See Figure 1.17). The curve indicated in Figure 1.16 does not represent a function from [0, 4] to R because a vertical line drawn through $x = 2.5$ in [0, 4] does not meet the curve (See Figure 1.18).

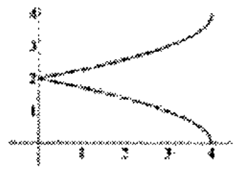


Figure 1.15

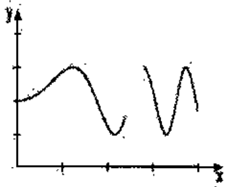


Figure 1.16

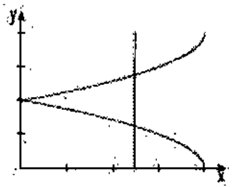


Figure 1.17

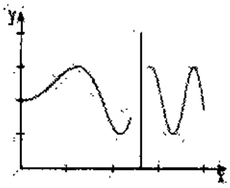


Figure 1.18

Notes

Testing whether a given curve represents a function or not by drawing vertical lines is called ***vertical line test*** or simply vertical test.

The third curve $y2 = x$ in Illustration 1.2 fails in the vertical line test and hence it is not a function from R to R.

2. Some Elementary Functions

Some frequently used functions are known by names. Let us list some of them.

(i) Let X be any non-empty set. The function $f : X \rightarrow X$ defined by $f(x) = x$ for all x □ X is called the ***identity function*** on X (See Figure 1.19). It is denoted by IX or I.

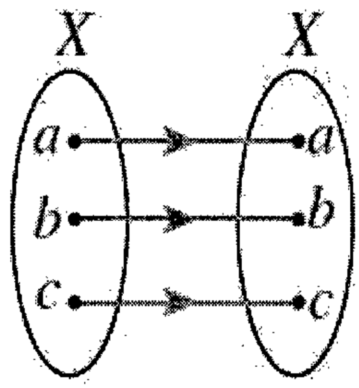


Figure 1.19

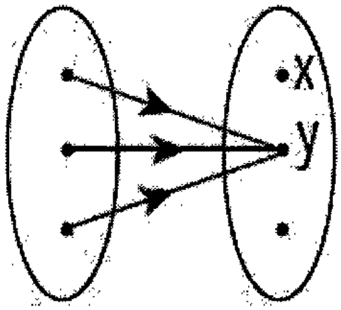


Figure 1.20

(ii) Let X and Y be two sets. Let c be a fixed element of Y. The function $f : X \rightarrow Y$ defined by $f(x) = c$ for all x □ X is called a ***constant function*** (See Figure 1.20).

The value of a constant function is same for all values of x throughout the domain.

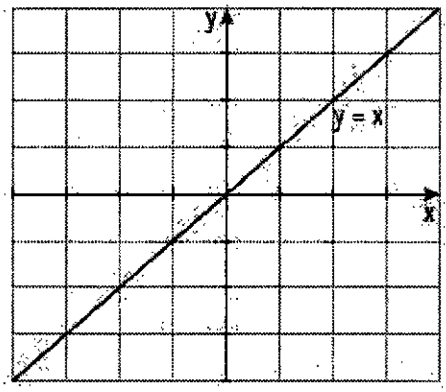


Figure 1.21

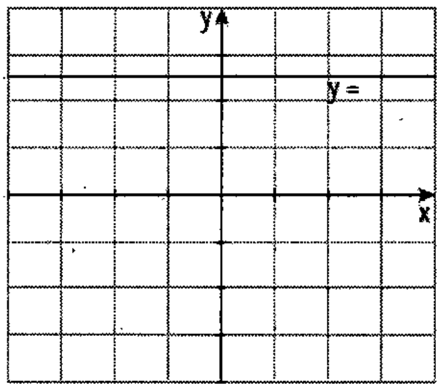


Figure 1.22

If X and Y are R, then the graph of the identity function and a constant function are as in Figures 1.21 and 1.22. If X is any set, then the constant function defined by $f(x) = 0$ for all x is called the ***zero function***. So zero function is a particular case of constant function.

(iii) The function f: R → R defined by $f(x) = |x|$, where $|x|$ is the modulus or absolute value of x, is called the ***modulus function*** or ***absolute value function***. (See Figure 1.23.)

Let us recall that $|x|$ is defined as

$$|x| = \begin{cases} -x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ x & \text{if} \quad x > 0 \end{cases} \text{ or } |x| = \begin{cases} -x & \text{if} \quad x \le 0 \\ x & \text{if} \quad x > 0 \end{cases} \text{ or } |x| = \begin{cases} -x & \text{if} \quad x < 0 \\ x & \text{if} \quad x \ge 0 \end{cases}$$

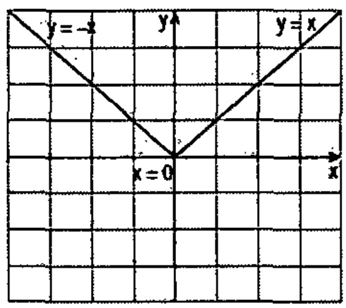


Figure 1.23

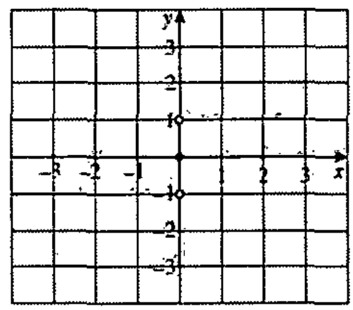


Figure 1.24

(iv) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \frac{x}{|x|} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ is called the *signum function*. This function is denoted by sgn. (See Figure 1.24)

(v) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x)$ is the greatest integer less than or equal to x is called the integral part function or the *greatest integer function* or the *floor function*. This function is denoted by $\lfloor x \rfloor$. (See Figure 1.25.)

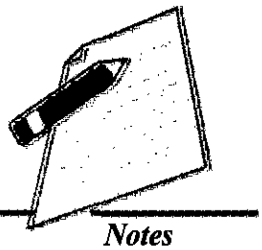
(vi) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x)$ is the smallest integer greater than or equal to x is called the *smallest integer function* or the *ceil function* (See Figure 1.26.). This function is denoted by $\lceil \cdot \rceil$: that is $f(x)$ is denoted by $\lceil x \rceil$.

The functions (v) and (vi) are also called *step functions*.

Let us note that $\lfloor 1\frac{1}{5} \rfloor = 1$, $\lfloor 7.23 \rfloor = 7$, $\lfloor -2\frac{1}{2} \rfloor = -3$ (not -2), $\lfloor 6 \rfloor = 6$ and $\lfloor -4 \rfloor = -4$.

Let us note that $\lceil 1\frac{1}{5} \rceil = 2$, $\lceil 7.23 \rceil = 8$, $\lceil -2\frac{1}{2} \rceil = -2$ (not -3), $\lceil 6 \rceil = 6$ and $\lceil -4 \rceil = -4$.

One may note the relations among the names of these functions, the symbols denoting the functions and the commonly used words ceiling and floor of a room and their graphs are given in Figures 1.25 and 1.26.



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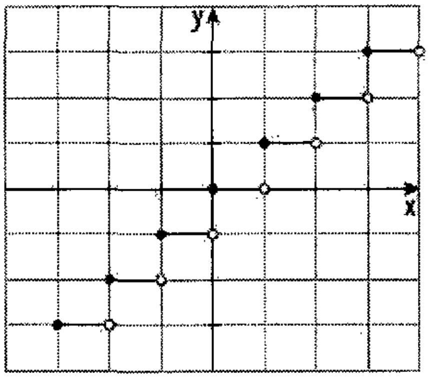


Figure 1.25

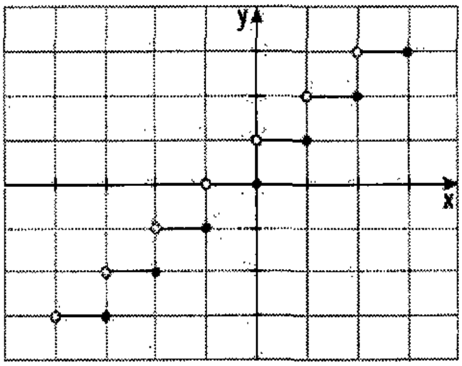


Figure 1.26

3. Types of Functions

Though functions can be classified into various types according to the need, we are going to concentrate on two basic types: ***one-to-one functions and onto functions.***

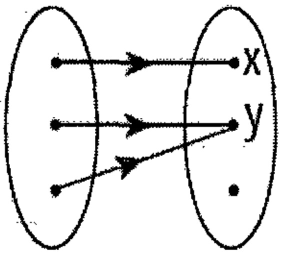


Figure 1.27

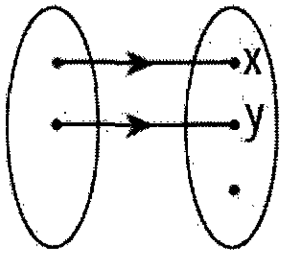


Figure 1.28

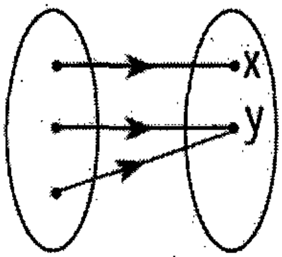


Figure 1.29

Let us look at the two simple functions given in Figure 1.27 and Figure 1.28. In the first function two elements of the domain, *b* and *c*, are mapped into the same element *y*, whereas it is not the case in the Figure 1.28. Functions like the second one are examples of one-to-one functions.

Let us look at the two functions given in Figures 1.28 and 1.29. In Figure 1.28 the element *z* in the co-domain has no pre-image, whereas it is not the case in Figure 1.29. Functions like in Figure 1.29 are example of onto functions. Now we define one-to-one and onto functions.

Definition 1.7

A function $f : A \to B$ is said to be *one-to-one* if $x, y \in A, x \neq y \Rightarrow f(x) \neq f(y)$ [or equivalently $f(x) = f(y) \Rightarrow x = y$]. A function $f : A \to B$ is said to be *onto*, if for each $b \in B$ there exists at least one element $a \in A$ such that $f(a) = b$. That is, the range of f is B.

Another name for one-to-one function is ***injective function***; onto function is ***surjective function***. A function $f : A \to B$ is said to be ***bijective*** if it is both one-to-one and onto.

To prove a function $f : A \to B$ to be one-to-one, it is enough to prove any one of the following:

if x ¹ y, then $f(x)$ ¹ $f(y)$, or equivalently if $f(x) = f(y)$, then $x = y$.

It is easy to observe that every identity function is one-to-one function as well as onto. A constant function is not onto unless the co-domain contains only one element. The following statements are some important simple results.

Let A and B be two sets with m and n elements.

i. There is no one-to-one function from A to B if $m > n$.

ii. If there is an one-to-one function from A to B, then $m \leq n$.

iii. There is no onto function from A to B if $m < n$.

iv. If there is an onto function from A to B, then $m \geq n$.

v. There is a bijection from A to B, if and only if, $m = n$.

vi. There is no bijection from A to B if and only if, $m! = n$.

A function which is not onto is called into function. That is, the range of the function is a proper subset of its co-domain. Let us see some illustrations.

(1) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d, e\}$ and $f = \{(1, a), (2, c), (3, e), (4, b)\}$.
This function is one-to-one but not onto.

(2) $X = \{1, 2, 3, 4\}, Y = \{a, b\}$ and $f = \{(1, a), (2, a), (3, a), (4, a)\}$.
This function is not one-to-one; it is not onto.

(3) $X = \{1, 2, 3, 4\}, Y = \{a\}$ and $f = \{(1, a), (2, a), (3, a), (4, a)\}$.
This function is not one-to-one but it is onto. It seems that this function is same as the previous one. The co-domain of the function is very important when deciding whether the function is onto or not.

(4) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d, e\}$ and $f = \{(1, a), (2, c), (3, b), (4, b)\}$.
This function is neither one-to-one nor onto.

(5) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d\}$ and $f = \{(1, a), (2, c), (3, d), (4, b)\}$.
This function is both one-to-one and onto.

(6) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d, e\}$ and $f = \{(1, a), (2, c), (3, e)\}$.
This is not at all a function, only a relation.

(7) Let X be a finite set with k elements. Then, we have a bijection from X to $\{1, 2, \ldots k\}$.

Let us consider functions defined on some known sets through a formula rule.

Example 1.14 Check whether the following functions are one-to-one and onto.

(i) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 2$.

(ii) $f : \mathbb{N} \cup \{-1, 0\} \to \mathbb{N}$ defined by $f(n) = n + 2$.

Solution:

(i) If $f(n) = f(m)$, then $n + 2 = m + 2$ and hence $m = n$. Thus f is one-to-one. As 1 has no pre-image, this function is not onto. (See Figure 1.30)

(ii) As above, this function is one-to-one. If m is in the co-domain, then $m - 2$ is in the domain and $f(m - 2) = (m - 2) + 2 = m$; thus m has a pre-image and hence this function is onto. (See Figure 1.31)

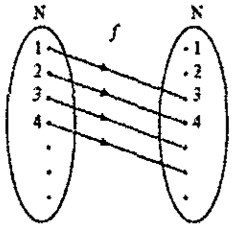


Figure 1.30

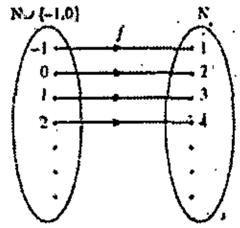


Figure 1.31

It seems that the second function (ii) is same as the first function (i). But the domains are different. From this we see that the domain of the function is also important in deciding whether the function is onto or not. The co-domain has no role in deciding whether the function is one-to-one or not. But it is important to decide whether the function is onto or not.

Example 1.15 Check the following functions for one-to-oneness and ontoness.

(i) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2$.

(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(n) = n^2$.

Solution:

(i) $f(m) = f(n) \Rightarrow m^2 = n^2 \Rightarrow m = n$ since $m, n \in \mathbb{N}$. Thus f is one-to-one. But. non-perfect square elements in the co-domain do not have pre-images and hence not onto.

(ii) Two different elements in the domain have same images and hence f is not one-to-one. Clearly the range of f is a proper subset of $\mathbb{R}$. Thus it is not onto.

Now, we recall Illustration 1.1. In this illustration the function f: $C \to D$ is defined by

$$f(L) = O, f(E) = H, f(T) = W, f(U) = X, f(S) = V, f(W) = Z, f(I) = L, f(N) = Q$$

Where $C = \{L, E, T, U, S, W, I, N\}$ and $D = \{O, H, W, X, V, Z, L, Q\}$, is an one-to-one and onto function.

In Illustration 1.3, the function $f : A \to \mathrm{N}$ is defined by $f(a) = 1, f(b) = 2, \ldots,$ $f(z) = 26$, where $A = \{a, b, \ldots z\}$. This function is one-to-one. If we take N as co-domain, the function is not onto. Instead of N if we take the co-domain as $\{1, 2, 3 \ldots, 26\}$ then it becomes onto.

Example 1.16 Check whether the following for one-to-oneness and ontoness.

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$.

(ii) $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$.

Solution:

(i) This is not at all a function because $f(x)$ is not defined for $x = 0$.

(ii) This function is one-to-one (verify) but not onto because 0 has no pre-image.

If we consider $\mathbb{R} - \{0\}$ as the co-domain for the second, then f will become a bijection.

Example 1.17 If $f : \mathbf{R} - \{-1, 1\} \to \mathbf{R}$ is defined by $f(x) = \frac{x}{x^2-1}$, verify whether f is one-to-one or not.

Solution:

We start with the assumption $f(x) = f(y)$. Then,

$$\begin{aligned} \frac{x}{x^2-1} &= \frac{y}{y^2-1} \\ \Rightarrow \quad x(y^2-1) &= y(x^2-1) \\ \Rightarrow \quad xy^2 - x - yx^2 + y &= 0 \\ \Rightarrow \quad (y-x)(xy+1) &= 0 \end{aligned}$$

This implies that $x = y$ or $xy = -1$. So if we select two numbers x and y so that $xy = -1$, then $f(x) = f(y)$. $(2, -\frac{1}{2}), (7, -\frac{1}{7}), (-2, \frac{1}{2})$ are some among the infinitely many possible pairs. Thus $f(2) = f(\frac{-1}{2}) = \frac{2}{3}$. That is, $f(x) = f(y)$ does not imply $x = y$. Hence it is not one-to-one.

Example 1.18 If $f : \mathbf{R} \to \mathbf{R}$ is defined as $f(x) = 2x^2 - 1$, find the pre-images of $17, 4$ and -2.

Solution:

To find the pre-image of 17, we solve the equation $2x^2 - 1 = 17$. The two solutions of this equation, 3 and -3 are the pre-images of 17 under f. The equation $2x^2 - 1 = 4$ yields $\sqrt{\frac{5}{2}}$ and $-\sqrt{\frac{5}{2}}$ as the two pre-images of 4. To find the pre-image of -2, we solve the equation $2x^2 - 1 = -2$. This shows that $x^2 = -\frac{1}{2}$ which has no solution in $\mathbb{R}$ because square of a number cannot be negative and hence -2 has no pre-image under f.

Example 1.19 If $f : [-2, 2] \to B$ is given by $f(x) = 2x^3$, then find B so that f is onto.

Solution:

The minimum value is $f(-2)$ and its maximum value is $f(2)$ which are equal to -16 and 16 respectively. So B is $[-16, 16]$.

As $f(x) = 2x^3$ is an increasing function on $[-2, 2]$, the minimum value is attained at the left end and the maximum value is attained at the right end. (For more about increasing / decreasing functions one may refer later chapters.)

Example 1.20 Check whether the function $f(x) = x|x|$ defined on $[-2, 2]$ is one-to-one or not. If it is one-to-one, find a suitable co-domain so that the function becomes a bijection.

Solution:

Let $x, y \in [-2, 2]$ such that $f(x) = f(y)$. If $y = 0$, then $x = 0$. So let $y \neq 0$ and hence $x \neq 0$. Now $x|x| = y|y|$ since $f(x) = f(y)$. This implies that $\frac{x}{y} = \frac{|y|}{|x|}$. Since $\frac{|y|}{|x|} > 0$, $\frac{x}{y} > 0$; thus x and y are either both positive or both negative and hence $x^2 = y^2$.

So if $f(x) = f(y)$, we must have $x^2 = y^2$. Also x and y are either both negative or both positive. This is possible only if $x = y$. Thus f is one-to-one. When $x < 0$, $f(x) = -x^2$ and when $x \geq 0$, $f(x) = x^2$. So the range is $[-4, 4]$. So f becomes a bijection from $[-2, 2]$ to $[-4, 4]$.

Horizontal Test

Similar to the vertical line test we have a test called horizontal test to check whether a function is one-to-one, onto or not. Let a function be given as a curve in the plane. If the horizontal line through a point y in the co-domain meets the curve at some points, then the x-coordinate of all the points give pre-images for y.

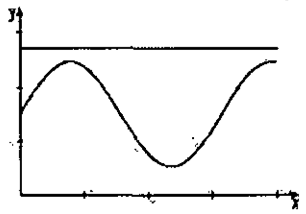


Figure 1.32

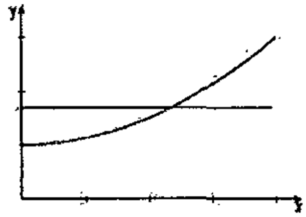


Figure 1.33

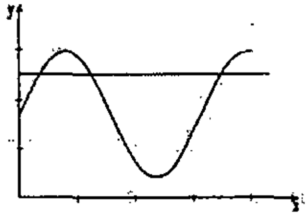


Figure 1.34

(i) If the horizontal line through a point y in the co-domain does not meet the curve, then there will be no pre-image for y and hence the function is not onto.

(ii) If the horizontal line through at least one point meets the curve at more than one point, then the function is not one-to-one.

(iii) If for all y in the range the horizontal line through y meets the curve at only one point, then the function is one-to-one.

So we may say, the function represented by a curve is one-to-one if and only if for all y in the range, the horizontal line through the point y meets the curve at exactly one point.

The function represented by a curve is onto if and only if for all y in the co-domain, the horizontal line through the point y meets the curve atleast one point.

The curve given in Figure 1.32 represents a function from [0, 4] which is not onto if the co-domain contains [1, 3]. The curve given in Figure 1.33 represents a one-to-one function from [0, 4] to R and the curve given in Figure 1.34 represents a function from [0, 4] to R which is not one-to-one.

Testing whether a given curve represents a one-to-one function, onto function or not by drawing horizontal lines is called ***horizontal line test*** or simply horizontal test.

Further by seeing the diagrams in Illustration 1.2 and Figures 1.5 to Figure 1.7, the function

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ is an one-to-one and onto function.
(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is neither one-to-one nor onto.
(iii) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = +\sqrt{x}$ is an one-to-one but not onto function.
(iv) $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = +\sqrt{x}$ is an one-to-one and onto function.
(v) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = -\sqrt{x}$ is one-to-one but not onto function.
(vi) $f : [0, \infty) \to (-\infty, 0]$ defined by $f(x) = -\sqrt{x}$ is one-to-one and onto function.

Example 1.21 Find the largest possible domain for the real valued function f defined by $f(x) = \sqrt{x^2 - 5x + 6}$.

Solution:

As we are finding the square root of $x^2 - 5x + 6$, we must have $x^2 - 5x + 6 \geq 0$ for all x in the domain. For this, follow the given procedure.

Solving $x^2 - 5x + 6 = 0$, we get $x = 2$ and 3. Now draw the number line as in Figure 1.35.

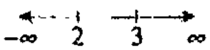


Figure 1.35

Now we have three intervals. $(-\infty, 2)$, $(2, 3)$ and $(3, \infty)$

(i) Take any point in $(-\infty, 2)$, say $x = 1$. Clearly $x^2 - 5x + 6$ is positive.
(ii) Take any point in $(2, 3)$, say $x = 2.5$. Clearly $x^2 - 5x + 6$ is negative.
(iii) Take any point in $(3, \infty)$, say $x = 4$. Clearly $x^2 - 5x + 6$ is positive.

For all x, in the intervals $(-\infty, 2)$ and $(3, \infty)$, $x^2 - 5x + 6$ is positive. At $x = 2, 3$ the value of $x^2 - 5x + 6$ is zero. Thus, $\sqrt{x^2 - 5x + 6}$ is defined for all x in $(-\infty, 2] \cup [3, \infty)$.
Hence the domain of $\sqrt{x^2 - 5x + 6}$ is $(-\infty, 2] \cup [3, \infty)$.

Example 1.22 Find the domain of $f(x) = \frac{1}{1-2\cos x}$.

Solution:
The function is defined for all $x \in \mathbb{R}$ except $1 - 2\cos x = 0$. That is, except $\cos x = \frac{1}{2}$. That is except $x = 2n\pi \pm \frac{\pi}{3}, n \in \mathbf{Z}$. Hence the domain is $\mathrm{R} - \{2n\pi \pm \frac{\pi}{3}\}, n \in \mathbf{Z}$

Example 1.23 Find the range of the function $f(x) = \frac{1}{1-3\cos x}$.

Solution:
Clearly.

$$\begin{array}{rrcll} & -1 & \leq & \cos x & \leq & 1 \\ \Rightarrow & 3 & \geq & -3\cos x & \geq & -3 \\ \Rightarrow & -3 & \leq & -3\cos x & \leq & 3 \\ \Rightarrow & 1-3 & \leq & 1-3\cos x & \leq & 1+3 \end{array}$$

Thus we get $-2 \leq 1 - 3\cos x$ and $1 - 3\cos x \leq 4$.
By taking reciprocals, we get $\frac{1}{1-3\cos x} \leq -\frac{1}{2}$ and $\frac{1}{1-3\cos x} \geq \frac{1}{4}$.
Hence the range of f is $(-\infty, -\frac{1}{2}] \cup [\frac{1}{4}, \infty)$.

Example 1.24 Find the largest possible domain for the real valued function given by $f(x) = \frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}$.

Solution:
If $x < -3$ or $x > 3$, then x^2 will be greater than 9 and hence $9 - x^2$ will become negative which has no square root in $\mathbb{R}$. So x must lie on the interval $[-3, 3]$.

Also if $x \geq -1$ and $x \leq 1$, then $x^2 - 1$ will become negative or zero. If it is negative, $x^2 - 1$ has no square root in $\mathbb{R}$. If it is zero, f is not defined. So x must lie outside $[-1, 1]$. That is, x must lie on $(-\infty, -1) \cup (1, \infty)$. Combining these two conditions, the largest possible domain for f is $[-3, 3] \cap ((-\infty, -1) \cup (1, \infty))$. That is, $[-3, -1) \cup (1, 3]$.

Draw the number line and plot the intervals to get the required domain interval.

4. Operations on Functions Composition

Let there be two functions f and g as given in the Figure 1.36 and Figure 1.37. Let us note that the co-domain of f and the domain of g are the same. Let us cut off Figure 1.37 of g and paste it on the Figure 1.36 of f so that the domain Y of g is pasted on co-domain Y of f. (See Figure 1.38.)

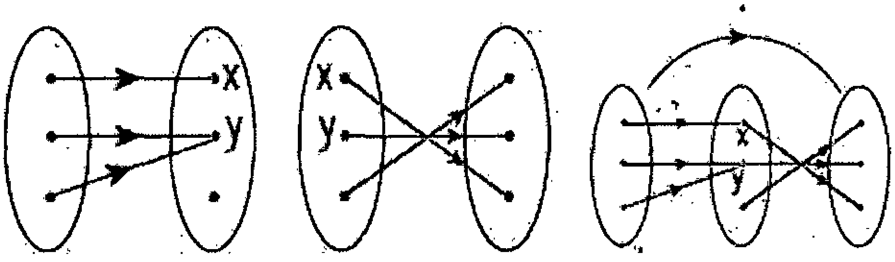


Figure 1.36 Figure 1.37 Figure 1.38

Now we can define a function $h : X \to Z$ in a natural way. To find the image of a under h, we first see the image of a under f; it is x; then we see the image of this x under g; this is r. That is, $h(a) = r$. Similarly, we declare $h(b) = q$ and $h(c) = q$. In this way we can define a new function h. This h is called the composition of f with g.

Definition 1.8

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then the function $h : X \to Z$ defined as $h(x) = g(f(x))$ for every $x \in X$ is called the *composition* of f with g. It is denoted by $g \circ f$ (Read this as f composite with g). (See Figures 1.38 and 1.39.)

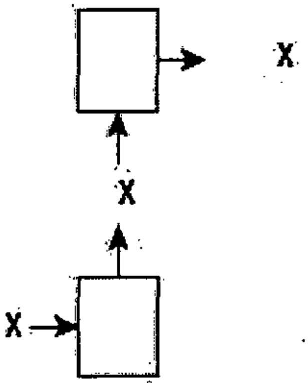


Figure 1.39

We can note that the range of f need not be Y. If $f : X \to Y_1, g : Y_2 \to Z$ and $Y_1 \subseteq Y_2$, then also we can define $g \circ f$; we can take Y_2 as the co-domain of f and use the same definition. So we can define $g \circ f$ if and only if the range of f is contained in the domain of g.

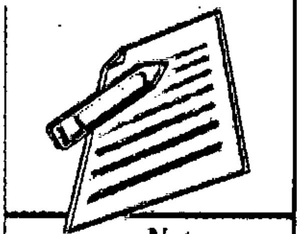
Example 1.25 Let $f = \{(1,2),(3,4),(2,2)\}$ and $g = \{(2,1),(3,1),(4,2)\}$. Find $g \circ f$ and $f \circ g$.

Solution:

To check whether compositions can be defined, let us find the domain and range of these functions.

Domain of $f = \{1,2,3\}$, Range of $f = \{2,4\}$, Domain of $g = \{2,3,4\}$ and Range of $g = \{1,2\}$. Since the range of f is contained in the domain of g we can define $g \circ f$; so as to find the image of 1 under $g \circ f$, we first find the image of 1 under f and then its image under g. The image of 1 under f is 2 and its image under g is 1. So $(g \circ f)(1) = g(f(1)) = g(2) = 1$.

Similarly we find that $(g \circ f)(2) = 1$ and $(g \circ f)(3) = 2$. So $g \circ f = \{(1,1),(2,1),(3,2)\}$. Similarly $f \circ g = \{(2,2),(3,2),(4,2)\}$.



Example 1.26 Let $f = \{(1,4),(2,5),(3,5)\}$ and $g = \{(4,1),(5,2),(6,4)\}$. Find $g \circ f$. Can you find $f \circ g$?

Solution:
Clearly, $g \circ f = \{(1,1),(2,2),(3,2)\}$. But $f \circ g$ is not defined because the range of $g = \{1,2,4\}$ is not contained in the domain of $f = \{1,2,3\}$.

Example 1.27 Let f and g be the two functions from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 3x - 4$ and $g(x) = x^2 + 3$. Find $g \circ f$ and $f \circ g$.

Solution:
We have,
$(g \circ f)(x) = g(f(x)) = g(3x-4) = (3x-4)^2 + 3 = 9x^2 - 24x + 19.$
$(f \circ g)(x) = f(g(x)) = f(x^2+3) = 3(x^2+3) - 4 = 3x^2 + 5.$

Here we have $f \circ g \neq g \circ f$. Thus the operation "composition of functions" is in general not commutative.

Theorem 1.2: Let $f : A \to B$ and $g : B \to C$ be two functions. If f and g are one-to-one, then $g \circ f$ is one-to-one.

Proof. Let $x \neq y$ in A. Since f is one-to-one, $f(x) \neq f(y)$. Since g is one-to-one, $g(f(x)) \neq g(f(y))$. That is, $x \neq y \Rightarrow (g \circ f)(x) \neq (f \circ g)(y)$. Hence $g \circ f$ is one-to-one. □

Example 1.28 Show that the statement,

"If f and $g \circ f$ are one-to-one, then g is one-to-one" is not true.

Solution:
To claim a statement is not true we have to prove by giving one counter example. Consider the diagram given in Figure 1.40.

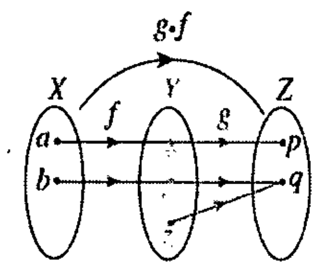


Figure 1.40

Clearly f and $g \circ f$ are one-to-one. But g is not one-to-one. Thus from the above diagram it shows that the statement is not true.

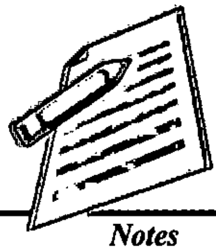
Example 1.29 Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x - |x|$ and $g(x) = 2x + |x|$. Find $f \circ g$.

Solution:
We know

$$|x| = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$$

So

$$f(x) = \begin{cases} 2x - (-x) & \text{if } x \leq 0 \\ 2x - x & \text{if } x > 0 \end{cases}$$



Thus

$$f(x)=\begin{cases}3x & \text{if } x\le 0\\ x & \text{if } x>0\end{cases}$$

Also

$$g(x)=\begin{cases}2x+(-x) & \text{if } x\le 0\\ 2x+x & \text{if } x>0\end{cases}$$

Thus

$$g(x)=\begin{cases}x & \text{if } x\le 0\\ 3x & \text{if } x>0\end{cases}$$

Let $x\le 0$. Then

$$(g\circ f)(x)=g(f(x))=g(3x)=3x.$$

The last equality is taken because $3x\le 0$ whenever $x\le 0$.

Let $x>0$. Then

$$(g\circ f)(x)=g(f(x))=g(x)=3x.$$

Thus $(g\circ f)(x)=3x$ for all x.

5. Inverse of a Function

Let there be a bijection $f: X\rightarrow Y$ as given in the Figure 1.41.

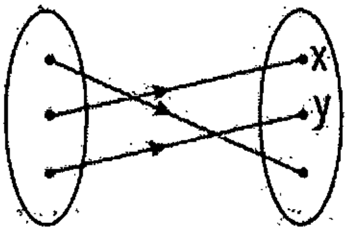


Figure 1.41

If we look this function in a mirror, we get a function from Y to X. Let us call that function as g.

Then g is a function from Y to X defined by $g(x) = b$, $g(y) = c$, $g(z) = a$.

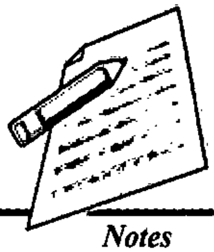
This function g is an example for the inverse of f. Now we define the inverse of a function.

Definition 1.9

Let $f : X \rightarrow Y$ be a bijection. The function $g : Y \rightarrow X$ defined by $g(y) = x$ if $f(x) = y$, is called the *inverse* of f and is denoted by f^{-1}.

If a function f has an inverse, then we say that f is ***invertible***. There is a nice relationship between composition of functions and inverse.

Let $f: X\rightarrow Y$ be a bijection and $g : Y\rightarrow X$ be its inverse.



Then $g \circ f = IX$ and $f \circ g = IY$ where IX and IY are identity functions on X and Y respectively. Moreover, if $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are functions such that $g \circ f = IX$ and $f \circ g = IY$, then both f and g are bijections and they are inverses to each other; that is $f-1 = g$ and $g-1 = f$.

Using the discussions above, the terms invertible and inverse can be defined in some other way as follows:

Definition 1.10

A function $f : X \rightarrow Y$ is said to be *invertible* if there exists a function $g : Y \rightarrow X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$ where I_X and I_Y are identity functions on X and Y respectively. In this case, g is called the inverse of f and g is denoted by f^{-1}.

We may use this concept to prove some functions are bijective.

If f is a bijection, then $f-1(y)$ is nothing but the pre-image of y under f. Let us note that the inverses are defined only for bijections. If f is not one-to-one, then there exists a and b such that a ' b and $f(a) = f(b)$. Let this value be y. Then we cannot define $f-1(y)$ because both a and b are pre-images of y under f, as $f-1$ cannot assume two different values for y. If f is not onto, then there will be a y in Y without a pre-image. In this case also we cannot assign any value to $f-1(y)$.

For example, if $A = \{1, 2, 3, 4\}$ and $f = \{(1, 2), (2, 4), (3, 1), (4, 3)\}$. Then the range of f is $\{1, 2, 3, 4\}$; the inverse of f is $\{(1, 3), (2, 1), (3, 4), (4, 2)\}$.

Working Rule to Find the Inverse of Functions from R to R:

Let f : R → R be the given function.

i. write y = f(x);

ii. write x in terms of y;

iii. write f −1(y) = the expression in y.

iv. replace y as x.

Example 1.30 If $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = 2x - 3$ prove that f is a bijection and find its inverse.

Solution:
Method 1:

One-to-one : Let $f(x) = f(y)$. Then $2x-3 = 2y-3$; this implies that $x = y$. That is, $f(x) = f(y)$ implies that $x = y$. Thus f is one-to-one.

Onto : Let $y \in \mathbb{R}$. Let $x = \frac{y+3}{2}$. Then $f(x) = 2(\frac{y+3}{2}) - 3 = y$. Thus f is onto. This also can be proved by saying the following statement. The range of f is $\mathbb{R}$ (how?) which is equal to the co-domain and hence f is onto.

Inverse Let $y = 2x - 3$. Then $y + 3 = 2x$ and hence $x = \dfrac{y+3}{2}$. Thus $f^{-1}(y) = \dfrac{y+3}{2}$. By replacing y as x, we get $f^{-1}(x) = \dfrac{x+3}{2}$.

Method 2:

Let $y = 2x - 3$. Then $x = \dfrac{y+3}{2}$. Let $g(y) = \dfrac{y+3}{2}$.

Now

$$(g \circ f)(x) = g(f(x)) = g(2x - 3) = \frac{(2x-3)+3}{2} = x.$$

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{y+3}{2}\right) = 2\left(\frac{y+3}{2}\right) - 3 = y.$$

Thus, $g \circ f = I_X$ and $f \circ g = I_Y$

This implies that f and g are bijections and inverses to each other. Hence f is a bijection and $f^{-1}(y) = \frac{y+3}{2}$. Replacing y by x we get, $f^{-1}(x) = \frac{x+3}{2}$.

6. Algebra of Functions

A function whose co-domain is R or a subset of R is called a real valued function. We can discuss many more operations on functions if it is real valued:

Let f and g be two real valued functions. Can we define addition of f and g? Naturally we expect the sum of two functions to be a function. The value of $f + g$ at a point x should be related to the values of f and g at x. So to define $f + g$ at a point x, we must know both $f(x)$ and $g(x)$. In other words x must be in the domain of f as well as in the domain of g. And the natural way of defining $+ g$ at x is $f(x) + g(x)$. So if we impose a condition that the domains of f and g to be the same, then we can define f + g. In the same way we can define subtraction, multiplication and many more algebraic operations available on the set R of the real numbers.

Definition 1.11

Let X be any set. Let f and g be real valued functions defined on X. Define, for all $x \in X$

- $(f+g)(x) = f(x) + g(x)$.
- $(f-g)(x) = f(x) - g(x)$.
- $(fg)(x) = f(x)g(x)$.
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$.
- $(cf)(x) = cf(x)$, where c is a real constant.
- $(-f)(x) = -f(x)$.

Note that the domain may be any set, not necessarily a set of numbers. For example if X is a set of students of a class, f and g functions representing the marks obtained by the students in two tests, then the function $f + g$ represent the total marks of the students in the two tests. It is easy to see that the operations addition, subtraction, multiplication and division defined above satisfy the following properties.

- (f + g) + h = f + (g + h)
- f + g = g + f
- 0 + f = f + 0, where 0 is the zero function defined by 0(x)=0 for all x.
- f + (−f)=(−f) + f = 0
- f(g + h) = fg + fh
- (c1 + c2)f = c1f + c2f where c1 and c2 are real constants.

We can list many more properties of these operations. The proofs are simple; however let us prove only one to show a way in which these properties can be proved.

Let us prove $f(g + h) = fg + fh$. To prove $f(g + h) = fg + fh$ we have to prove that $(f(g + h))(x) = (fg + fh)(x)$ for all x in the domain.

Theorem 1.3: If f and g are real-valued functions, then f(g + h) = fg + fh.

Proof. Let X be any set and f and g be real-valued functions defined on X. Let x □ X.

(f(g + h))(x)

= f(x)(g + h)(x) (by the definition of product)

= f(x)[g(x) + h(x)] (by the definition of addition)

= f(x)g(x) + f(x)h(x) (by the distributivity of reals)

= (fg)(x)+(fh)(x) (by the definition of product)

= (fg + fh)(x) (by the definition of addition)

Thus (f(g + h))(x)=(fg + fh)(x) for all x □ X; hence f(g + h) = fg + fh.

7. Some Special Functions

Now let us see some special functions.

(i) The function f : R → R defined by

f(x) = a0xn + a1xn−1 + a2xn−2 + ... + an−1x + an,

where *ai* are constants, is called a ***polynomial function***. Since the right hand side of the equality defining the function is a polynomial, this function is called a polynomial function.

(ii) The function *f* : R → R defined by *f*(*x*) = *ax* + *b* where *a* = 0 and *b* are constants, is called a ***linear function***. A function which is not linear is called a ***non-linear function***. Clearly a linear function is a polynomial function. The graph of this function is a straight line; a straight line is called a linear curve; so this function is called a linear function. (one may come across different definitions for linear functions in higher study of mathematics.)

(iii) Let *a* be a non-negative constant. Consider the function *f* : R → R defined by *f*(*x*) = *ax*. If *a* = 0, x^{1} 0 then the function becomes the zero function and if *a* = 1, then function *f* : R → R defined by *f*(*x*) = *ax* is the constant function *f*(*x*) = 1. [See, Figures 1.42 and 1.43].

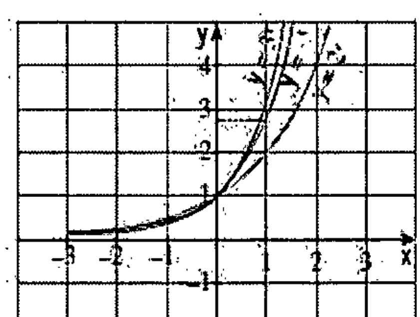


Figure 1.42

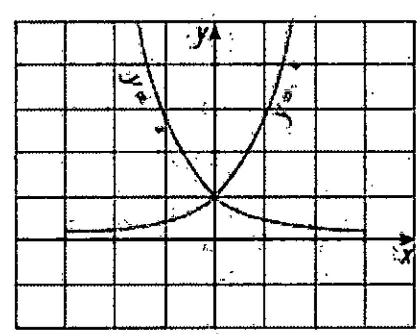


Figure 1.43

When *a* > 1, the function *f*(*x*) = *ax* is called an ***exponential function***. Moreover, any function having *x* in the "power" is called as an exponential function.

(iv) Let $a > 1$ be a constant. The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \log_a x$ is called a ***logarithmic function***. In fact, the inverse of an exponential function $f(x) = a^x$ on a suitable domain is called a logarithmic function. [See, Figure 1.44].

(v) The real valued function f defined by $f(x) = \frac{p(x)}{q(x)}$ on a suitable domain, where $p(x)$ and $q(x)$ are polynomials, $q(x) \neq 0$, is called a ***rational function***. In fact, the domain of these function are the sets obtained from $\mathbb{R}$ by removing the real numbers at which $q(x) = 0$.

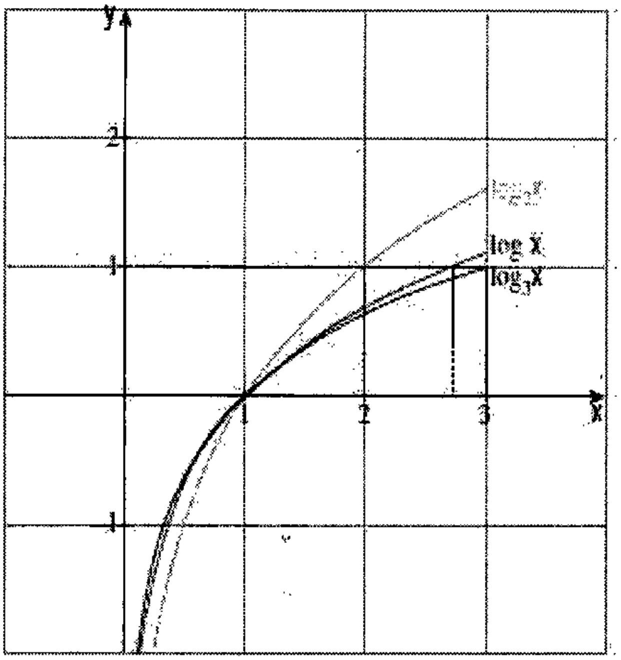


Figure 1.44

(vi) If f is a real valued function such that $f(x) \neq 0$, then the real valued function g defined by $g(x) = \frac{1}{f(x)}$ on a suitable domain is called the ***reciprocal function*** of f. The domain of g is the set obtained from $\mathbb{R}$ by removing the real numbers at which $f(x) = 0$. For example, the largest possible domain of $f(x) = \frac{1}{x-1}$ is $\mathbb{R} - \{1\}$.

Let us see two more categories of functions.

> **Definition 1.12**
>
> A function $f : \mathbb{R} \to \mathbb{R}$ is said to be an ***odd function*** if $f(-x) = -f(x)$ for all $\in \mathbb{R}$. It is said to be an ***even function*** if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. [See. Figures 1.45 and 1.46].

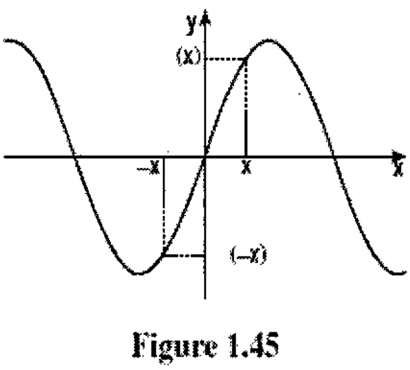


Figure 1.45

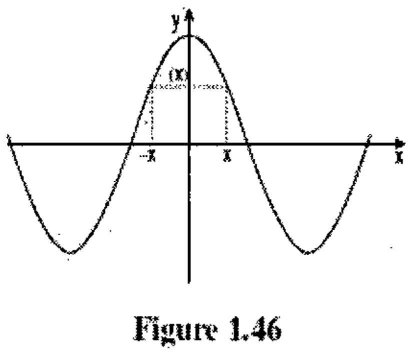


Figure 1.46

The function defined by $f(x) = x$, $f(x) = 2x$ and $f(x) = x3 + 2x$ are some examples for odd functions. The functions defined by $f(x) = x2$, $f(x) = 3$, $f(x) = x4 + x2$ and $f(x) = |x|$ are some examples for even functions. Note that the function $f(x) = x + x2$ is neither even nor odd.

We can prove the following results.

i. The sum of two odd functions is an odd function.

ii. The sum of two even functions is an even function

iii. The product of two odd functions is an even function.

iv. The product of two even functions is an even function.

v. The product of an odd function and an even function is an odd function.

vi. The only function which is both odd and even function is the zero function.

vii. The product of a positive constant and an even function is an even function.

viii. The product of a negative constant and an even function is also an even function.

ix. The product of a constant and an odd function is an odd function.

x. There are functions which are neither odd nor even.

Let us prove one of the above properties. The other properties can be proved similarly.

Theorem 1.4: The product of an odd function and an even function is an odd function.

Proof. Let f be an odd function and g be an even function. Let $h = fg$. Now

$h(-x) = (fg)(-x) = f(-x)g(-x) = -f(x)g(x)$ (as f is odd and g is even) $= -h(x)$

Thus h is an odd function. This shows that fg is an odd function.

Summary of the Chapter

Suppose that a particle is moving in the space. We assume the physical particle as a point. As time varies, the particle changes its position. Mathematically at any time the point occupies a position in the three-dimensional space R3. Let us assume that the time varies from 0 to 1. So the movement or *functioning* of the particle decides the position of the particle at any given time t between 0 and 1. In other words, for each t □ [0, 1], the functioning of the particle gives a point in R3. Let us denote the position of the particle at time t as $f(t)$. Let us see another simple example. We know that the equation $2x-y = 0$ describes a straight line. Here whenever x assumes a value, y assumes some value accordingly. The movement or *functioning* of y is decided by that of x. Let us denote y by $f(x)$. We may see many situations like this in nature. In the study of natural phenomena, we find that it is necessary to consider the variation of one quantity depending on the variation of another. Clearly a linear function is a polynomial function. The graph of this function is a straight line; a straight line is called a linear curve; so this function is called a linear function. (one may come across different definitions for linear functions in higher study of mathematics.)

Review Questions

Exercise - 1.3

1. Suppose that 120 students are studying in 4 sections of eleventh standard in a school. Let A denote the set of students and B denote the set of the sections. Define a relation from A to B as "*x related to y if the student x belongs to the section y*". Is this relation a function? What can you say about the inverse relation? Explain your answer.
2. Write the values of f at $-4, 1, -2, 7, 0$ if

$$f(x) = \begin{cases} -x+4 & \text{if } -\infty < x \leq -3 \\ x+4 & \text{if } -3 < x < -2 \\ x^2 - x & \text{if } -2 \leq x < 1 \\ x - x^2 & \text{if } 1 \leq x < 7 \\ 0 & \text{otherwise} \end{cases}$$

3. Write the values of f at $-3, 5, 2, -1, 0$ if

$$f(x) = \begin{cases} x^2+x-5 & \text{if } x \in (-\infty, 0) \\ x^2+3x-2 & \text{if } x \in (3, \infty) \\ x^2 & \text{if } x \in (0, 2) \\ x^2-3 & \text{otherwise} \end{cases}$$

4. State whether the following relations are functions or not. If it is a function check for one-to-oneness and ontoness. If it is not a function, state why?

   (i) If $A = \{a, b, c\}$ and $f = \{(a, c), (b, c), (c, b)\}; (f : A \to A)$.
   (ii) If $X = \{x, y, z\}$ and $f = \{(x, y), (x, z), (z, x)\}; (f : X \to X)$.

5. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Give a function from $A \to B$ for each of the following:

   (i) neither one-to-one nor onto. (ii) not one-to-one but onto.
   (iii) one-to-one but not onto. (iv) one-to-one and onto.

6. Find the domain of $\dfrac{1}{1 - 2\sin x}$.
7. Find the largest possible domain of the real valued function $f(x) = \dfrac{\sqrt{4 - x^2}}{\sqrt{x^2 - 9}}$.
8. Find the range of the function $\dfrac{1}{2\cos x - 1}$.
9. Show that the relation $xy = -2$ is a function for a suitable domain. Find the domain and the range of the function.
10. If $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = |x| + x$ and $g(x) = |x| - x$, find $g \circ f$ and $f \circ g$.
11. If f, g, h are real valued functions defined on $\mathbb{R}$, then prove that $(f + g) \circ h = f \circ h + g \circ h$. What can you say about $f \circ (g + h)$? Justify your answer.
12. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x - 5$, prove that f is a bijection and find its inverse.
13. The weight of the muscles of a man is a function of his body weight x and can be expressed as $W(x) = 0.35x$. Determine the domain of this function.
14. The distance of an object falling is a function of time t and can be expressed as $s(t) = -16t^2$. Graph the function and determine if it is one-to-one.
15. The total cost of airfare on a given route is comprised of the base cost C and the fuel surcharge S in rupee. Both C and S are functions of the mileage m; $C(m) = 0.4m + 50$ and $S(m) = 0.03m$. Determine a function for the total cost of a ticket in terms of the mileage and find the airfare for flying 1600 miles.
16. A salesperson whose annual earnings can be represented by the function $A(x) = 30,000 + 0.04x$, where x is the rupee value of the merchandise he sells. His son is also in sales and his earnings are represented by the function $S(x) = 25,000 + 0.05x$. Find $(A + S)(x)$ and determine the total family income if they each sell Rupees $1,50,00,000$ worth of merchandise.

17. The function for exchanging American dollars for Singapore Dollar on a given day is $f(x) = 1.23x$, where x represents the number of American dollars. On the same day the function for exchanging Singapore Dollar to Indian Rupee is $g(y) = 50.50y$, where y represents the number of Singapore dollars. Write a function which will give the exchange rate of American dollars in terms of Indian rupee.
18. The owner of a small restaurant can prepare a particular meal at a cost of Rupees 100. He estimates that if the menu price of the meal is x rupees, then the number of customers who will order that meal at that price in an evening is given by the function $D(x) = 200 - x$. Express his day revenue, total cost and profit on this meal as functions of x.
19. The formula for converting from Fahrenheit to Celsius temperatures is $y = \frac{5x}{9} - \frac{160}{9}$. Find the inverse of this function and determine whether the inverse is also a function.
20. A simple cipher takes a number and codes it, using the function $f(x) = 3x - 4$. Find the inverse of this function, determine whether the inverse is also a function and verify the symmetrical property about the line $y = x$ (by drawing the lines).

4 TRIGONOMETRIC FUNCTIONS

- Understand the concept of trigonometric Functions.
- Discuss the properties of trigonometric Functions.
- Describe the types of angles.
- Understand the concept of angle.
- Discuss the measurement of angles

Objective of the chapter:

The basic objective of this chapter is to through some light on the initial concepts of trigonometric functions so that the fundamentals of trigonometric functions can be learned.

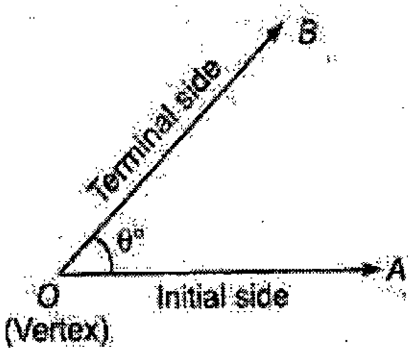
Introduction

Angle

When a ray OA starting from its initial position OA rotates about its end point 0 and takes the final position OB, we say that angle

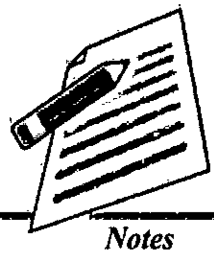
AOB (written as □ AOB) has been formed. The amount of rotation from the initial side to the terminal side is called the measure of

The angle.

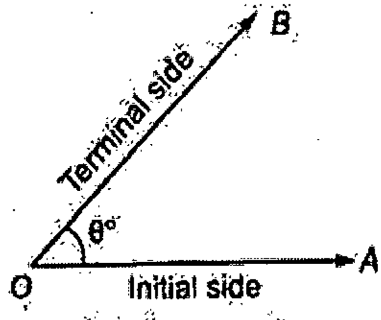


Positive and Negative Angles

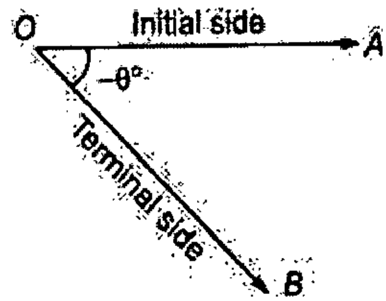
An angle formed by a rotating ray is said to be positive or negative depending on whether it moves in an anti-clockwise or a clockwise direction, respectively.



Notes



(Positive angle)



(Negative angle)

Measurement of Angles

There are three systems for measuring angles,

Sexagesimal System/Degree Measure (English System)

In this system, a right angle is divided into 90 equal parts, called degrees. The symbol $1°$ is used to denote one degree. Each degree is divided into 60 equal parts, called minutes and one minute is divided into 60 equal parts, called seconds. Symbols $1'$ and $1''$ are used to denote one minute and one second, respectively.

i.e., 1 right angle = $90°$

$1° = 60'$

$1' = 60''$

Centesimal System (French System)

In this system, a right angle is divided into 100 equal parts, called 'grades. Each grade is subdivided into 100 min and each minute is divided into 100 s.

i.e., 1 right angle = 100 grades = 100g 1g = $100'$

$1' = 100''$

1. Circular System (Radian System)

In this system, angle is measured in radian.

A radian is the angle subtended at the centre of a circle by an arc, whose length is equal to the radius of the circle.

The number of radians in an angle subtended by an arc of circle at the centre is equal to arc/radius.

Relationships

(i) π radian = $180°$

or 1 radian $(180°/\pi)$= $57°16'22''$ where, $\pi = 22/7 = 3.14159$

(ii) $1° = (\pi/180)$ rad = 0.01746 rad

1. If D is the number of degrees, R is the number of radians and G is the number of grades in an angle θ, then

$$\frac{D}{90}=\frac{G}{100}=\frac{2R}{\pi}$$

1. $\theta = l/r$ where θ = angle subtended by arc of length l at the centre of the circle, r = radius of the circle.

Trigonometric Ratios

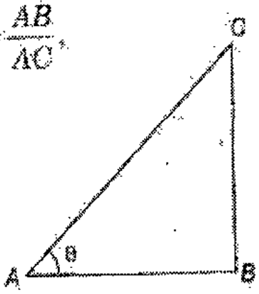
Relation between different sides and angles of a right-angled triangle are called trigonometric ratios or T-ratios

Trigonometric ratios can be represented as

$$\sin\theta=\frac{\text{Perpendicular}}{\text{Hypotenuse}}=\frac{BC}{AC},\ \cos\theta=\frac{\text{Base}}{\text{Hypotenuse}}=\frac{AB}{AC},$$

$$\tan\theta=\frac{\text{Perpendicular}}{\text{Base}}=\frac{BC}{AB},\ \operatorname{cosec}\theta=\frac{1}{\sin\theta}$$

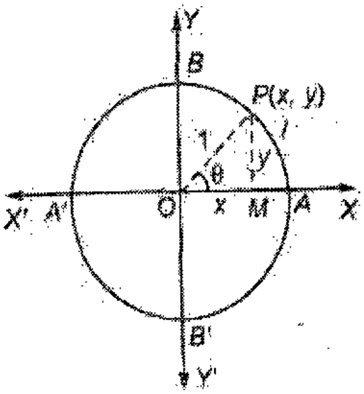
$$\sec\theta=\frac{1}{\cos\theta},\ \cot\theta=\frac{\cos\theta}{\sin\theta}=\frac{1}{\tan\theta}$$



Trigonometric (or Circular) Functions

Let X'OX and YOY' be the coordinate axes. Taking 0 as the centre and a unit radius, draw a circle, cutting the coordinate axes at A, B, A' and B', as shown in the figure.

$$\left[\because \angle AOP=\frac{\text{arc } AP}{\text{radius } OP}=\frac{\theta}{1}=\theta^c,\ \text{using } \theta=\frac{l}{r}\right]$$



Now, the six circular functions may be defined as under

1. $\cos\theta = x$
2. $\sin\theta = y$
3. $\sec\theta = 1/x,\ x \neq 0$
4. $\operatorname{cosec}\theta = 1/y,\ y \neq 0$
5. $\tan\theta = y/x,\ x \neq 0$
6. $\cot\theta = x/y,\ y \neq 0$

Domain and Range

Trigonometric Ratios	Domain	Range
$\sin\theta$	R	$[-1, 1]$
$\cos\theta$	R	$[-1, 1]$
$\tan\theta$	$R - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	R
$\operatorname{cosec}\theta$	$R - \{n\pi : n \in I\}$	$R - (-1, 1)$
$\sec\theta$	$R - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	$R - (-1, 1)$
$\cot\theta$	$R - \{n\pi : n \in I\}$	R

Range of Modulus Functions

$|\sin\theta| \le 1$, $|\cos\theta| \le 1$, $|\sec\theta| \ge 1$, $|\operatorname{Cosec}\theta| \ge 1$ for all values of 0, for which the functions are defined.

Trigonometric Identities

An equation involving trigonometric functions which is true for all those angles for which the functions are defined is called trigonometrical identity. Some identities are

(i) $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ or $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

(ii) $\cos\theta = \dfrac{1}{\sec\theta}$ or $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$ or $\tan\theta = \dfrac{1}{\cot\theta} = \dfrac{\sin\theta}{\cos\theta}$

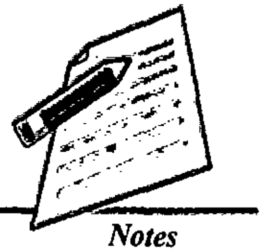
(iv) $\cos^2\theta + \sin^2\theta = 1$ or $1 - \cos^2\theta = \sin^2\theta$ or $1 - \sin^2\theta = \cos^2\theta$

(v) $1 + \tan^2\theta = \sec^2\theta$

(vi) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Sign of Trigonometric Ratios

Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
I $(0, 90^\circ)$	+	+	+	+	+	+
II $(90^\circ, 180^\circ)$	+	–	–	–	–	+
III $(180^\circ, 270^\circ)$	–	–	+	+	–	–
IV $(270^\circ, 360^\circ)$	–	+	–	–	+	–



Trigonometric Ratios of Some Standard Angles

Angle	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	$-\infty$
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

Trigonometric Ratios of Some Special Angles

Angle	$7\frac{1}{2}°$	15°	$22\frac{1}{2}°$	18°	36°
$\sin\theta$	$\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{1}{2}\sqrt{2-\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{4}\sqrt{10-2\sqrt{5}}$
$\cos\theta$	$\frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{1}{2}\sqrt{2+\sqrt{2}}$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$	$\frac{\sqrt{5}+1}{4}$
$\tan\theta$	$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$	$2-\sqrt{3}$	$\sqrt{2}-1$	$\frac{\sqrt{25-10\sqrt{15}}}{5}$	$\sqrt{5-2\sqrt{5}}$

Trigonometric Ratios of Allied Angles

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90°. The angles — θ, $90° \pm \theta$, $180° \pm \theta$, $270° + \theta$, $360° - \theta$ etc., are angles allied to the angle θ, if θ is measured in degrees.

θ	$\sin\theta$	$\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$\tan\theta$	$\cot\theta$
$-\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$
$90°-\theta$	$\cos\theta$	$\sec\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$\cot\theta$	$\tan\theta$
$90°+\theta$	$\cos\theta$	$\sec\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$-\cot\theta$	$-\tan\theta$
$180°-\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$-\cos\theta$	$-\sec\theta$	$-\tan\theta$	$-\cot\theta$
$180°+\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$-\cos\theta$	$-\sec\theta$	$\tan\theta$	$\cot\theta$
$270°-\theta$	$-\cos\theta$	$-\sec\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cot\theta$	$\tan\theta$
$270°+\theta$	$-\cos\theta$	$-\sec\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$-\cot\theta$	$-\tan\theta$
$360°-\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$

Trigonometric Periodic Functions

A function f(x) is said to be periodic, if there exists a real number $T > 0$ such that $f(x + T) = f(x)$ for all x. T is called the period of the function, all trigonometric functions are periodic.

θ	$\sin\theta$	$\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$\tan\theta$	$\cot\theta$
$-\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$
$90°-\theta$	$\cos\theta$	$\sec\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$\cot\theta$	$\tan\theta$
$90°+\theta$	$\cos\theta$	$\sec\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$-\cot\theta$	$-\tan\theta$
$180°-\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$-\cos\theta$	$-\sec\theta$	$-\tan\theta$	$-\cot\theta$
$180°+\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$-\cos\theta$	$-\sec\theta$	$\tan\theta$	$\cot\theta$
$270°-\theta$	$-\cos\theta$	$-\sec\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cot\theta$	$\tan\theta$
$270°+\theta$	$-\cos\theta$	$-\sec\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$-\cot\theta$	$-\tan\theta$
$360°-\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$

Maximum and Minimum Values of Trigonometric Expressions

(i) Maximum value of $a\cos\theta \pm b\sin\theta = \sqrt{a^2+b^2}$

Minimum value of $a\cos\theta \pm b\sin\theta = -\sqrt{a^2+b^2}$

(ii) Maximum value of $a\cos\theta \pm b\sin\theta + c = c + \sqrt{a^2+b^2}$

Minimum value of $a\cos\theta \pm b\sin\theta + c = c - \sqrt{a^2+b^2}$

Trigonometric Ratios of Compound Angles

The algebraic sum of two or more angles is generally called compound angles and the angles are known as the constituent angle. Some standard formulas of compound angles have been given below.

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(vii) $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot B + \cot A}$

(viii) $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$

(ix) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(x) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(xi) $\sin(A+B+C) = \cos A\cos B\sin C + \cos A\sin B\cos C$
$\qquad + \sin A\cos B\cos C - \sin A\sin B\sin C$

or $\sin(A+B+C) = \cos A\cos B\cos C(\tan A + \tan B + \tan C$
$\qquad - \tan A\tan B\tan C)$

(xii) $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C$
$\qquad - \sin A\cos B\sin C - \cos A\sin B\sin C$

or $\cos(A+B+C) = \cos A\cos B\cos C(1 - \tan A\tan B - \tan B\tan C$
$\qquad - \tan C\tan A)$

(xiii) $$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$$

If $A+B+C=0$, then $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

(xiv) (a) $\sin(A_1 + A_2 + \ldots + A_n) = (\cos A_1\cos A_2\cos A_3\ldots\cos A_n)$
$\qquad \times (S_1 - S_3 + S_5 - S_7 + \ldots)$

(b) $\cos(A_1 + A_2 + \ldots + A_n) = (\cos A_1\cos A_2\cos A_3\ldots\cos A_n)$
$\qquad \times (1 - S_2 + S_4 - S_6 + \ldots)$

(c) $$\tan(A_1 + A_2 + \ldots + A_n) = \frac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}$$

where,

$S_1 = \tan A_1 + \tan A_2 + \ldots + \tan A_n$

$S_2 = \tan A_1\tan A_2 + \tan A_2\tan A_3 + \ldots$

$S_3 = \tan A_1\tan A_2\tan A_3 + \tan A_2\tan A_3\tan A_4 + \ldots$

Transformation Formulae

(i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(ii) $2\cos A\cos B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(v) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

(vi) $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

(vii) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(viii) $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$$= \sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$$

Trigonometric Ratios of Multiple Angles

(i) $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$

$$= 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

(iv) $\sin 3A = 3\sin A - 4\sin^3 A$

(v) $\cos 3A = 4\cos^3 A - 3\cos A$

(vi) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

(vii) $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2} = \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(viii) $\cos A = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(ix) $1 - \cos A = 2\sin^2\dfrac{A}{2}$

(x) $1 + \cos A = 2\cos^2\dfrac{A}{2}$

(xi) $\dfrac{1-\cos A}{1+\cos A} = \tan^2\dfrac{A}{2}$

(xii) $\sin\left(\dfrac{A}{2}\right) + \cos\left(\dfrac{A}{2}\right) = \pm\sqrt{1+\sin A}$

(xiii) $\sin\left(\dfrac{A}{2}\right) - \cos\left(\dfrac{A}{2}\right) = \pm\sqrt{1-\sin A}$

Trigonometric Ratios of Some Useful Angles

(i) $\sin 75^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \cos 15^\circ$

(ii) $\cos 75^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \sin 15^\circ$

(iii) $\tan 75^\circ = 2+\sqrt{3} = \cot 15^\circ$

(iv) $\cot 75^\circ = 2-\sqrt{3} = \tan 15^\circ$

(v) $\sin 9^\circ = \dfrac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4} = \cos 81^\circ$

(vi) $\cos 9^\circ = \dfrac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4} = \sin 81^\circ$

Important Results

1. Product of Trigonometric Ratio

(i) $\sin\theta \sin(60^\circ-\theta)\sin(60^\circ+\theta) = \dfrac{1}{4}\sin 3\theta$

(ii) $\cos\theta \cos(60^\circ-\theta)\cos(60^\circ+\theta) = \dfrac{1}{4}\cos 3\theta$

(iii) $\tan\theta \tan(60^\circ-\theta)\tan(60^\circ+\theta) = \tan 3\theta$

(iv) $\cos 36^\circ \cos 72^\circ = \dfrac{1}{4}$

(v) $\cos A \cos 2A \cos 4A \ldots \cos 2^{n-1}A = \dfrac{1}{2^n \sin A}\sin(2^n A)$

2. Sum of Trigonometric Ratios

(i) $\sin A + \sin(A+B) + \sin(A+2B) + \ldots + \sin(A+nB)$

$$= \frac{\sin\left\{A+(n-1)\frac{B}{2}\right\}\sin\frac{nB}{2}}{\sin\frac{B}{2}}$$

(ii) $\cos A + \cos(A+B) + \cos(A+2B) + \ldots + \cos(A+nB)$

$$= \frac{\sin\frac{nB}{2}}{\sin\frac{B}{2}}\cos\left\{A+\frac{(n-1)B}{2}\right\}$$

3. *A*, *B* and *C* are Angles of a Triangle

(i) (a) $\sin(B+C) = \sin A$

(b) $\cos(B+C) = -\cos A$

(c) $\sin\left(\dfrac{B+C}{2}\right) = \cos\dfrac{A}{2}$

(d) $\cos\left(\frac{B+C}{2}\right)=\sin\frac{A}{2}$

(ii) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

(iii) $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$

(iv) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(v) $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(vi) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(vii) $\cot B\cot C+\cot C\cot A+\cot A\cot B=1$

(viii) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(ix) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

4. Trigonometric Equations

(i) $\sin n\pi=0$ and $\cos n\pi=(-1)^n$

(ii) $\sin\theta=\sin\alpha\Rightarrow\theta=n\pi+(-1)^n\alpha, n\in I$

Characteristics of Trigonometric Functions

Trigonometric functions have some nice properties. For example,

(i) Sine and cosine functions are complementary to each other in the sense that $\sin(90°-\theta)=\cos\theta$ and $\cos(90°-\theta)=\sin\theta$.

(ii) As $\cos\theta$ and $\sin\theta$ are obtained as coordinates of a point on the unit circle, they satisfy the inequalities $-1\le\cos\theta\le 1$ and $-1\le\sin\theta\le 1$. Hence, $\cos\theta, \sin\theta\in[-1,1]$

(iii) Trigonometric function repeats its values in regular intervals.

(iv) Sine and cosine functions have an interesting property that $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$

Let us discuss the last two properties.

Periodicity of Trigonometric Functions

We know that a function f is said to be a periodic function with period p, if there exists a smallest positive number p such that $f(x+p)=f(x)$ for all x in the domain.

For example, $\sin(x+2n\pi)=\sin x$, n □ Z.

i.e., $\sin(x+2\pi)=\sin(x+4\pi)=\sin(x+6\pi)=\ldots=\sin x$ Thus, $\sin x$ is a periodic function with period 2π.

Similarly, $\cos x$, $\operatorname{cosec} x$ and $\sec x$ are periodic functions with period 2π.

But $\tan x$ and $\cot x$ are periodic functions with period π.

The periodicity of $\sin x$ and $\cos x$ can be viewed best using their graphs.

(i) The graph of the sine function

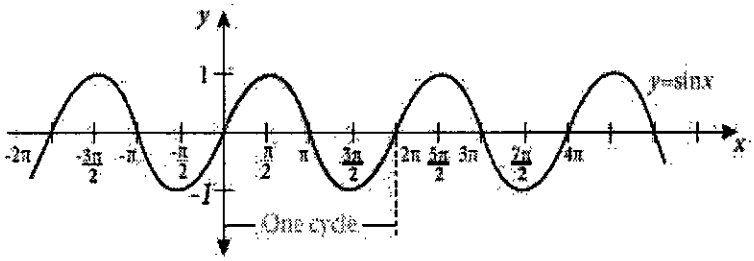


Figure 3.13: y =sinx

Here x represents a variable angle. Take the horizontal axis to be the x-axis and vertical axis to be the y-axis. Graph of the function $y = \sin x$ is shown in the Figure 3.13. First, note that it is periodic of period 2π. Geometrically it means that if you take the curve and slide it 2π either left or right, then the curve falls back on itself. Second, note that the graph is within one unit of the y-axis. The graph increases and decreases periodically. For instance, increases from $-\pi/2$ to $\pi/2$ and decreases from $\pi/2$ to $3\pi/2$.

(ii) The graph of the cosine function

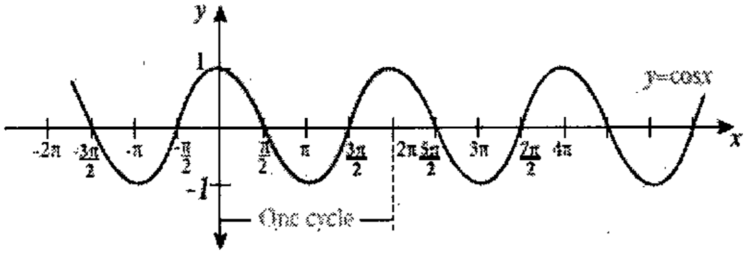


Figure 3.14: y = cosx

Observe that the graph of $y = \cos x$ looks just like the graph of $y = \sin x$ except it is being translated to the left by $\frac{\pi}{2}$. This is because of the identity $\cos x = \sin\left(\frac{\pi}{2} + x\right)$. It easily follows from the graph that $\cos x = \cos(-x) = \sin\left(\frac{\pi}{2} + x\right)$

(i) The sine and cosine functions are useful for one very important reason, since they repeat in a regular pattern (*i.e.*, they are periodic). There are a vast array of things in and around us that repeat periodically. For example, the rising and setting of the sun, the motion of a spring up and down, the tides of the ocean and so on, are repeating at regular intervals of time. All periodic behaviour can be studied through combinations of the sine and cosine functions.

(ii) Periodic functions are used throughout science to describe oscillations, waves and other phenomena that occur periodically.

Odd and Even trigonometric functions

Even and odd functions are functions satisfying certain symmetries. A real valued function $f(x)$ is an even function if it satisfies $f(-x) = f(x)$ for all real number x and an odd function if it satisfies $f(-x) = -f(x)$ for all real number x.

Basic trigonometric functions are examples of non-polynomial even and odd functions because $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$ for all x, it follows that $\cos x$ is an even function and $\sin x$ is an odd function.

Also note that $\sec x$ is an even function while $\tan x$, $\operatorname{cosec} x$ and $\cot x$ are all odd functions. However, $f(t) = t - \cos t$ is neither even function nor odd function (why?)

Example 3.14 Determine whether the following functions are even, odd or neither.

(i) $\sin^2 x - 2\cos^2 x - \cos x$ (ii) $\sin(\cos(x))$ (iii) $\cos(\sin(x))$ (iv) $\sin x + \cos x$

Solution:

(i) Let $f(x) = \sin^2 x - 2\cos^2 x - \cos x$

$f(-x) = f(x)$ [since $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$]

Thus, $f(x)$ is even.

(ii) Let $f(x) = \sin(\cos(x))$

$f(-x) = f(x)$. $f(x)$ is an even function.

(iii) $f(x) = \cos(\sin(x))$. $f(-x) = f(x)$; Thus, $f(x)$ is an even function.

(iv) Let $f(x) = \sin x + \cos x$

$f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

Thus, $f(x) = \sin x + \cos x$ is neither even nor odd.

(i) In general, a function is an even function if its graph is unchanged under reflection about the y-axis. A function is odd if its graph is symmetric about the origin.

(ii) The properties of even and odd functions are useful in analyzing trigonometric functions particularly in the sum and difference formula.

(iii) The properties of even and odd functions are useful in evaluating some definite integrals, which we will see in calculus.

Summary of the Chapter

We know that a function f is said to be a periodic function with period p, if there exists a smallest positive number p such that $f(x + p) = f(x)$ for all x in the domain.

For example, $\sin(x + 2n\pi) = \sin x$, n □ Z. *i.e.*, $\sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) = \ldots = \sin x$ Thus, $\sin x$ is a periodic function with period 2π. Similarly, $\cos x$, cosec$x$ and $\sec x$ are periodic functions with period 2π. But $\tan x$ and $\cot x$ are periodic functions with period π. The periodicity of $\sin x$ and $\cos x$ can be viewed best using their graphs. Even and odd functions are functions satisfying certain symmetries. A real valued function $f(x)$ is an even function if it satisfies $f(-x) = f(x)$ for all real number x and an odd function if it satisfies $f(-x) = -f(x)$ for all real number x. Basic trigonometric functions are examples of non-polynomial even and odd functions Because $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$ for all x, it follows that $\cos x$ is an even function and $\sin x$ is an odd function. Also note that $\sec x$ is an even function while $\tan x$, $\operatorname{cosec} x$ and $\cot x$ are all odd functions.

Review Questions

Exercise - 3.3

1. Find the values of (i) $\sin(480^\circ)$ (ii) $\sin(-1110^\circ)$ (iii) $\cos(300^\circ)$ (iv) $\tan(1050^\circ)$
   (v) $\cot(660^\circ)$. (vi) $\tan\left(\frac{19\pi}{3}\right)$ (vii) $\sin\left(-\frac{11\pi}{3}\right)$.
2. $\left(\frac{5}{7}, \frac{2\sqrt{6}}{7}\right)$ is a point on the terminal side of an angle θ in standard position. Determine the trigonometric function values of angle θ.
3. Find the values of other five trigonometric functions for the following:

   (i) $\cos\theta = -\frac{1}{2}$, θ lies in the III quadrant.
   (ii) $\cos\theta = \frac{2}{3}$, θ lies in the I quadrant.
   (iii) $\sin\theta = -\frac{2}{3}$, θ lies in the IV quadrant.
   (iv) $\tan\theta = -2$, θ lies in the II quadrant.
   (v) $\sec\theta = \frac{13}{5}$, θ lies in the IV quadrant.

4. Prove that $\dfrac{\cot(180^\circ + \theta)\sin(90^\circ - \theta)\cos(-\theta)}{\sin(270^\circ + \theta)\tan(-\theta)\text{cosec}(360^\circ + \theta)} = \cos^2\theta\cot\theta$.
5. Find all the angles between 0° and 360° which satisfy the equation $\sin^2\theta = \frac{3}{4}$.
6. Show that $\sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9} = 2$.

1 SEQUENCE AND SERIES

- Understand the concept of sequence.
- Discuss the types of sequence.
- Understand the concept of series.
- Understand the formulas of sequence and series.
- Understand the application of these formulae to solve problems.

Objective of the chapter:

The basic objective of this chapter is to through some light on the initial concepts of sequence and series so that the application of these formulae to solve problems can be learned.

Introduction

Meaning of Sequence

A sequence is a group of objects which follow some particular pattern. If we have some objects listed in some order so that it has 1st term, 2nd term and so on, then it is a sequence.

Example

In this image we can see the group of sticks with a certain pattern, so we can observe that what could come next

As we can see that the next figure in the sequence will have 9 sticks.

What is a sequence in Math?

In Mathematics, it is a group of numbers in an ordered form which follow a certain pattern is called **Sequence**.

$$\overset{+3}{} \quad \overset{+3}{} \quad \overset{+3}{} \quad \overset{+3}{}$$
$$5,\ 8,\ 11,\ 14,\ 17,\ \ldots$$

Sequence A

These numbers are called **'Term'** or **'Members'** or **'Elements'**. It is similar as a set of numbers.

As in the above figure 5,8,11,14,17 are the terms of the sequence.

Finite Sequence

The sequence which has limited or finite number of terms is called **Finite Sequence.**

Example

{1, 3, 9, 27} is the sequence of multiples of 3.

{m, o, n, k, e, y} is the sequence of letters in the word "monkey".

Infinite Sequence

The sequence which has unlimited or infinite number of terms or it has no end is called **Infinite Sequence.**

Example

2, 4, 6, 8, …

This is the infinite sequence of even numbers. It is the three dots here which show that it is an infinite sequence, with no end. So its last term will be represented by n∞.

In the other way we can write it as

$\{a_n\}_{n=1}^{\infty}$

Here ∞ the infinity means infinite sequence.

Types of Sequence

There are three types of sequence

- Arithmetic Sequence
- Geometric Sequence
- Fibonacci Sequence

Arithmetic Sequence

Any sequence in which the difference between every successive terms is constant then it is called **Arithmetic Sequence.** It could be in ascending or descending form according to the **constant number.**

Example

$$0, 3, 6, 9, 12, \ldots$$

$$+3 \quad +3 \quad +3 \quad +3$$

Here we are getting the terms by adding 3 every time. this is the difference between the two successive terms so it is called the difference.

The difference is represented by "d".

In the above example we can see that $a1 = 0$ and $a2 = 3$.

The difference between the two successive terms is

$a2 - a1 = 3$

$a3 - a2 = 3$

If the first term of an arithmetic sequence is a1 and the common difference is d, then the nth term of the sequence is given by:

$an = a1 + (n-1)\, d$

Geometric Sequence

Any sequence in which the ratio between every successive term is constant then it is called **Geometric Sequence**. It could be in ascending or descending form according to the **constant ratio.**

Example

$$1, 4, 16, 64, \ldots$$

Here

$a1 = 1$

$a2 = 4 = a1(4)$

$a3 = 16 = a2(4)$

Here we are multiplying it with 4 every time to get the next term. Here the ratio is 4. The ratio is denoted by "r".

an = an−1□r or an = a1□rn−1

Fibonacci sequence

By adding the value of the two terms before the required term, we will get the next term. Such type of sequence is called Fibonacci sequence. There is no visible pattern.

Example

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

In the above sequence, we can see

a1 =0 and a2 = 1

a3 = a2 + a1 = 0 + 1 =1

a4 = a3 + a2 = 1 + 1 =2 and so on.

So, the formula of the Fibonacci sequence is

an = an – 2 + an – 1, n > 2

This is also called the **Recursive Formula**.

Meaning of Series

The summation of all the numbers of the sequence is called **Series.** Generally, it is written as Sn.

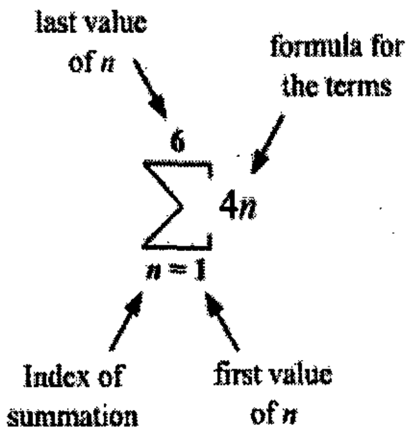
Example

If we have a sequence 1, 4, 7, 10, …

Then the series of this sequence is 1 + 4 + 7 + 10 +…

Notation of Series

We use the sigma notation that is, **the Greek symbol "Σ"** for the series which means **"sum up"**.



The series 4 + 8 + 12 + 16 + 20 + 24 can be expressed as $\sum 6n = 1\ 4n$. We read this expression as the sum of 4n as n goes from 1 to 6.

Finite and Infinite Series

A series with finite number of terms is called **Finite Series.**

2 + 4 + 6 + 8 + 10

A series infinite number of terms is called **Infinite Series.**

2 + 4 + 6 + 8 + 10 + …

Types of Series

There are different types of series-

Arithmetic Series

Arithmetic series is the summation of the terms of the arithmetic sequence that is, if the difference between the every term to its preceding term is always constant then it is said to be an

The arithmetic series is in the form of

{a + (a + d) + (a + 2d) + (a + 3d) + ...}

Where a is the first term of the series and d is the difference of it which is known as the common difference of the given series.

Formula of Arithmetic Series

If a is the first term, d is the difference and n is the total number of the terms, then the formula for nth term is given by

an = a + (n - 1) d

Sum of an Arithmetic Series

$$S_n = \frac{n}{2}\,[2a + (n-1)d]$$

Geometric Series

Geometric series is the summation of the terms of the geometric sequence i.e. if the ratio between the every term to its preceding term is always constant then it is said to be a geometric series.

Formula of Geometric Series

In general, we can define geometric series as

$$\sum_{n=1}^{\infty} ar^n = a + ar + ar^2 + ar^3 + .. + ar^n$$

Where, a is the first term of the series and r is the common ratio for it.

Formula for nth term of the geometric series

an = a1 r n - 1

Where, n is the number of the term.

Sum of Geometric Series

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Meaning of Arithmetic Progression (A.P.)

Arithmetic Progression is the sequence of numbers such that the difference between the two successive terms is always constant. And that difference is called the common difference. It is also known as **Arithmetic Sequence.**

Meaning of Geometric Progression (G.P.)

Geometric Progression is the sequence of numbers such that the next term of the sequence comes by multiplying or dividing the preceding number with the constant (non-zero) number. And that constant number is called the **Common Ratio.** It is also known as **Geometric Sequence.**

a, ar, ar2, ar3, ..., arn

Arithmetic Mean

Arithmetic mean is basically the average of two numbers. If we have two numbers n and m, then we can include a number L in between these numbers so that the three numbers will form an arithmetic sequence like n, L, m.

In that case the number L is the arithmetic mean of the numbers n and m.

According to the property of Arithmetic progression, we can say that-

$L - n = m - L$ that is, the common difference of the given AP.

$$L = \frac{n + m}{2}$$

This is generally used to find the missing number of the sequence between the two given numbers.

Example

What will be the 6th number of the sequence if the 5th term is 12 and the 7th term is 24?

As the two numbers are given so the 6th number will be the Arithmetic mean of the two given numbers.

$$AM = \frac{12 + 24}{2}$$
$$= \frac{36}{2}$$
$$= 18$$

Hence the 6th term will be 18.

Geometric Mean

Geometric Mean is the Special type of average of two numbers. If a and b are the two numbers then the geometric mean will be

$$GM = \sqrt{ab}$$

Example

Find the geometric mean of 2 and 18.

Solution:

We can use the above formula to calculate the geometric mean.
a = 2 and b = 18

$$\begin{aligned} GM &= \sqrt{ab} \\ &= \sqrt{2\times18} \\ &= 6 \end{aligned}$$

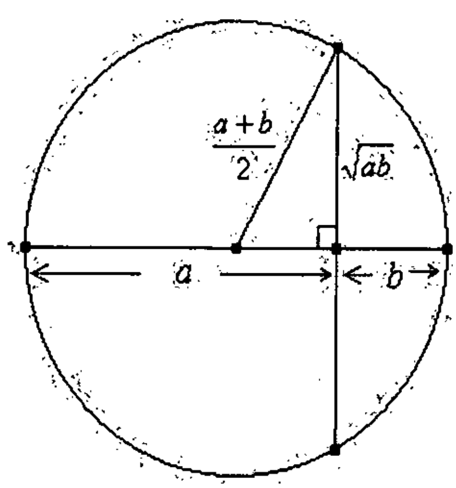
Relation between A.M. and G.M.

Here we can see that the sequence 2, 6, 18 is a geometric progression.

As we have seen above the formula for the Arithmetic mean and the Geometric mean are as follows:

$$A.M. = \frac{a+b}{2} \text{ and } GM = \sqrt{ab}$$

Where a and b are the two given positive numbers.



Let A and G be A.M. and G.M.

So

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

Now let's subtract the two means with each other

$$A - G = \frac{a+b}{2} - \sqrt{ab}$$

$$= \frac{a + b - 2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2} \geq 0$$

This shows that A ≥ G

Example

Find the two numbers, If Arithmetic mean and Geometric mean of two positive real numbers are 20 and 16, respectively.

Solution:

Given

$$A.M = \frac{a+b}{2} = 20$$

$$= a + b = 20 \qquad \ldots.(1)$$

$$G.M. = \sqrt{ab} = 16$$

$$= ab = 256 \qquad \ldots.(2)$$

Now we will put these values of a and b in

(a – b)2 = (a + b)2 – 4ab

(a – b)2 = (40)2 – 4(256)

= 1600 – 1024

= 576

a – b = ± 24 (by taking the square root) …(3)

By solving (1) and (3), we get

a + b = 40

a – b = 24

a = 8, b = 32 or a = 32, b = 8

Special Series

Special Series are the series which are special in some way. It could be arithmetic or geometric.

Some of the special series are:

(i) 1 + 2 + 3 +… + n (sum of first n natural numbers)

(ii) 12 + 22 + 32 +… + n2(sum of squares of the first n natural numbers)

(iii) 13 + 23 + 33 +… + n3(sum of cubes of the first n natural numbers).

Sum to n terms of Special Series

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^2 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Summary of the chapter

A sequence is a group of objects which follow some particular pattern. If we have some objects listed in some order so that it has 1st term, 2nd term and so on, then it is a sequence. In Mathematics, it is a group of numbers in an ordered form which follow a certain pattern is called Sequence. These numbers are called 'Term' or 'Members' or 'Elements'. It is similar as a set of numbers. The sequence which has limited or finite number of terms is called Finite Sequence. {1, 3, 9, 27} is the sequence of multiples of 3. {m, o, n, k, e, y} is the sequence of letters in the word "monkey". The sequence which has unlimited or infinite number of terms or it has no end is called Infinite Sequence. Special Series are the series which are special in some way. It could be arithmetic or geometric. Some of the special series are: (i) 1 + 2 + 3 +… + n (sum of first n natural numbers) (ii) 12 + 22 + 32 +… + n2(sum of squares of the first n natural numbers) (iii) 13 + 23 + 33 +… + n3(sum of cubes of the first n natural numbers).

Review Questions

EXERCISE 5.2

1. **Write the first 6 terms of the sequences whose n^{th} terms are given below and classify them as arithmetic progression, geometric progression, arithmetico-geometric progression, harmonic progression and none of them.**

(i) $\dfrac{1}{2^{n+1}}$ (ii) $\dfrac{(n+1)(n+2)}{(n+3)(n+4)}$

(iii) $4\left(\dfrac{1}{2}\right)^n$ (iv) $\dfrac{(-1)^n}{n}$

(v) $\dfrac{2n+3}{3n+4}$ (vi) 2018 (vii) $\dfrac{3n-2}{3^{n-1}}$

Solution :

(i) $\dfrac{1}{2^{n+1}}$

$$\text{Let } a_n = \frac{1}{2^{n+1}}$$

$$a_1 = \frac{1}{2^{1+1}} = \frac{1}{2^2},\ a_2 = \frac{1}{2^{2+1}} = \frac{1}{2^3}$$

$$a_3 = \frac{1}{2^{3+1}} = \frac{1}{2^4}$$

$$a_4 = \frac{1}{2^{4+1}} = \frac{1}{2^5},\ a_5 = \frac{1}{2^{5+1}} = \frac{1}{2^6},\ a_6 = \frac{1}{2^{6+1}} = \frac{1}{2^7}$$

$\therefore$ The first 6 terms of the sequence are $\dfrac{1}{2^2}, \dfrac{1}{2^3}, \dfrac{1}{2^4}, \dfrac{1}{2^5}, \dfrac{1}{2^6}$ and $\dfrac{1}{2^7}$.

$$\text{Since } a_1 = \frac{1}{2^2} \text{ and } r = \frac{1}{2^3} \div \frac{1}{2^2} = \frac{1}{2^3} \times 2^2 = \frac{1}{2}$$

$$r = \frac{1}{2^4} \div \frac{1}{2^3} = \frac{1}{2^4} \times 2^3 = \frac{1}{2}$$

$\therefore$ The given sequence is a geometric progression.

(ii) $\dfrac{(n+1)(n+2)}{(n+3)(n+4)}$

Let $a_n = \dfrac{(n+1)(n+2)}{(n+3)(n+4)}$

$$a_1 = \frac{(1+1)(1+2)}{(1+3)(1+4)} = \frac{2(3)}{4(5)} = \frac{\overset{3}{\cancel{6}}}{\underset{10}{\cancel{20}}} = \frac{3}{10}$$

$$a_2 = \frac{(2+1)(2+2)}{(2+3)(2+4)} = \frac{3(4)}{5(6)} = \frac{\overset{2}{\cancel{12}}}{\underset{5}{\cancel{30}}} = \frac{2}{5}$$

$$a_3 = \frac{(3+1)(3+2)}{(3+3)(3+4)} = \frac{(\overset{2}{\cancel{4}})(5)}{\underset{3}{\cancel{6}}(7)} = \frac{10}{21}$$

$$a_4 = \frac{5\,(\overset{3}{\cancel{6}})}{7\,(\cancel{8})} = \frac{15}{28}$$

$$a_5 = \frac{\overset{5}{\cancel{6}}(7)}{\underset{4}{\cancel{8}}(\underset{3}{\cancel{9}})} = \frac{7}{12}$$

$$a_6 = \frac{7(\overset{4}{\cancel{8}})}{9(\underset{5}{\cancel{10}})} = \frac{28}{45}$$

$\therefore$ The sequence is $\dfrac{3}{10}, \dfrac{2}{5}, \dfrac{10}{21}, \dfrac{15}{28}, \dfrac{7}{12}, \dfrac{28}{45}, \ldots$

This is neither a A.P, G.P nor AGP.

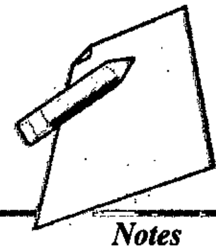
(iii) $4\left(\dfrac{1}{2}\right)^n$

Let $a_n = 4\left(\dfrac{1}{2}\right)^n$

$$a_1 = 4\left(\frac{1}{2}\right)^1 = \frac{\overset{2}{\cancel{4}}}{\cancel{2}} = 2$$

$$a_2 = 4\left(\frac{1}{2}\right)^2 = \frac{\cancel{4}}{\cancel{4}} = 1$$

$$a_3 = 4\left(\frac{1}{2}\right)^3 = \frac{\cancel{4}}{\cancel{8}} = \frac{1}{2}$$



$$a_4 = 4\left(\frac{1}{2}\right)^4 = \frac{\cancel{4}}{\cancel{16}} = \frac{1}{4}$$

$$a_5 = 4\left(\frac{1}{2}\right)^5 = \frac{4}{\underset{8}{\cancel{32}}} = \frac{1}{8}$$

$$a_6 = 4\left(\frac{1}{2}\right)^6 = \frac{\cancel{4}}{\underset{16}{\cancel{64}}} = \frac{1}{16}.$$

$\therefore$ The sequence is $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...$

Here $a = 1$ and $r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{1}{2}$

Hence the given sequence is a G.P.

(iv) $\frac{(-1)^n}{n}$

Let $a_n = \frac{(-1)^n}{n}$

$$a_1 = \frac{(-1)^1}{1} = -1,\ a_2 = \frac{(-1)^2}{2} = \frac{1}{2},\ a_3 = \frac{(-1)^3}{3} = \frac{-1}{3},$$

$$a_4 = \frac{(-1)^4}{4} = \frac{1}{4},\ a_5 = \frac{(-1)^5}{5} = -\frac{1}{5},\ a_6 = \frac{(-1)^6}{6} = \frac{1}{6}.$$

$\therefore$ The sequence is $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, ...$

That is $-\frac{1}{1}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, -\frac{1}{6}...$

Consider 1, 2, 3, 4, ... which is an A.P.

Since $d = 2 - 1 = 3 - 2 = 1$

and $-1, 1, -1, 1, ...$ is a G.P. where

$$r = \frac{1}{-1} = \frac{-1}{1} = -1$$

Hence this is an arithmetico-geometric progression.

(v) $\dfrac{2n+3}{3n+4}$.

Let

$a_n = \dfrac{2n+3}{3n+4}$

$a_1 = \dfrac{2+3}{3+4} = \dfrac{5}{7}$, $\quad a_2 = \dfrac{4+3}{6+4} = \dfrac{7}{10}$, $\quad a_3 = \dfrac{6+3}{9+4} = \dfrac{9}{13}$,

$a_4 = \dfrac{8+3}{12+4} = \dfrac{11}{16}$, $\quad a_5 = \dfrac{10+3}{15+4} = \dfrac{13}{19}$, $\quad a_6 = \dfrac{12+3}{18+4} = \dfrac{15}{22}$

$\therefore$ The sequence is $\dfrac{5}{7}, \dfrac{7}{10}, \dfrac{9}{13}, \dfrac{11}{16}, \dfrac{13}{19}, \dfrac{15}{22}, \ldots$

This is neither A.P. G.P nor AGP.

(vi) 2018

Let $a_n = 2018$.

Then the first 6 terms are 2018, 2018, 2018, 2018, 2018, 2018,....

This is a constant sequence which has same common ratio and same common difference. Hence this is an A.P., G.P and AGP.

(vii) $\dfrac{3n-2}{3^{n-1}}$

$$\text{Let } a_n = \frac{3n-2}{3^{n-1}}$$

$$a_1 = \frac{1}{3^0} = 1$$

$$a_2 = \frac{3(2)-2}{3^1} = \frac{4}{3}$$

$$a_3 = \frac{3(3)-2}{3^2} = \frac{7}{9}$$

$$a_4 = \frac{3(4)-2}{3^3} = \frac{10}{27}$$

$\therefore$ The sequence is $\dfrac{1}{1}, \dfrac{4}{3}, \dfrac{7}{9}, \dfrac{10}{27}, \ldots$

$= 1,\ 4\left(\dfrac{1}{3}\right), 7\left(\dfrac{1}{3}\right)^2, 10\left(\dfrac{1}{3}\right)^3 \ldots$

$\therefore$ 1, 4, 7, 10, ... is an A.P and $\left(\dfrac{1}{3}\right)^0, \left(\dfrac{1}{3}\right)^1, \left(\dfrac{1}{3}\right)^2$ an G.P.

Hence the given sequence is an arithmetico-geometric progression.

2. **Write the first 6 terms of the sequences whose n^{th} term a_n is given below**

(i) $a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$

(ii) $a_n = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ a_{n-1} + a_{n-2} & \text{if } n > 2 \end{cases}$

(iii) $a_n = \begin{cases} n & \text{if } n \text{ is } 1, 2, \text{ or } 3 \\ a_{n-1} + a_{n-2} + a_{n-3} & \text{if } n > 3 \end{cases}$

Solution :

(i) $a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$

$a_1 = 1 + 1 = 2, \quad a_2 = 2, \quad a_3 = 3 + 1 = 4$

$a_4 = 4, \quad a_5 = 5 + 1 = 6, \quad a_6 = 6$

Hence the first 6 terms are 2, 2, 4, 4, 6, 6, ...

(ii) $a_n = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ a_{n-1} + a_{n-2} & \text{if } n > 2 \end{cases}$

$a_1 = 1, a_2 = 2$

$a_3 = a_2 + a_1 = 2 + 1 = 3$

$a_4 = a_3 + a_2 = 3 + 2 = 5$

$a_5 = a_4 + a_3 = 5 + 3 = 8$

$a_6 = a_5 + a_4 = 8 + 5 = 13$

Hence the first 6 terms are 1, 2, 3, 5, 8, 13, ...

(iii) $a_n = \begin{cases} n & \text{if } n \text{ is } 1, 2, \text{ or } 3 \\ a_{n-1} + a_{n-2} + a_{n-3} & \text{if } n > 3 \end{cases}$

Solution : $a_1 = 1, a_2 = 2, a_3 = 3$

$a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6$

$a_5 = a_4 + a_3 + a_2 = 6 + 3 + 2 = 11$

$a_6 = a_5 + a_4 + a_3 = 11 + 6 + 3 = 20.$

$\therefore$ The first 6 terms are 1, 2, 3, 6, 11, 20.

3. **Write the n^{th} term of the following sequences.**

(i) **2, 2, 4, 4, 6, 6, ...** (ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

(iii) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10} \ldots$

(iv) **6, 10, 4, 12, 2, 14, 0, 16, –2, ...**

Solution :

(i) **2, 2, 4, 4, 6, 6, ...**

Given sequence is 2, 2, 4, 4, 6, 6, ...

The odd terms are 2, 4, 6, ... and even terms are also 2, 4, 6 ...

$$\therefore a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$$

(ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

Consider the terms in the numerator 1, 2, 3, ...

$$a = 1, d = 2 - 1 = 1 \Rightarrow a_n = a + (n-1)d$$

$$a_n = 1 + (n-1)(1) = \cancel{1} + n - \cancel{1} = n$$

The terms in the denominator are 2, 3, 4, 5, 6 ...

$$\text{Here } a = 2, d = 1$$

$$\therefore a_n = 2 + (n-1)1 = 2 + n - 1 = n + 1.$$

Hence n^{th} term of the given sequence is $\frac{n}{n+1}$ $\forall n \in \mathbb{N}$

(iii) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10} \ldots$

Numerators are 1, 3, 5, 7, 9,...

$$a = 1, d = 3 - 1$$

$$\therefore a_n = 1 + (n-1)2$$

$$= 1 + 2n - 2 = 2n - 1$$

Denominators are 2, 4, 6 , 8, 10, ...

$$a = 2, d = 2$$

$$\therefore a_n = 2 + (n-1)2$$

$$= \cancel{2} + 2n - \cancel{2} = 2n$$

Hence n^{th} term of the given sequence is $\frac{2n-1}{2n}$ $\forall n \in \mathbb{N}$

(iv) **6, 10, 4, 12, 2, 14, 0, 16, –2, ...**

Odd terms are 6, 4, 2, 0,...

$$\therefore t_n = 6 + (n-1)(-2) = 6 - 2n + 2$$

$$= 8 - 2n$$

Even terms are 10, 12, 14, 16, ...

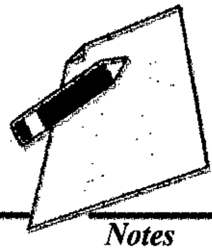
$$\text{Here } a = 1, d = 2$$

$$\therefore t_n = 10 + (n-1)(2) = 10 + 2n - 2$$

$$= 8 + 2n$$

$\therefore$ n^{th} term of the given sequence is

$$\begin{cases} 8 - 2n \text{ if } n \text{ is odd} \\ 8 + 2n \text{ if } n \text{ is even} \end{cases}$$



4. The product of three increasing numbers in GP is 5832. If we add 6 to the second number and 9 to the third number, then resulting numbers form an AP. Find the numbers in GP.

Solution : Let the increasing numbers in G.P be $\frac{a}{r}$, a, ar

$$
\begin{aligned}
\text{Given } \frac{a}{r} \cdot a \cdot ar &= 5832 \\
\Rightarrow \qquad a^3 &= 5832 = 18^3 \\
\Rightarrow \qquad a &= 18
\end{aligned}
$$

Also given $\frac{a}{r}$, $a+6$, $ar+9$ form an A.P.

$$
\begin{aligned}
\therefore \quad a+6-\frac{a}{r} &= ar+9-a-6 \\
\Rightarrow \quad 18+6-\frac{18}{r} &= 18r+9-18-6 \ [\because a=18] \\
\Rightarrow \quad 24-\frac{18}{r} &= 18r-15 \\
\Rightarrow \quad 24+15 &= 18r+\frac{18}{r} \\
\Rightarrow \quad 39 &= \frac{18r^2+18}{r} \\
\Rightarrow \quad 39r &= 18r^2+18 \\
\Rightarrow \quad 18r^2-39r+18 &= 0
\end{aligned}
$$

-324

-39

$\frac{-27}{18}$ $\quad$ $\frac{-12}{18}$

$\frac{-3}{3}$ $\quad$ $\frac{-2}{3}$

$2r-3$ $\quad$ $3r-2$

$$
\begin{aligned}
\Rightarrow \quad (2r-3)(3r-2) &= 0 \\
\Rightarrow \qquad r &= \frac{3}{2}, \frac{2}{3}
\end{aligned}
$$

Case (i) When $a = 18$, $r=\frac{3}{2}$, the numbers in G.P. are $\frac{18}{\frac{3}{2}}$, 18, $18\left(\frac{3}{2}\right) \Rightarrow 12, 18, 27$

Case (ii) When $a = 18$, $r=\frac{2}{3}$, the numbers in G.P. are $\frac{18}{\frac{2}{3}}$, 18, $18\left(\frac{2}{3}\right)$

$\Rightarrow 27, 18, 12.$

5. **Write the n^{th} term of the sequence $\frac{3}{1^2 2^2}, \frac{5}{2^2 3^2}, \frac{7}{3^2 4^2}$, ...as a difference of two terms.**

Solution : The terms in the numerator are 3, 5, 7,.... which forms an A.P.

$$\therefore t_n = 3 + (n-1)2$$
$$= 3 + 2n - 2 = 2n + 1$$

$[a = 3, d = 2]$

The terms in the denominator are $1^2 2^2$, $2^2 3^2$, $3^2 4^2$, ...

$$\therefore t_n = [n.(n+1)]^2$$

$\therefore n^{th}$ term of the given sequence is

$$= \frac{(2n+1)}{[n(n+1)]^2} = \frac{n^2 + 2n + 1 - n^2}{n^2 (n+1)^2}$$

[Adding and subtracting n^2]

$$= \frac{(n^2 + 2n + 1) - n^2}{n^2 (n+1)^2} = \frac{(n+1)^2 - n^2}{n^2 (n+1)^2}$$

$$= \frac{\cancel{(n+1)^2}}{n^2 \cancel{(n+1)^2}} - \frac{\cancel{n^2}}{\cancel{n^2} (n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$\therefore t_n = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

6. **If t_k is the k^{th} term of a G.P, then show that t_{n-k}, t_n, t_{n+k} also form a GP for any positive integer k.**

Solution : Given t_k is the k^{th} term of a G.P.

$$\therefore t_n = a.r^{n-1},\ t_{n-k} = a.r^{n-k-1},\ t_{n+k} = a.r^{n+k-1}$$

Common ratio

$$= \frac{t_n}{t_{n-k}} = \frac{\cancel{a}.r^{n-1}}{\cancel{a}.r^{n-k-1}}\ r^{\cancel{n}-\cancel{1}-\cancel{n}+k+\cancel{1}} = r^k$$

$$\text{Also} = \frac{t_{n+k}}{t_n} = \frac{\cancel{a}.r^{n+k-1}}{\cancel{a}.r^{n-1}}\ r^{\cancel{n}+k-\cancel{1}-\cancel{n}+\cancel{1}} = r^k$$

Since the common ratio is same, t_{n-k}, t_n, t_{n+k} form a G.P.

7. **If a, b, c are in geometric progression, and if $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, then prove that x, y, z are in arithmetic progression.** **[Hy - 2018]**

Solution : Given $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}} = k$

$$\Rightarrow \quad a = k^x,\ b = k^y \text{ and}$$
$$c = k^z \quad ...(1)$$

Also, given that a, b, c are in G.P

$$\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac$$

$$(k^y)^2 = k^x \cdot k^z \qquad \text{[using (1)]}$$

$$\Rightarrow \quad k^{2y} = k^{x+z}$$

$$\Rightarrow \quad 2y = x+z$$

$$\Rightarrow \quad y+y = x+z$$

$$\Rightarrow \quad y-x = z-y$$

$\Rightarrow$ common difference is same for x, y, z

$\therefore$ x, y, z are in arithmetic progression.

8. The AM of two numbers exceeds their GM by 10 and HM by 16. Find the numbers.

Solution : Let the numbers be a and b

$$\therefore \text{A} = \frac{a+b}{2}, \text{G} = \sqrt{ab} \text{ and } \text{H} = \frac{2ab}{a+b}$$

Given $\text{A} - \text{G} = 10$ and $\text{A} - \text{H} = 16$

$$\text{G} = \text{A} - 10 \text{ and } \text{H} = \text{A} - 16$$

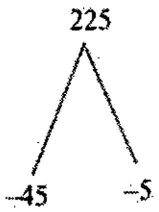
We know $\quad \text{G}^2 = \text{AH}$

$$\Rightarrow \quad (\text{A}-10)^2 = \text{A}(\text{A}-16)$$

$$\Rightarrow \cancel{\text{A}^2} + 100 - 20\text{A} = \cancel{\text{A}^2} - 16\text{A}$$

$$\Rightarrow \quad 100 = 4\text{A}$$

$$\Rightarrow \text{A} = 25 \Rightarrow \frac{a+b}{2} = 25$$



$$\Rightarrow \quad a+b = 50 \qquad \text{... (1)}$$

$$\therefore \text{G} = \text{A} - 10 = 25 - 10 = 15$$

$$\therefore \sqrt{ab} = 15 \Rightarrow ab = 225$$

$$\Rightarrow \quad b = \frac{225}{a} \qquad \text{... (2)}$$

Substituting (2) in (1) we get,

$$a + \frac{225}{a} = 50$$

$$\Rightarrow \quad \frac{a^2+225}{a} = 50$$

$$\Rightarrow \quad a^2 + 225 = 50a$$

$$\Rightarrow a^2 - 50a + 225 = 0$$

$$\Rightarrow (a-45)(a-5) = 0$$

$$\Rightarrow \quad a = 5, 45$$

$$\text{If } a = 5, b = \frac{225}{5} = 45$$

$$\text{If } a = 45, b = \frac{225}{45} = 5$$

Hence the numbers are 5, 45

9. If the roots of the equation $(q-r)x^2 + (r-p)x + p - q = 0$ are equal, then show that p, q and r are in A.P.

Solution : Given equation is $(q-r)x^2 + (r-p)x + p - q = 0$

$a = q - r, \quad b = r - p, \quad c = p - q$

Since the roots of the quadratic equation are equal, $b^2 - 4ac = 0$

$$\Rightarrow (r-p)^2 - 4(q-r)(p-q) = 0$$

$$\Rightarrow r^2 + p^2 - 2rp - 4(pq - q^2 - rp + rq) = 0$$

$$\Rightarrow r^2 + p^2 - 2rp - 4pq + 4q^2 + 4rp - 4rq = 0$$

$$\Rightarrow r^2 + p^2 + 4q^2 + 2rp - 4pq - 4rq = 0$$

$$\Rightarrow (r + p - 2q)^2 = 0$$

$$[\because (a+b+c)^2 = a^2 + b^2 + c^2 + 2abc + 2abc + 2ca]$$

$$\Rightarrow r + p - 2q = 0$$

$$\Rightarrow 2q = r + p$$

$$\Rightarrow q - p = r - q$$

$\Rightarrow$ common difference is equal for p, q, r

Hence, p, q, r are in A.P.

10. If a, b, c are respectively the p^{th} q^{th} and r^{th} terms of a GP. show that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$.

Solution : Let A be the first term and R be the common ratio of the given G.P.

$$\text{Then } a = p^{th} \text{ term} \Rightarrow a = AR^{p-1}$$

$$\Rightarrow \log a = \log A + (p-1)\log R \quad ...(1)$$

$$b = q^{th} \text{ term} \Rightarrow b = AR^{q-1}$$

$$\Rightarrow \log b = \log A + (q-1)\log R \quad ...(2)$$

$$c = r^{th} \text{ term} \Rightarrow c = AR^{r-1}$$

$$\Rightarrow \log c = \log A + (r-1)\log R \quad ...(3)$$

$$\text{Now, LHS} = (q-r)\log a + (r-p)\log b + (p-q)\log c$$

$$= (q-r)[\log A + (p-1)\log R] + (r-p)[\log A + (q-1)\log R] + (p-q)[\log A + (r-1)\log R]$$

$$= \log A\,[\cancel{q} - \cancel{r} + \cancel{r} - \cancel{p} + \cancel{p} - \cancel{q}] +$$
$$\log R\,[(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)]$$

$$= \log A(0) + \log R\left[\cancel{pq} - \cancel{pr} - \cancel{q} + \cancel{r} + \cancel{qr} - \cancel{pq} - \cancel{r} + \cancel{p} + \cancel{rp} - \cancel{rq} - \cancel{p} + \cancel{q}\right]$$

$$= \log R\,[0] = 0$$

$$\therefore (q-r)\log a + (r-p)\log b + (p-q)\log c = 0.$$

1 COMPLEX NUMBERS AND QUADRATIC EQUATIONS

- Understand the concept of complex numbers.
- Discuss the types of complex numbers.
- Understand the concept of quadratic equations.
- Understand the formulas of quadratic equations.
- Understand the application of these formulae to solve problems.

Objective of the chapter:

The basic objective of this chapter is to through some light on the initial concepts of complex numbers and quadratic equations so that the practical application of these to solve the problems can be learned.

Introduction

Definition of Complex Numbers

Before Defining complex numbers, assume that $\sqrt{-1} = i$ or $i2 = -1$ which means i can be assumed as the solution of the equation $x2 + 1 = 0$. ***i*** is called as **Iota** in complex numbers.

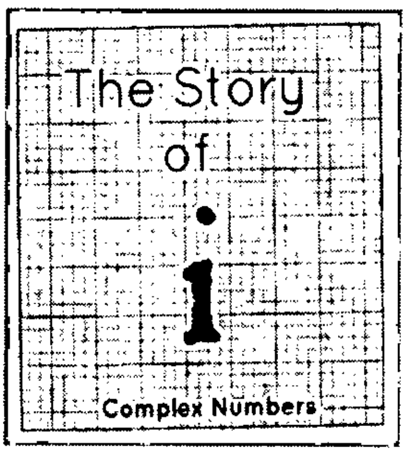
We can further formulate as,

$i2 = -1$

$i3 = i2 * i = -i$

$i4 = i2 * i2 = 1$

So, we can say now, $i4n = 1$, where n is any positive integer.



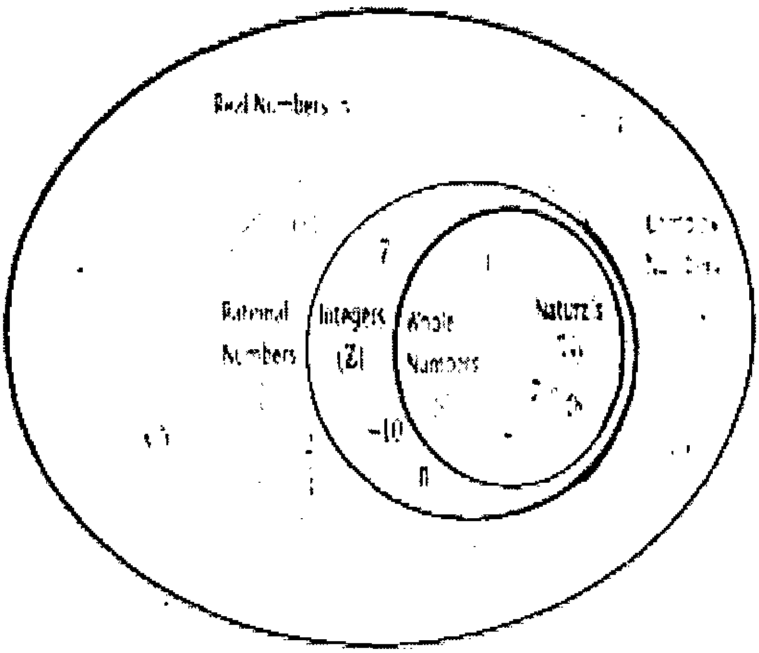
Also, note that i + i2 + i3 + i4 = 0 or in + i2n + i3n + i4n = 0

Now, any number of the form **a + ib**, where a and b are real numbers are defined as complex numbers. For example: 4 + 5i, -3 +23i, -3 + i√5 are complex numbers.

Complex numbers are denoted by z or w (= a + ib) which has two parts, one is called as the **real part** and another as **imaginary part**. z= a (or b = 0) is called as the purely real number while z = ib (or a = 0) is called as the purely imaginary number. Also note that zero or 0 + i0 is both purely real and purely imaginary but not imaginary.

Real part of z is a, denoted by Re(z) and Imaginary part of Z is b, denoted by Imag (z). For z = 1+2i, re(z) = 1 and Imag (z) = 2

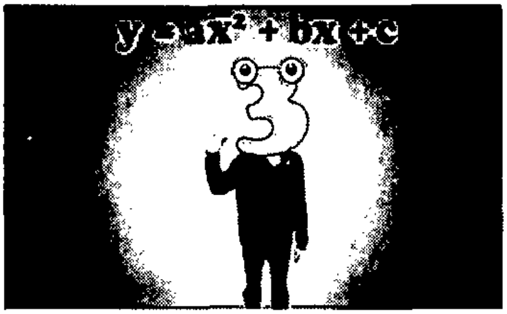
Complex number is considered as the super-set of all the other possible numbers.
N□ W □ I□Q □R □C



Definition of Quadratic Equation

An Algebraic Equation with degree n is called as the **Polynomial Equation**. Quadratic equation is the special case of it where the degree is equals to 2. So we can say that a polynomial with degree 2 or two roots (or solution) is called as the quadratic equation.

The general form of any Quadratic equation is as **y = ax2+ bx + c**, where **a ≠ 0**, b and c can be any real (or complex) number. Here, a, b and c are called as the **Coefficients**. a is also called as the **Leading Coefficient**.



Shape of the quadratic equation is parabolic in nature. For y = **ax2+ bx + c**, the parabola is open upwards if a is positive and open downwards if a is negative. While for x = **ay2+ by + c**, parabola is open rightwards if a is positive while open open leftwards if a is negative.

Quadratic equation has two roots or solution because of its degree 2. These two roots are given with the help of **Quadratic Formula** as:

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } x = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Roots of any equation indicate the point on the x-axis where the curve for any function cuts the x - axis. The number of points where the curve cut the x-axis denoted the number of roots or solution or degree of the given polynomial.

The graph for the quadratic equation is in **Parabolic Shape**. If a is positive, i.e. a > 0, then the parabola will be open-upwards and hence will only have a least value or minimum value of this function while for a to be negative, i.e. a < 0, the parabola will be open-downwards and will have a definite maximum value but not minimum value.

What is Complex Root? Or what is Complex solution?

Roots for any equation are the set of points where the graph for the given function intersects or touches the x–axis. The number of roots can be directly related with degree of a polynomial.

But if the graph of a polynomial does not intersect, in such cases we assume as the complex root (or solution). This means that, there no such real number but the complex number which will satisfy the given functional equation.

While solving the Quadratic equation, if b2 - 4ac < 0, then the roots will not be real roots and the roots in this case will be called as the **Complex Roots**.

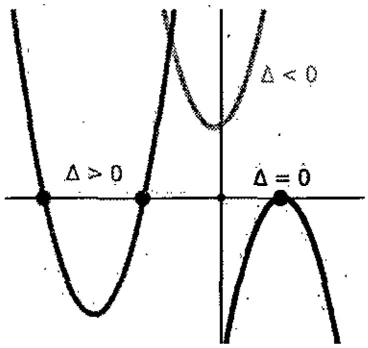
What are the different forms of representation for complex numbers?

Complex numbers can be represented into mainly four forms. They are:

- Cartesian or algebraic or rectangular form
- Trigonometric or polar form
- Exponential form
- Vector form

What is the discriminant of a quadratic equation?

The value of D = b2 – 4ac, in quadratic means **discriminant**. Discriminant plays most important role in deciding the nature of the roots. It is also represented by delta or Δ.



If D = 0, means equal roots. Graphically, parabola touches the x-axis at a single point.

If D > 0, means real and distinct roots, graphically, parabola intersects the x-axis at two distinct points.

And if D < 0, means non-real or imaginary or complex roots. In this case the parabola never touches the x-axis.

What are the different ways of solving quadratic equation?

Solving Quadratic equation means finding the values of the variable which satisfies the given equation. Quadratic equation can be solved in two ways. The first method is called the **Factorization Method.** While the second method is called as the Hindu Method or Sri dharachary method. This method (Hindu) is formula-based method.

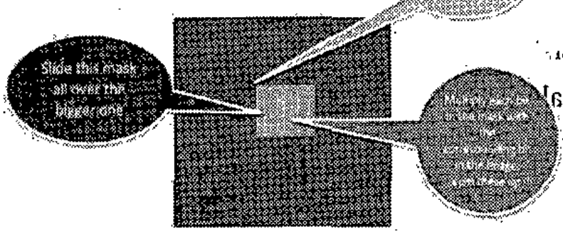
Few Examples of Complex Numbers and Quadratic Equations

Complex Numbers:

Complex Numbers has wide range of applications but most importantly used in the field of electrical engineering. In Electrical engineering, there are many applications which the complex numbers come into the picture. Such as, design of circuits using Capacitor and inductors, electromagnetism, oscillations and many more.

Other Applications

- Image Processing



Quadratic Equations:

A quadratic equation traces the parabolic cubic curve. We come across many real situations in which the path or the equation follows the parabolic curve i.e. they can be easily related with the help of quadratic equation. A boll thrown in air or jumping of any person from the building are some of the examples where the path traced by them in their journey follows the parabolic curve. Understanding quadratic equation helps us in understanding their instantaneous location during their journey.

It also helps in figuring out the profit of any business using quadratic equations.

Summary of the Chapter

Before Defining complex numbers, assume that √-1 = i or i2 = -1 which means *i* can be assumed as the solution of the equation x2 + 1 = 0. *i* is called as Iota in complex numbers. Complex numbers are denoted by z or w (= a + ib) which has two parts, one is called as the real part and another as imaginary part. z= a (or b = 0) is called as the purely real number while z = ib (or a = 0) is called as the purely imaginary number. Also note that zero or 0 + i0 is both purely real and purely imaginary but not imaginary. An Algebraic Equation with degree n is called as the Polynomial Equation. Quadratic equation is the special case of it where the degree is equals to 2. So we can say that a polynomial with degree 2 or two roots (or solution) is called as the quadratic equation. The general form of any Quadratic equation is as y = ax2+ bx + c, where a ≠ 0, b and c can be any real (or complex) number. Here, a, b and c are called as the Coefficients. a is also called as the Leading Coefficient.

Review Questions

Exercise - 2.4

1. Construct a quadratic equation with roots 7 and -3.
2. A quadratic polynomial has one of its zeros $1 + \sqrt{5}$ and it satisfies $p(1) = 2$. Find the quadratic polynomial.
3. If α and β are the roots of the quadratic equation $x^2 + \sqrt{2}x + 3 = 0$, form a quadratic polynomial with zeroes $\frac{1}{\alpha}, \frac{1}{\beta}$.
4. If one root of $k(x-1)^2 = 5x - 7$ is double the other root, show that $k = 2$ or -25.
5. If the difference of the roots of the equation $2x^3 - (a+1)x + a - 1 = 0$ is equal to their product, then prove that $a = 2$.
6. Find the condition that one of the roots of $ax^2 + bx + c$ may be (i) negative of the other. (ii) thrice the other, (iii) reciprocal of the other.
7. If the equations $x^2 - ax + b = 0$ and $x^2 - ex + f = 0$ have one root in common and if the second equation has equal roots, then prove that $ae = 2(b + f)$.
8. Discuss the nature of roots of (i) $-x^2 + 3x + 1 = 0$; (ii) $4x^2 - x - 2 = 0$, (iii) $9x^2 + 5x = 0$.
9. Without sketching the graphs, find whether the graphs of the following functions will intersect the x-axis and if so in how many points.
   (i) $y = x^2 + x + 2$, (ii) $y = x^2 - 3x - 7$, (iii) $y = x^2 + 6x + 9$.
10. Write $f(x) = x^2 + 5x + 4$ in completed square form.

2 PERMUTATION AND COMBINATION

- Understand the concept of Permutation.
- Understand the concept of Combination.
- Discuss the concept of factorial notation.
- Understand the similarities between permutation and combination.
- Understand the difference between permutation and combination.

Objective of the chapter:

The basic objective of this chapter is to through some light on the initial concepts of permutation and combinations so that the practical application of permutation and combination can be examined in detailed.

Introduction

What do you mean by Permutation and Combination?

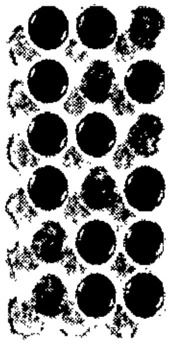
These two words permutation and combination, at the initial level are very confusing and are generally used interchangeably. So let's take them one by one and understand them.

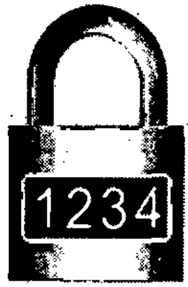
Combination means from the given certain objects (may be alike or different) **selecting** one or more objects. Combination can also be replaced by the words – selection, collection or committee.

For Example – Combination of top 5 cricket players from the team of 11 players is the selection of 5 players (in any order).

The sequence in which they have to be selected is not important here. Also we can say that the order of selection is not the concern in the case of combination

The word ***permutation*** means **arrangement** of the alike or different objects taken some or all at a time. So we can observe the word 'arrangement' used in the definition of permutation. Here the arrangement means selection as well as ordering. That means the order in which the objects are selected have also been taken care of in this case.





Permutation Vs Combination

For Example – The number of 5-digit numbers which can be formed using the digits 0, 1, 2, 3, 4 and 5.

In this example, we just not have to select the 5 digits out of given 6 digits but also have to see the number of possible cases for the different arrangement. So, the numbers 34251, 21034, 42351 are all different cases.

We will try to explore these definitions in the upcoming heading.

What is Factorial Notation in mathematics?

Since the definition or the formula of both, permutation and combination, requires the use of factorial notation, so let's first understand this here before learning any further.

In Mathematics, the factorial is represented by the symbol '!' i.e., if we have to write 5 factorials, so it will be written as 5! So in general factorial of any positive number n will be represented by n!

Mathematically,

$$n! = \begin{cases} 1 & if\, n = 0 \\ (n-1)!\, n & if\, n > 0 \end{cases}$$

Where n is any positive integer.

So, 4! = 3! X 4 = 2! X 3 x 4 = 1! x 2 x 3 x 4 = 0! x 1 x 2 x 3 x 4 = 1 x 2 x 3 x 4

Similarly, we can say for any positive integer 'n'

n! = n x (n – 1) x (n – 2) x .... x 3 x 2 x 1.

Thus, we seeing the above equation, we may also define the factorial of any positive integer n as '*the product of all the positive integers less than or equal to n*'.

Just see below for the factorial of few frequently used numbers.

0! = 1

1! = 1

2! = 2 x 1 = 2

3! = 3 x 2 x 1 = 6

4! = 4 x 3 x 2 x 1 = 24

5! = 5 x 4 x 3 x 2 x 1 = 120 and so on.

Differentiate Permutation and Combination

The very basic difference in permutation and combination is the **order** of the objects considered. In combination, the order is not considered at all while for permutation it is must. So the permutation is the ordered arrangement while the combination is the unordered selection.

From the three alphabets A, B and C, the permutation of these 3 letters will be ABC, ACB, BAC, BCA, CBA and CAB. While the combination of 3 letters will be just (A, B, C).

Permutations & Combinations

A combination is an arrangement of items in which **ORDER DOES NOT MATTER.**	**A permutation is** an arrangement of items in a particular order. Notice, **ORDER MATTERS!**

Permutation gives the answer to the number of arrangements while the combination explains the possible number of selections.

Permutation of a single combination can be multiple but the combination of a single permutation is unique (considering all at a time).

Explain Permutation with some practical examples

Permutation refers to the situation where the arrangement of objects is being considered.

In general, the permutation of n distinct objects taken r at a time is represented and calculated as:

$$^nP_r = \frac{n!}{(n-r)!}$$

This can also be represented as **P(n, r) or Pnr.** Here, in the definition of permutation, r can be any positive integer less then or equals to n. So on the basis of the values of r whether it is less then or equals to n we can have two different conditions or theorems.

Theorem – 1

The number of permutations or arrangement of n distinct things taken all at a time can be represented by:

$$^nP_n = P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1.$$

Example

Consider the 5 seats in a car on which 5 persons are to be seated.

So to find the number of cases in which 5 persons can be seated will be the case of permutation of 5 persons taking all 5 seats at a time (*5P5)*.

Remember:

n ⟶ **total**
r ⟶ **want**

Theorem - 2

The number of permutations or arrangement of n distinct things taken only r (r < n) at a time can be represented by:

$$ {}^nP_r = P(n, r) = \frac{n!}{(n-r)!} = \frac{(n-r)!\,[n.(n-1).(n-2)......(n-(r-1))]}{(n-r)!} = n.(n-1).(n-2)......(n-(r-1)) $$

Example

Consider the 10 chairs in a room for which 15 persons are supposed to be seated.

In this case, here we have 15 persons to be arranged but only on 10 chairs. So this can be calculated by*15P10.*

One more thing which we can learn by observing the above example is that, we first need to choose or select 10 out of 15 persons which can be arranged on 10 available chairs.

So, every problem on permutation is broken down into selection and then arrangement.

That is, **Permutation = Selection x Arrangement**

What do you mean by Combination? Give some examples

Combination is the selection or collection of one or more things from the given list of alike or distinct objects taken all or some at a time.

In general, the combination of n distinct objects taken r at a time, is represented and calculated as:

$$ {}^nC_r = \frac{n!}{(n-r)!\,r!} $$

This can also be represented as $C(n, r)$ or C_r^n or $\binom{n}{r}$. Here, in the definition of combination, r can be any positive integer less then or equals to n.

$$C_{(n,r)} = \frac{n!}{r!\,(n-r)!}$$

$$P_{(n,r)} = \frac{n!}{(n-r)!}$$

n = set size: the total number of items in the sample

r = subset size the number of items to be selected from the sample

Let's discuss few important cases here in combination.

Theorem - III

The number of combinations of n different things taken r at a time.

$$^nC_r = C(n, r) = \frac{n!}{(n-r)!\; r!}$$

Example

Calculate the number of selections of 3 different coloured pens from the available 5 pens of all different coloured pens

Here, we just need to select 3 pens (in any order) from the available 5 pens. This can be calculated as:

$$^5C_3 = C(5, 3) = \frac{5!}{(5-3)!\; 3!}$$

Here, in this theorem, note that if r = n that is, all the things have to be selected.

So,

$$^nC_n = C(n, n) = \frac{n!}{(n-n)!\; n!} = 1$$

, which is very obvious.

Theorem – IV

Number of combinations of n different things taken r at a time when p particular things are always **included** will be calculated as

n-pCr-p

Example

Calculate the number of ways of combination or selection of 11 players out of 20 players when virat kohli, M.S. Dhoni and Y. Singh are always included.

Here we have been giving 20 players of which only 11 players are to be selected. We are also given the 3 players out of 20 which must be included in any case. So actually we can understand that, out of 11 we have already 3 players so we just need to select 8 addition players from the remaining 17 players.

Thus, Total number of ways for the above problem = 20-3 C11-3 = 17C8

Theorem - V

The number of combinations of n different things taken r at a time when p particular things are always **excluded** can be calculated as:

n-pCr

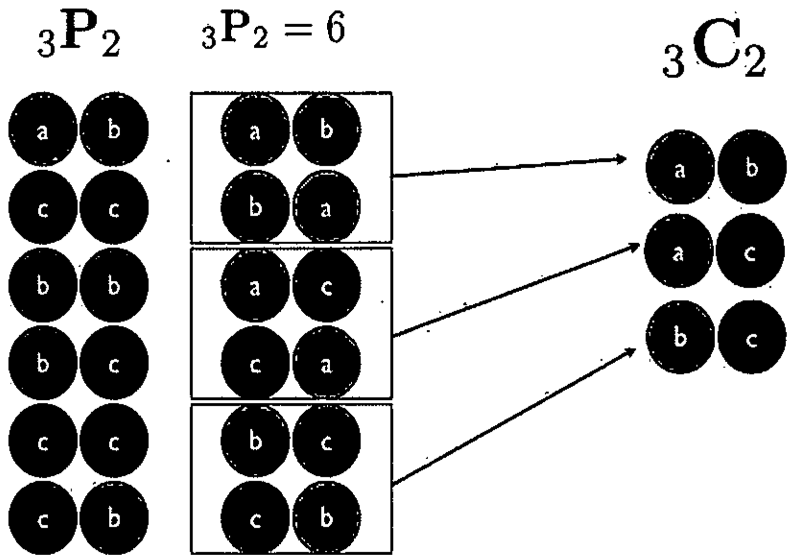
Let's take the same kind of example which we have discussed in previous theorem.

Example

Calculate the number of ways of combination or selection of 11 players out of 20 players ravindra jadeja and balaji are always excluded.

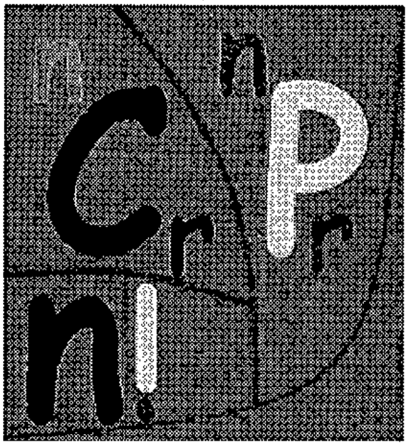
Again, we have been given 20 players of which 11 players to be selected but this time 2 specific players are to be excluded. Thus, actually we have the option of 18 players effectively for selecting 11 players.

Thus, The total number of selections = 20-2C11 = 18C11



Is there any relation between Permutation and Combination?

As discussed in the previous sections, permutation is the combination (or selection) and the arrangement as well.



Thus, while calculating the permutation, we first need to choose or selecting the thing before their arrangement.

So,

Permutation = Selection x Arrangement

This can also be understood from their mathematical relation. Since we know that,

$$^nP_r = \frac{n!}{(n-r)!} \quad \text{and} \quad ^nC_r = \frac{n!}{(n-r)!\,r!}$$

Thus, from the above two formulas, this is very clear that

nPr = nCr r!

where, **nCr** denotes the selection and **r!** Denotes the arrangement of r objects for the r places.

Summary of the Chapter

These two words permutation and combination, at the initial level are very confusing and are generally used interchangeably. So let's take them one by one and understand them. Combination means from the given certain objects (may be alike or different) selecting one or more objects. Combination can also be replaced by the words – selection, collection or committee. For Example – Combination of top 5 cricket players from the team of 11 players is the selection of 5 players (in any order). The sequence in which they have to be selected is not important here. Also we can say that the order of selection is not the concern in the case of combination. The word *permutation* means arrangement of the alike or different objects taken some or all at a time. So we can observe the word 'arrangement' used in the definition of permutation. Here the arrangement means selection as well as ordering. That means the order in which the

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Mathematics

Notes

objects are selected have also been taken care of in this case. The very basic difference in permutation and combination is the order of the objects considered. In combination, the order is not considered at all while for permutation it is must. So the permutation is the ordered arrangement while the combination is the unordered selection.

Review Question

Exercise 4.2

1. If $^{(n-1)}P_3 : {}^nP_4 = 1 : 10$, find n.

Solution : Given $^{(n-1)}P_3 : {}^nP_4 = 1 : 10$

$$\Rightarrow \quad \frac{^{(n-1)}P_3}{^nP_4} = \frac{1}{10}$$

$$\Rightarrow \quad 10.\,{}^{(n-1)}P_3 = 1.{}^nP_4$$

$$\left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$

$$\Rightarrow \quad 10 \times \frac{(n-1)!}{(n-1-3)!} = \frac{n!}{(n-4)!}$$

$$\Rightarrow \quad \frac{10 \times \cancel{(n-1)!}}{\cancel{(n-4)!}}$$

$$\Rightarrow \quad \frac{10 \times \cancel{(n-1)!}}{\cancel{(n-4)!}} = \frac{n\cancel{(n-1)!}}{\cancel{(n-4)!}}$$

$$\Rightarrow \quad 10 = n$$

$$\therefore n = 10.$$

2. If $^{10}P_{r-1} = 2 \times {}^6P_r$, find r

Solution : Given $^{10}P_{r-1} = 2 \times {}^6P_r$

$$\Rightarrow \quad \frac{10!}{(10-r+1)!} = 2 \times \frac{6!}{(6-r)!} \quad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$

$$\Rightarrow \quad \frac{10 \times 9 \times 8 \times 7 \times \cancel{6!}}{(11-r)!} = \frac{2 \times \cancel{6!}}{(6-r)!}$$

$$\Rightarrow \quad \frac{10 \times 9 \times 8 \times 7}{(11-r)(10-r)(9-r)(8-r)(7-r)(6-r)!}$$

$$= \frac{2}{(6-r)!}$$

$$\Rightarrow \quad \frac{10\times9\times8\times7}{(11-r)(10-r)(9-r)(8-r)(7-r)}=2$$

$$\Rightarrow \quad (11-r)(10-r)(9-r)(8-r)(7-r)$$
$$= 5\times9\times8\times7$$

$$\Rightarrow \quad (11-r)(10-r)(9-r)(8-r)(7-r)$$
$$= 7\times6\times5\times4\times3$$

$$\Rightarrow \quad (11-r)(10-r)(9-r)(8-r)(7-r)$$
$$= (11-4)(10-4)(9-4)(8-4)(7-4)$$

$$\Rightarrow \quad r = 4$$

3. (i) Suppose 8 people enter an event in a swimming meet. In how many ways could the gold, silver and bronze prizes be awarded?

Solution : (i) Gold medal can be awarded to any one of the 8 candidates in 8 ways.

Silver medal can be awarded to any one of the remaining 7 candidates in 7 ways

Bronze medal can be awarded to any one of the remaining 6 candidates in 6 ways.

$\therefore$ Total numbers of ways of awarding the prize

$= 8\times7\times6 = 336$

(ii) Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them?

Solution : 4 coats can be given to 3 men in ${}^4P_3$ ways. 5 waist coats can be given to 3 men in ${}^5P_3$ ways 6 caps can be given to 3 men in ${}^6P_3$ ways.

$\therefore$ Total number of ways of wearing them

$= {}^4P_3 \times {}^5P_3 \times {}^6P_3.$

$= \dfrac{4!}{1!}\times\dfrac{5!}{2!}\times\dfrac{6!}{3!}$

$= 4\times3\times2\times\dfrac{5\times4\times3\times\cancel{2!}}{\cancel{2!}}\times\dfrac{6\times5\times4\times\cancel{3!}}{\cancel{3!}}$

$= 24\times60\times120 = 172800$

4. Determine the number of permutations of the letters of the word SIMPLE if all are taken at a time?

Solution : There are 6 letters in the word 'SIMPLE'. So, total number of words is equal to the number of arrangements of these letters, taken all at a time. Sum order of such arrangements is ${}^6P_6 = 6! = 720$

5. A test consists of 10 multiple choice questions. In how many ways can the test be answered if

(i) Each question has four choices?

(ii) The first four questions have three choices and the remaining have five choices?

(iii) Question number n has $n + 1$ choices?

Solution :

(i) Each question has four choices?

Since each question can be answered in 4 ways, the total number of ways of answering 10 questions is $4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^{10}$

(ii) The first four questions have three choices and the remaining have five choices?

Since first four question have three choices, the number of ways of answering first four questions is $3 \times 3 \times 3 \times 3 = 3^4$.

Remaining 6 questions have 5 choices each

$\therefore$ Number of ways of answering the remaining 6 question

$= 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$.

$\therefore$ Total number of ways of answering the questions $= 3^4 \times 5^6$

(iii) Question number n has $n + 1$ choices?

Since 10 questions can be answered in 10 ways the total number of ways of answering (10+1) question is 11!

6. A student appears in an objective test which contain 5 multiple choice questions. Each question has 4 choices out of which one correct answer.

(i) What is the maximum number of different answers can the students give?

(ii) How will the answer change if each question may have more than one correct answers?

Solution :

(i) What is the maximum number of different answers can the students give?

Since each question can be answered in 4 ways, the maximum number of different answers.

$= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$

(ii) How will the answer change if each question may have more than one correct answers?

The question may have 1 correct answer, or 2 correct answers or 3 or 4 or 5 correct answers.

$\therefore$ Number of correct answers

$= 1 + 2 + 3 + 4 + 5 = 15$

$\therefore$ Maximum number of answers $= 15^5$

7. How many strings can be formed from the letters of the word ARTICLE, so that vowels occupy the even places?

Solution : In the letters of the word, ARTICLE, there are three vowels namely A, I, E.

There are 3 even places.

3 vowels can occupy the even places in ${}^3P_3 = 3!$ ways.

Remaining 4 letters can occupy 4 places in 4! ways.

Hence, total number of ways of arrangement $= 4! \times 3! = 4 \times 3 \times 2 \times 3 \times 2 = 144$.

8. 8 women and 6 men are standing in a line.

(i) How many arrangements are possible if any individual can stand in any position?

(ii) In how many arrangements will all 6 men be standing next to one another?

(iii) In how many arrangements will no two men be standing next to one another?

Solution :

(i) How many arrangements are possible if any individual can stand in any position?

Since any individual can stand in any position, 8 women and 6 men can be arranged in ${}^{14}P_{14} = 14!$ ways

(ii) In how many arrangements will all 6 men be standing next to one another?

Considering 6 men as one unit, we have 9 people and they can be arranged in 9! ways. These 6 men can arrange among themselves in 6! ways.

$\therefore$ Total number of arrangement $= 9! \times 6!$

(iii) In how many arrangements will no two men be standing next to one another?

8 women can be arranged in 8 places in 8! ways.

$\times \boxed{W} \times \boxed{W} \times \boxed{W} \times \boxed{W} \times \boxed{W} \times \boxed{W} \times \boxed{W} \times$

There are 9 (× marked) places for 6 men.

They can be arranged in ${}^9P_6$ ways.

$\therefore$ Total number of ways $= {}^9P_6 \times 8!$

9. Find the distinct permutations of the letters of the word MISSISSIPPI?

Solution : There are 11 letters in the given word of which 4 are S's, 4 are I's and 2 are P's and 1 M.

Hence total number of distinct words

$$= \frac{11!}{4!4!2!1!} = \frac{11!}{4!4!2!}$$

$$= \frac{11 \times 10 \times 9 \times \cancel{8} \times 7 \times \cancel{6} \times 5 \times \cancel{4!}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1 \times \cancel{4!} \times \cancel{2}} = 34650$$

10. How many ways can the product $a^2b^3c^4$ be expressed without exponents?

Solution : Given factors are two a's, 3 b's and 4 c's

Total number of exponents = 9.

Hence, number of ways the product can be expressed without exponents

$$= \frac{9!}{2!3!4!} = \frac{9 \times \overset{4}{\cancel{8}} \times 7 \times \cancel{6} \times 5 \times \cancel{4!}}{\cancel{2} \times \cancel{3} \times \cancel{2} \times \cancel{4!}} = 1260$$

11. In how many ways 4 mathematics books, 3 physics books, 2 chemistry books and 1 biology book can be arranged on a shelf so that all books of the same subjects are together.

Solution : Four subjects can be arranged on the shelf in 4! ways. The books on mathematics can be arranged in 4! ways, physics on 3! ways, chemistry on 2! ways and Biology on 1! ways.

Hence, total number of ways of arranging the books

$= 4! \times 4! \times 3! \times 2! \times 1!$
$= (4 \times 3 \times 2 \times 1)(4 \times 3 \times 2 \times 1)(3 \times 2)(2 \times 1)$
$= (24)(24)(6)(2) = 6912.$

12. In how many ways can the letters of the word SUCCESS be arranged so that all S's are together?

Solution : Considering all S as one letter there are 5 letters containing 2 C's, one U, and one E which can be arranged in $\frac{5!}{2!1!1!} = \frac{5 \times 4 \times 3 \times \cancel{2}}{\cancel{2}} = 60$ ways

13. A coin is tossed 8 times.

(i) How many different sequences of heads and tails are possible?

(ii) How many different sequences containing six heads and two tails are possible?

Solution :

(i) How many different sequences of heads and tails are possible?

A coin has got only 2 out comes. When 8 coin is tossed 8 times, the possible number of sequences of heads and tails are 2^8.

(ii) **How many different sequences containing six heads and two tails are possible?**

Since there are 6 heads of one kind and 4 tails of other kind, required number of sequences

$$= \frac{\lfloor 8}{\lfloor 6 \lfloor 2} = 28.$$

14. How many strings are there using the letters of the word INTERMEDIATE, if

(i) **The vowels and consonants are alternative**

(ii) **All the vowels are together**

(iii) **Vowels are never together**

(iv) **No two vowels are together.**

Solution :

(i)

V	C	V	C	V	C	V	C	V	C	V	C

Case (i)

When the first place is a vowel, number of permutation $= \frac{6!}{2!3!}$

Remaining 6 places have consonants

$\therefore$ Number of permutation $= \frac{6!}{2!}$

$\therefore$ Number of ways $= \left(\frac{6!}{2!3!}\right)\left(\frac{6!}{2!}\right)$

Case(ii)

When the first place is a consonant,

Number of ways $= \left(\frac{6!}{2!3!}\right)\left(\frac{6!}{2!}\right)$

$\therefore$ Total number of ways $= 2\left(\frac{6!}{2!3!}\right)\left(\frac{6!}{2!}\right)$

$= 2\,(60)\,(360) = 43200$

(ii) 6 consonants out of which 2 are alike can be place in $\frac{6!}{2!}$ ways and 6 vowels out of which 3E's are alike and 2I's are alike can be arranged in 7 places in $7P_6 \times \frac{1}{3!} \times \frac{1}{2!}$ ways

$\therefore$ Total number of words $= \frac{6!}{2!} \times 7P_6 \times \frac{1}{3!} \times \frac{1}{2!}$

$$= \frac{6 \times 5 \times \cancel{4} \times \cancel{3!}}{\cancel{2}} \times \frac{7!}{1!} \times \frac{1}{\cancel{3!}} \times \frac{1}{\cancel{2}}$$

$= 30 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 151200.$

(iii) **Vowels are never together**

Total number of arrangements $= \dfrac{12!}{2!\,2!\,3!}$

By (ii), number of arrangements, when all the vowels are together $= \dfrac{7!}{2!}\left(\dfrac{6!}{2!\,3!}\right)$

$\therefore$ Number of arrangements, when all the vowels are never together

$$= \frac{12!}{2!\,2!\,3!} - \frac{7!}{2!}\left(\frac{6!}{2!\,3!}\right)$$

$$= 19958400 - 151200 = 19807200.$$

(iv) No two vowels are together is the same as (i) 43200.

15. Each of the digits 1, 1, 2, 3, 3 and 4 is written on a separate card. The six cards are then laid out in a row to form a 6-digit number.

(i) How many distinct 6-digit numbers are there?

(ii) How many of these 6-digit numbers are even?

(iii) How many of these 6-digit numbers are divisible by 4?

Solution :

(i) **How many distinct 6-digit numbers are there?**

Given numbers are 1, 1, 2, 3, 3, 4

Here 1 occur twice

3 occur twice

$\therefore$ Number of distinct 6-digit numbers

$$= \frac{6!}{2!2!} = \frac{6\times5\times\cancel{4}\times3\times2!}{\cancel{2}\times\cancel{2}} = 30\times6 = 180$$

(ii) **How many of these 6-digit numbers are even?**

Unit place can be filled in 2 ways by 2 or 4.

Remaining 5 digits can be filled in $\dfrac{5!}{2!2!}$

$\therefore$ Number of even 6 digit numbers

$$= \frac{5!}{2!2!}\times2 = \frac{5\times4\times3\times\cancel{2}}{\cancel{2}} = 60$$

(iii) **How many of these 6-digit numbers are divisible by 4?**

Unit place can be filled in one ways by 4.

Remaining 5 digits can be filled in $\frac{5!}{2!2!}$ ways

$\therefore$ Number of even 6 digit numbers divisible by 4

$$=\frac{5!}{2!2!}\times 1=\frac{5\times\cancel{4}\times 3\times 2\times 1}{\cancel{2}\times\cancel{2}}=30$$

16. If the letters of the word GARDEN are permuted in all possible ways and the strings thus formed are arranged in the dictionary order, then find the ranks of the words (i) GARDEN (ii) DANGER.

Solution :

(i) **GARDEN**

The lexicographic order of the letters of the given word is A, D, E, G, N, R.

The word GARDEN has 6 letters in which no letters are repeating.

Number of words starting with $A = 5! = 120$

Number of words starting with $D = 5! = 120$

Number of words starting with $E = 5! = 120$

Number of words starting with $GAD = 3! = 6$

Number of words starting with $GAE = 3! = 6$

Number of words starting with $GAN = 3! = 6$

Number of words starting with $GARDE = 1!$

Next word is GARDEN

$\therefore$ Rank $= 120 + 120 + 120 + 6 + 6 + 6 + 1$

$= 379.$

(ii) **DANGER**

The lexicographic order of the letters of the given word is A, D, E, G, N, R.

The word GARDEN has 6 letters in which no letters are repeating.

Number of words starting with $A = 5!$

Number of words starting with $DAE = 3!$

Number of words starting with $DAG = 3!$

Number of words starting with $DANE = 2$

Next word is $DANGER = 1$

$\therefore$ *Rank* $= 120 + 6 + 6 + 2 + 1 = 135$

17. Find the number of strings that can be made using all letters of the word THING. If these words are written as in a dictionary, what will be the 85th string?

Solution : In the word THING, there are 5 letters

The lexicographic order of the word is G, H, I, N, T

Number of words starting with $G = 4! = 24$

Number of words starting with $H = 4! = 24$

Number of words starting with $I = 4! = 24$

Number of words starting with $NG = 3! = 6$

Number of words starting with $NGH = 2! = 2$

Number of words starting with NIGHT= 1!

85th string is NIGHT

18. If the letters of the word FUNNY are permuted in all possible ways and the strings thus formed are arranged in the dictionary order, find the rank of the word FUNNY.

Solution : Lexicographic order of the word is F, N, N, U, Y

Number of words starting with $FN = 3! = 6$

Number of words starting with FUNNY= 1

$\therefore$ Rank of FUNNY $= 6 + 1 = 7$.

19. Find the sum of all 4-digit numbers that can be formed using digits 1, 2, 3, 4, and 5 repetitions not allowed?

Solution : The number of 4-digit numbers that can be formed using the 5 digits is ${}^5P_4 = 120$

Let us find the sum of the digits in the unit place.

By filling 1 in the unit place, remaining 3 places can be filled with remaining 4 digits is ${}^4P_3 = 24$ ways.

Similarly, each of the digits 2, 3, 4, 5 appear 24 times in unit place.

$\therefore$ Sum of all these 120 numbers are

$({}^4P_3 \times 1) + ({}^4P_3 \times 2) + ({}^4P_3 \times 3)\ ({}^4P_3 \times 4) + ({}^4P_3 \times 5)$

$= {}^4P_3\,(1 + 2 + 3 + 4 + 5) = {}^4P_3 \times 15 = 24 \times 15 = 360$

Similarly sum of the digits in the ten's place is 3600

Sum of the digits in the hundreds place is 36000

Sum of the digits in the thousands place is 360000

Hence, sum of all the 4 – digit numbers formed by using the digits 1, 2, 3, 4 and 5 is

$= 360 + 3600 + 36000 + 360000 = 399960.$

20. Find the sum of all 4-digit numbers that can be formed using digits 0, 2, 5, 7, 8 *without* repetition?

Solution :

tho	hun	tens	unit
4	4	3	2

The gives digits are 0, 2, 5, 7, 8

Number of 4 digit numbers that can be formed is $4 \times 4 \times 3 \times 2 = 96$.

Out of 96, there will be 24 numbers ending with 0

18 numbers ending with 2

18 numbers ending with 5

18 numbers ending with 7

18 numbers ending with 8

$\therefore$ Total for unit place is $(24 \times 0) + (18 \times 2) + (18 \times 5) + (18 \times 7) + (18 \times 8)$

$= 18\,(2 + 5 + 7 + 8) = 18 \times 22 = 396$

$\therefore$ Sum of all the 4 digit numbers $= 396 + 3960 + 39600 + (24 \times 22) \times 1000 = 571956$

EXERCISE 4.3

1. If $^nC_{12} = {}^nC_9$ find $^{21}C_n$

Solution : We have $^nCx = {}^nCy \Rightarrow x = y$ or $x + y = n$

$\Rightarrow 12 + 9 = n$

$\Rightarrow \quad n = 21$

$\Rightarrow \quad {}^{21}C_n = {}^{21}C_{21} = 1 \qquad [\because nC_n = 1]$

2. If $^{15}C_{2r-1} = {}^{15}C_{2r+4}$, find r.

Solution : Given $^{15}C_{2r-1} = {}^{15}C_{2r+4}$

Since $^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

We get $2r - 1 = 2r + 4$

$\Rightarrow \quad -1 \neq 4$ which is not possible (or)

$$2r - 1 + 2r + 4 = 15$$

$$4r + 3 = 15$$

$$4r = 12$$

$$\Rightarrow \quad r = \frac{\cancel{12}^{3}}{\cancel{4}} = 3$$

$$\therefore r = 3$$

3. If $^nP_r = 720$. If $^nC_r = 120$, find n, r

Solution : Given $^nP_r = 720$ and $^nC_r = 120$.

$$\Rightarrow \quad \frac{n!}{(n-r)!} = 720 \qquad ...(1)$$

$$\frac{n!}{r!(n-r)!} = 120 \qquad ...(2)$$

$$\Rightarrow \quad \frac{\frac{n!}{(n-r)!}}{\frac{n!}{r!(n-r)!}} = \frac{\cancel{720}^{60}}{\cancel{120}}$$

[Dividing (1) by (2)]

$$\Rightarrow \frac{\cancel{n!}}{\cancel{(n-r)!}} \times \frac{r!\,\cancel{(n-r)!}}{\cancel{n!}} = 6$$

$$\Rightarrow \qquad r! = 6$$

$$\Rightarrow \qquad r! = 3 \times 2 \times 1 = 3!$$

$$\Rightarrow \qquad r = 3.$$

Substituting $r = 3$ in (1) we get,

$$\frac{n!}{(n-r)!} = 720$$

$$\Rightarrow \qquad \frac{n!}{(n-3)!} = 720$$

$$\Rightarrow \frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = 720$$

$$\Rightarrow \qquad n(n-1)(n-2) = 720$$

$$\Rightarrow \qquad n(n-1)(n-2) = 10 \times 9 \times 8$$

$$\Rightarrow \qquad n = 10.$$

4. **Prove that $^{15}C_3 + 2 \times {}^{15}C_4 + {}^{15}C_5 = {}^{17}C_5$**

Solution :

$$\text{LHS} = \frac{15\times14\times13}{1\times2\times3} + 2\times\frac{15\times14\times13\times12}{1\times2\times3\times4} + \frac{15\times14\times13\times12\times11}{1\times2\times3\times4\times5}$$

$$= \frac{15\times14\times13}{1\times2\times3}\left[1 + \frac{24}{4} + \frac{132}{20}\right]$$

$$= \frac{15\times14\times13}{1\times2\times3}\left[\frac{20+120+132}{20}\right]$$

$$= \frac{15\times14\times13\times272}{1\times2\times3\times4\times5} = \frac{17\times16\times15\times14\times13}{1\times2\times3\times4\times5}$$

$$= {}^{17}C_5 = \text{RHS}$$

Hence proved.

5. **Prove that $^{35}C_5 + \sum_{r=0}^{4} {}^{(39-r)}C_4 = {}^{40}C_5$**

Solution : $\text{LHS} = {}^{35}C_5 + \sum_{r=0}^{4} {}^{(39-r)}C_4$

$$= {}^{35}C_5 + {}^{39}C_4 + {}^{38}C_4 + {}^{37}C_4 + {}^{36}C_4 + {}^{35}C_4$$

$$= ({}^{35}C_5 + {}^{35}C_4) + {}^{39}C_4 + {}^{38}C_4 + {}^{37}C_4 + {}^{36}C_4$$

$$= {}^{36}C_5 + {}^{39}C_4 + {}^{38}C_4 + {}^{37}C_4 + {}^{36}C_4$$

$$[\because nC_{r-1} + nC_r = n+1C_r]$$

$$= ({}^{36}C_5 + {}^{36}C_4) + {}^{39}C_4 + {}^{38}C_4 + {}^{37}C_4$$

$$= {}^{37}C_5 + {}^{39}C_4 + {}^{38}C_4 + {}^{37}C_4$$

$= ({}^{37}C_5 + {}^{37}C_4) + {}^{39}C_4 + {}^{38}C_4$
$= {}^{38}C_5 + {}^{39}C_4 + {}^{38}C_4 = {}^{38}C_5 + {}^{38}C_4 + {}^{39}C_4$
$= {}^{39}C_5 + {}^{39}C_4 = {}^{40}C_5 = \text{RHS}$

6. If ${}^{(n+2)}C_8 : {}^{(n-2)}P_4 = 57:16$, find the value of n.

Solution : Given ${}^{(n+2)}C_8 : {}^{(n-3)}P_4 = 57 : 16$

$$\Rightarrow \quad \frac{(n+2)C_8}{(n-3)P_4} = \frac{57}{16}$$

$$\Rightarrow \quad 16 \cdot (n+2)\, C_8 = 57 \cdot (n-3)P_4$$

$$\frac{16(n+1)!}{8!(n+1-8)!} = \frac{57(n-3)!}{(n-3-4)!}$$

$$\left[\because nP_r = \frac{n!}{(n-r)!};\ nC_r = \frac{n!}{r!(n-r)!}\right]$$

$$\frac{16(n+1)n(n-1)(n-2)}{8!\cancel{(n-7)}!} = \frac{57\cancel{(n-3)}!}{\cancel{(n-7)}!}$$

$$(n+1)\,n\,(n-1)\,(n-2) = \frac{57 \times 8!}{16}$$

$$= \frac{57 \times \cancel{8} \times 7 \times 6 \times 5 \times 4 \times 3 \times \cancel{2} \times 1}{\cancel{16}}$$

$$= 3 \times 19 \times 7 \times 6 \times 5 \times 4 \times 3$$

$$(n+1)\,n\,(n-1)\,(n-2) = 21 \times 20 \times 19 \times 18$$

$$\boxed{n = 20}$$

7. Prove that ${}^{2n}C_n = \dfrac{2^n \times 1 \times 3 \times ...(2n-1)}{n!}$.

Solution : LHS $= {}^{2n}C_n$

$$= \frac{2n!}{n!(2n-n)!} = \frac{2n!}{n!n!}$$

$$= \frac{(2n)(2n-1)(2n-2)...4 \cdot 3 \cdot 2 \cdot 1}{n!n!}$$

$$= \frac{(2n)(2n-2)...4 \cdot 2[(2n-1)(2n-3)...3 \cdot 1]}{n!n!}$$

[Separate the even and odd terms]

$$= \frac{2^n\, n(n-1)\cdots 2 \cdot 1 \cdot [(2n-1)(2n-3)\cdots 3 \cdot 1]}{n!n!}$$

[Taking 2 common from each bracket]

$$= \frac{2^n \cdot \cancel{(n)!}\,[1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1)]}{\cancel{n!}\,n!}$$

$$= \frac{2^n[1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1)]}{n!} = \text{RHS}$$

8. Prove that if $1 \le r \le n$, then $n \times {}^{(n-1)}C_{r-1} = (n-r+1)\cdot {}^{n}C_{r-1}$

Solution : LHS $= n \times {}^{(n-1)}C_{r-1}$

$$= n \times \frac{(n-1)!}{(r-1)!(n-\cancel{1}-r+\cancel{1})!}$$

$$= n \times \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$= n \times \frac{(n-1)!(n-r+1)}{(n-r+1)(r-1)!(n-r)!}$$

[Multiplying and dividing by $(n-r+1)$]

$$= \frac{n!(n-r+1)}{(n-r+1)!(r-1)!}$$

$[n(n-1)! = n!$ and $(n-r+1)(n-r)! = (n-r+1)!]$

$$= (n-r+1)\cdot\frac{n!}{(r-1)!(n-r+1)!}$$

$$= (n-r+1)\times {}^{n}C_{r-1} = \text{RHS}$$

$$\left[\because nC_r = \frac{n!}{r!(n-r)!}\right]$$

Hence proved.

9. (i) A Kabaddi coach has 14 players ready to play. How many different teams of 7 players could the coach put on the court?

Solution : Here 7 players must be selected from 14 players. This can be done in ${}^{14}C_7$ ways.

Hence, number of different teams of players

$$= {}^{14}C_7 = \frac{14!}{7!7!}$$

$$= \frac{\overset{2}{\cancel{14}}\times 13\times \cancel{12}\times 11\times \cancel{10}\times \overset{3}{\cancel{9}}\times \overset{4}{\cancel{8}}\times \cancel{7!}}{\cancel{7!}\,\cancel{7}\times\cancel{6}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times 1}$$

$$= 13\times 11\times 2\times 3\times 4 = 3432$$

(ii) There are 15 persons in a party and if each 2 of them shakes hands with each other, how many handshakes happen in the party?

The total number of handshakes is same as the number of ways of selecting 2 persons among 15 persons.

This can be done in ${}^{15}C_2$ ways

$\therefore$ Number of handshakes $= {}^{15}C_2 = \dfrac{15!}{2!13!}$

$$= \frac{15\times 14\times \cancel{13!}}{2!\cancel{13!}} = \frac{15\times \overset{7}{\cancel{14}}}{\cancel{2}} = 15\times 7 = 105.$$

(iii) **How many chords can be drawn through 20 points on a circle?**

A chord is obtained by joining any two points on a circle.

Number of chords drawn though 20 points is same as the number of ways of selecting 2 points out of 20 points.

This can be done in ${}^{20}C_2$ ways.

Hence, total number of chords is ${}^{20}C_2$

$$= \frac{20!}{2!18!} = \frac{20 \times 19 \times \cancel{18!}}{2 \times \cancel{18!}} = \frac{\overset{10}{\cancel{20}} \times 19}{\cancel{2}}$$

$= 10 \times 19 = 190.$

(iv) **In a parking lot one hundred one year old cars, are parked. Out of them five are to be chosen at random for to check its pollution devices. How many different set of five cars can be chosen?**

5 cars can be chosen out of 100 cars in ${}^{100}C_5$ ways.

$$= \frac{100!}{5!(100-5)!} = \frac{\overset{5}{\cancel{100}} \times 99 \times 98 \times 97 \times 96 \times \cancel{95!}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{95!}}$$

$= 15057504$

(v) **How many ways can a team of 3 boys, 2 girls and 1 transgender be selected from 5 boys, 4 girls and 2 transgenders?**

Solution : 3 boys can be selected from 5 boys in ${}^5C_3$ ways 2 girls can be selected from 4 girls in ${}^4C_2$ ways and 1 transgender can be selected from 2 transgenders in ${}^2C_1$ ways.

$\therefore$ Total number of selections $= {}^5C_3 \times {}^4C_2 \times {}^2C_1$

$$= \frac{5!}{2!3!} \times \frac{4!}{2!2!} \times 2 = \frac{5 \times \overset{2}{\cancel{4}} \times \cancel{3!}}{\cancel{2} \times \cancel{3!}} \times \frac{\overset{2}{\cancel{4}} \times 3 \times \cancel{2!}}{\cancel{2!}\,\cancel{2!}} \times 2$$

$= 10 \times 6 \times 2 = 120.$

10. Find the total number of subsets of a set with [Hint: ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$]

(i) 4 elements **(ii) 5 elements**

(iii) n elements.

Solution :

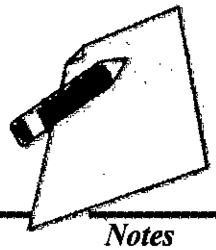
(i) **4 elements**

Number of subsets with no element $= {}^4C_0$

Number of subsets with 1 element $= {}^4C_1$

Number of subsets with 2 elements $= {}^4C_2$

Number of subsets with 3 elements $= {}^4C_3$



Number of subsets with 4 elements = ${}^4C_4$

$\therefore$ Total number of subsets

$= {}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$

$$\left[\begin{array}{l} \because nC_0 = 1,\ nC_n = 1 \\ nC_r = \dfrac{n(n-1)\ldots r \text{ elements}}{1.2.3\ldots r} \end{array}\right]$$

$$= 1 + 4 + \frac{\overset{2}{\cancel{4}} \times 3}{\cancel{2} \times 1} + \frac{4 \times \cancel{3} \times \cancel{2}}{\cancel{3} \times \cancel{2} \times 1} + 1$$

$= 1 + 4 + 6 + 4 + 1 = 16.$

(ii) 5 elements

Number of subsets with no element ${}^5C_0$

Number of subsets with 1, 2, 3, 4 and 5 elements are ${}^5C_0$, ${}^5C_1$, ${}^5C_2$, ${}^5C_3$, ${}^5C_4$ and ${}^5C_5$ respectively.

$\therefore$ Total number of subsets

$= {}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$$= 1 + 5 + \frac{5 \times \overset{2}{\cancel{4}}}{\cancel{2} \times 1} + \frac{5 \times \overset{2}{\cancel{4}}}{\cancel{2} \times 1} + 5 + 1.$$

$[\because {}^5C_4 = {}^5C_1 \ \&\ {}^5C_3 = {}^5C_2]$

$= 6 + 10 + 10 + 6 = 32.$

(iii) *n* elements.

Number of subsets with no element = ${}^nC_0$

Number of subsets with 1, 2, 3, 4, 5... n elements are ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$ respectively.

$\therefore$ Total number of subsets

$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$ = Sum of the co-efficient in the binomial expansion $(x + a)^n$

$= 2^n.$

11. A trust has 25 members.

(i) How many ways 3 officers can be selected?

(ii) In how many ways can a President, Vice President and a Secretary be selected?

Solution :

(i) How many ways 3 officers can be selected?

3 officers can be selected from 25 members in ${}^{25}C_3$ ways

(ii) In how many ways can a President, Vice President and a Secretary be selected?

A President can be selected from 25 members in 25 ways A Vice-President can be selected from the remaining 24 members in 24 ways and a Secretary can be selected in 23 ways.

$\therefore$ Total number of ways of selection ${}^{25}P_3$ ways.

1 CARTESIAN SYSTEM OF RECTANGULAR CO-ORDINATES

- Understand the concept of co-ordinates system.
- Understand the concept of ordinate plane.
- Discuss the concept of circle.
- Discuss the concept of parabola.
- Understand the practical application of co-ordinate geometry.
- Discuss the concept of circle.
- Understand the concept of conic sections.

Objective of the Module:

The basic objective of this chapter is to through some light on the initial concepts of co-ordinate geometry so that the practical application of co-ordinate geometry can be examined in detailed.

Introduction

Co-ordinate Geometry is a method of analysing geometrical shapes. It is one of the most scoring topics of the mathematics syllabus of IIT JEE and other engineering exams. Besides calculus, this is the only topic that can fetch your maximum marks. It is a vast topic and can further be divided into various parts like:

- Circle
- Parabola
- Ellipse
- Hyperbola
- Straight Lines

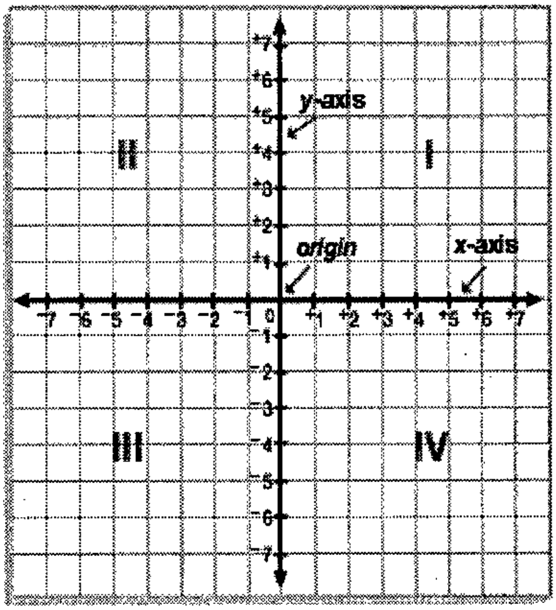
All these topics hold great importance from examination point of view but the Straight Line and the Circle are the most important. These topics together fetch maximum questions in the JEE and moreover they are a pre requisite to conic sections as well.

The Coordinate Plane

In Coordinate geometry, points are placed on the coordinate plane. The horizontal line is the x-axis while the vertical line is the y-axis. The point where they cross each other is called the origin.

A point's location on the plane is given by two numbers, the first tells where it is on the x-axis and the second tells where it is on the y-axis. Together, they define a single, unique position on the plane.

The figure given below illustrates the coordinate axis. The position of a point is given by an ordered pair in which the order is important as the first in the pair always stands for the x coordinate. Sometimes these are also referred to as the rectangular coordinates.



We have listed some of the important facts here, but the rest have been covered in detail in the later sections.

The Midpoint of a Line Joining Two Points

The midpoint of the line joining the points (x1, y1) and (x2, y2) is:
[½(x1 + x2), ½(y1 + y2)

Illustration:

Find the coordinates of the midpoint of the line joining (11, 2) and (3, 4).
Midpoint = [½(11+ 3), ½(2 + 4)] = (7, 3)

The Gradient of a Line Joining Two Points

The gradient of a line joining two points is given by
(y2-y1) / (x2-x1)

Parallel and Perpendicular Lines

Two parallel lines have the same gradient while if two lines are perpendicular then the product of their gradient is -1.

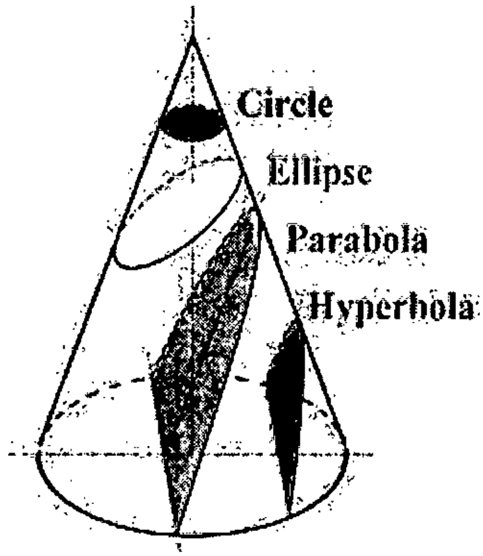
Example

a) $y = 4x + 1$

b) $y = -1/4\ x + 12$

c) $\frac{1}{2}y = 2x - 3$

The gradients of the lines are 4, -1/4 and 4 respectively. Hence, as stated above lines (a) and (b) are perpendicular, (b) and (c) are perpendicular and (a) and (c) are parallel.



Generally, conic section includes ellipse, parabola and hyperbola but sometimes circle is also included in conic sections. Circle can actually be considered as a type of ellipse. When a cone and a plane intersect and their intersection is in the form of a closed curve, it leads to the formation of circle and ellipse. As it is visible from the figure above, the circle is obtained when the cutting plane is parallel to the plane of generating circle of the cone. Similarly, in case the cutting plane is parallel to one generating line of the cone, the resulting conic is unbounded and is called parabola. The last case in which both the halves of the cone are intersected by the plane, produce two distinct unbounded curves called hyperbola.

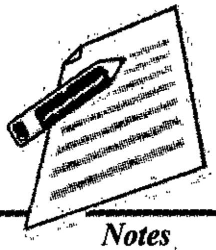
Conic section is a vital organ of coordinate geometry. It is easy to gain marks in this section as there are some standard questions asked from this section so they can be easily dealt with. Some topics no doubt are difficult but can be mastered with continuous practice. All these sections including circles, straight lines, ellipse, parabola and hyperbola have been discussed in detail in the coming sections.

Illustration:

If the circle $x2 + y2 + 2x + 2ky + 6 = 0$ and $x2 + y2 + 2ky + k = 0$ intersect orthogonally, then k is

(a) 2 or -3/2 (b) -2 or -3/2

(c) 2 or 3/2 (d) -2 or 3/2



Solution:

For the circles to intersect orthogonally, we must have

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$

Here, $g_1 = 1$, $g_2 = 0$, $f_1 = k$, $f_2 = k$, $c_1 = 6$, $c_2 = k$

Hence, we have the equation

$2(1)(0) + 2k.k = 6 + k$

This gives $2k^2 - k - 6 = 0$ which yields $k = -3/2, 2$.

Hence, the correct option is (a).

Illustration:

The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is… ?

Solution:

Tangent to the curve $y^2 = 8x$ is $y = mx + 2/m$.

This must satisfy $xy = -1$.

Hence, $x(mx + 2/m) = -1$

This gives $mx^2 + 2/m\, x + 1 = 0$.

Now, since it has equal roots so $D = 0$.

Hence, $4/m^2 - 4m = 0$

This gives $m^3 = 1$ which gives $m = 1$.

Hence, the equation of common tangent is $y = x + 2$.

Illustration:

If $P = (x, y)$, $F_1 = (3, 0)$, $F_2 = (-3, 0)$ and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals

(a) 8 (b) 6 (c) 10 (d) 12

Solution:

Given $16x^2 + 25y^2 = 400$

This can be written as $x^2/25 + y^2/16 = 1$

Hence, here we have $a^2 = 25$ and $b^2 = 16$.

But, $b^2 = a^2(1-e^2)$

$16 = 25(1-e^2)$

This gives $16/25 = (1-e^2)$

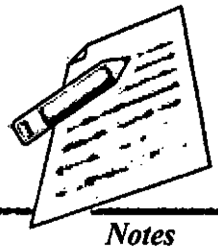
Hence, $e^2 = 9/25$.

The foci of the ellipse are $(\pm ae, 0)$.

$3 = a.3/5$

Hence, $a = 5$.

$PF_1 + PF_2 =$ major axis $= 2a = 10$.



Cartesian Coordinates

Cartesian coordinates can be used to pinpoint where we are on a map or graph. Using Cartesian Coordinates we mark a point on a graph by how far along and how far up it is:

The point (12,5) is 12 units along, and 5 units up. They are also called Rectangular Coordinates because it is like we are forming a rectangle.

X and Y Axis

The left-right (horizontal) direction is commonly called X.

The up-down (vertical) direction is commonly called Y.

Put them together on a graph ...

... and we are ready to go

Where they cross over is the "0" point,

we measure everything from there.

The X Axis runs horizontally through zero

The Y Axis runs vertically through zero Axis: The reference line from which distances are measured.

The plural of Axis is Axes, and is pronounced ax-eez

Example:

Point (6,4) is

6 units across (in the x direction), and

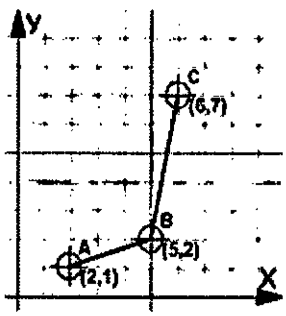
4 units up (in the y direction)

So (6,4) means:

Go along 6 and then go up 4 then "plot the dot".

And you can remember which axis is which by:

x is A CROSS, so x is ACROSS the page.



Play With It ! Now is a good time to play with Interactive Cartesian Coordinates to see for yourself how it all works.

Like 2 Number Lines Put Together

It is like we put two Number Lines together, one going left-right, and the other going down-up.

Direction

As x increases, the point moves further right.
When x decreases, the point moves further to the left.

As y increases, the point moves further up.
When y decreases, the point moves further down.

Writing Coordinates

The coordinates are always written in a certain order:

the horizontal distance first,

then the vertical distance.

This is called an "ordered pair" (a pair of numbers in a special order)

And usually the numbers are separated by a comma, and parentheses are put around the whole thing like this:

(3,2)

Example: (3,2) means 3 units to the right, and 2 units up

Example: (0,5) means 0 units to the right, and 5 units up.

In other words, only 5 units up.

The Origin

The point (0, 0) is given the special name "The Origin", and is sometimes given the letter "O".

Abscissa and Ordinate

You may hear the words "Abscissa" and "Ordinate" ... they are just the x and y values:

Abscissa: the horizontal ("x") value in a pair of coordinates: how far along the point is

Ordinate: the vertical ("y") value in a pair of coordinates: how far up or down the point is

"Cartesian" ... ?

They are called Cartesian because the idea was developed by the mathematician and philosopher Rene Descartes who was also known as Cartesius.

He is also famous for saying "I think, therefore I am".

What About Negative Values of X and Y?

Just like with the Number Line, we can also have negative values.

Negative: start at zero and head in the opposite direction:

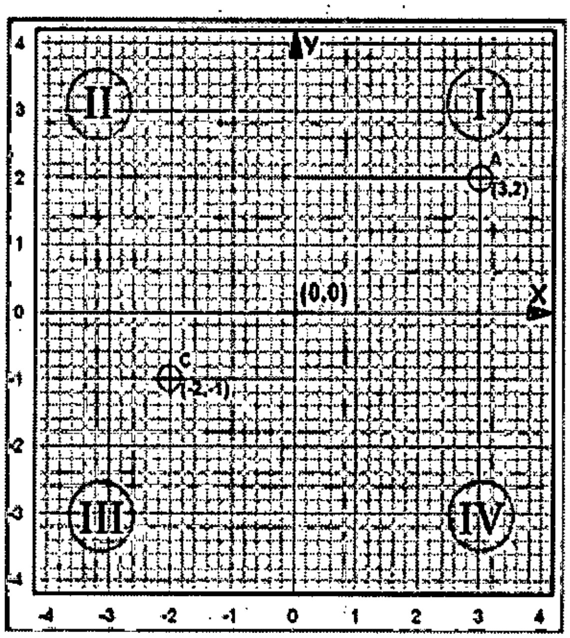
Negative x goes to the left

Negative y goes down

So, for a negative number:
go backwards for x
go down for y
For example (-6,4) means:
go back along the x axis 6 then go up 4.
And (-6,-4) means:
go back along the x axis 6 then go down 4.

Four Quadrants

When we include negative values, the x and y axes divide the space up into 4 pieces:
Quadrants I, II, III and IV
(They are numbered in a counter clockwise direction)
In Quadrant I both x and y are positive, but ...
in Quadrant II x is negative (y is still positive),
in Quadrant III both x and y are negative, and
in Quadrant IV x is positive again, while y is negative.
Like this:



Quadrant	X (horizontal)	Y (vertical)	Example
I	Positive	Positive	(3,2)
II	Negative	Positive	
III	Negative	Negative	(-2,-1)
IV	Positive	Negative	

Example: The point "A" (3,2) is 3 units along, and 2 units up.

Both x and y are positive, so that point is in "Quadrant I"

Example: The point "C" (-2,-1) is 2 units along in the negative direction, and 1 unit *down (i.e. negative direction).*

Both x and y are negative, so that point is in "Quadrant III"

Note: The word Quadrant comes form quad meaning four. For example, four babies born at one birth are called quadruplets, a four-legged animal is a quadruped. and a quadrilateral is a four-sided polygon.

Dimensions: 1, 2, 3 and more ...

Think about this:

1	The Number Line can only go: left-right so any position needs just one number
2	Cartesian coordinates can go: left-right, and up-down so any position needs two numbers
3	How do we locate a spot in the real world (such as the tip of your nose)? We need to know: left-right, up-down, and forward-backward, that is three numbers, or 3 dimensions!

3 Dimensions

Cartesian coordinates can be used for locating points in 3 dimensions as in this example:

Here the point (2, 4, 5) is shown in
three-dimensional Cartesian coordinates.

In fact, this idea can be continued into four dimensions and more - I just can't work out how to illustrate that for you!

2 STRAIGHT LINES

Straight line

Any first degree equation in two variables x and y of the form $ax + by + c = 0$ ...(1)

where a, b, c are real numbers and at least one of a, b is non-zero is called "Straight line" in xy plane.

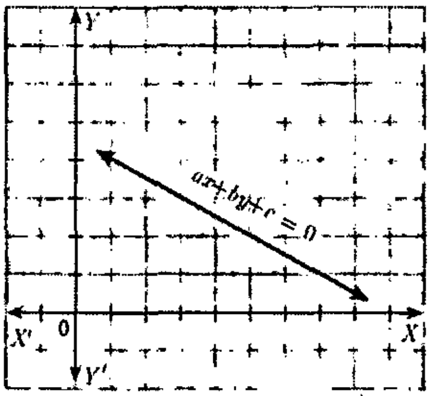


Fig. 5.26

1. Equation of coordinate axes

The X axis and Y axis together are called coordinate axes. The x coordinate of every point on OY (Y axis) is 0. Therefore equation of OY (Y axis) is $x = 0$ (fig 5.27)

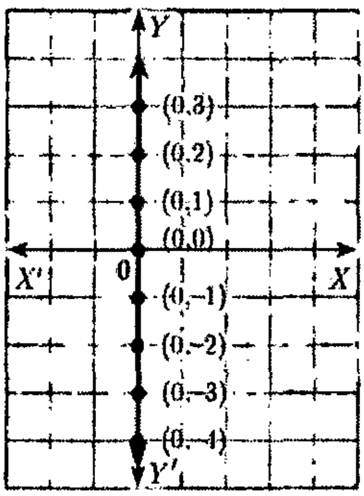


Fig. 5.27

The y coordinate of every point on OX (X axis) is 0. Therefore the equation of OX (X axis) is y = 0 (fig 5.28)

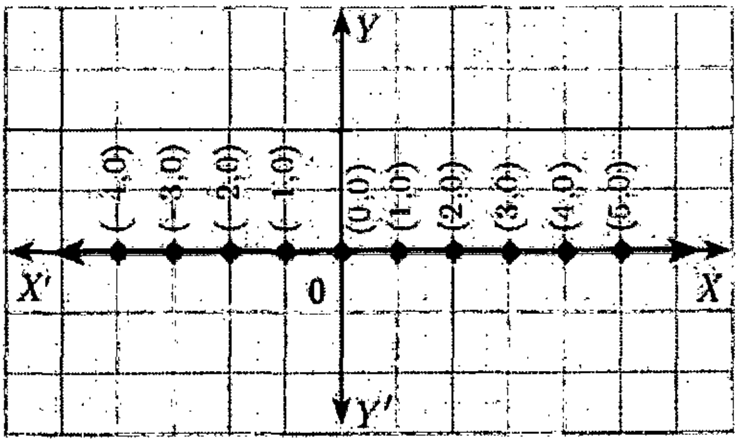


Fig. 5.28

2. Equation of a straight line parallel to X axis

Let AB be a straight line parallel to X axis, which is at a distance 'b'. Then y coordinate of every point on 'AB' is 'b'. (fig 5.29)

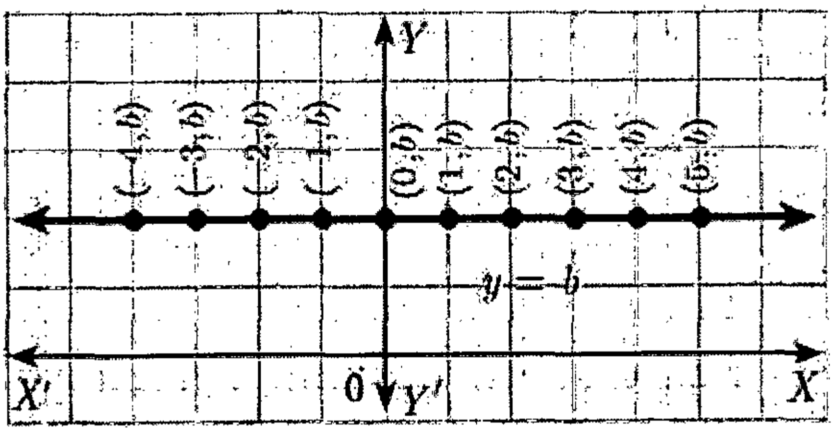


Fig. 5.29

Therefore, the equation of AB is y = b

Note

- If b > 0 , then the line y=b lies above the X axis
- If b < 0 , then the line y=b lies below the X axis
- If b = 0 , then the line y=b is the X axis itself.

3. Equation of a Straight line parallel to the Y axis

Let CD be a straight line parallel to Y axis, which is at a distance 'c'. Then x coordinate of every point on CD is 'c'. The equation of CD is x = c. (fig 5.30)

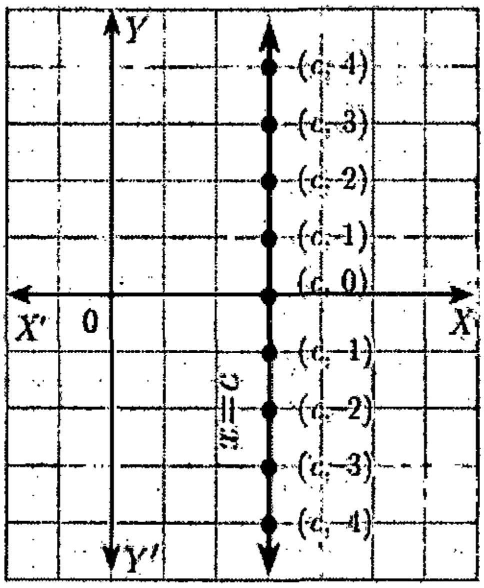


Fig. 5.30

Note

- If $c>0$, then the line $x=c$ lies right to the side of the Y axis
- If $c < 0$, then the line $x=c$ lies left to the side of the Y axis
- If $c=0$, then the line $x=c$ is the Y axis itself.

Example 1

Find the equation of a straight line passing through (5,7) and is (i) parallel to X axis (ii) parallel to Y axis.

Solution

(i) The equation of any straight line parallel to X axis is $y=b$.

Since it passes through (5,7), $b = 7$.

Therefore, the required equation of the line is $y=7$.

(ii) The equation of any straight line parallel to Y axis is $x=c$

Since it passes through (5,7), $c = 5$

Therefore, the required equation of the line is $x=5$.

4. Slope-Intercept Form

Every straight line that is not vertical will cut the Y axis at a single point. The coordinate of this point is called y intercept of the line.

A line with slope m and y intercept c can be expressed through the equation $y=mx+c$

We call this equation as the slope-intercept form of the equation of a line.

Note

- If a line with slope m, m`0 makes x intercept d, then the equation of the straight line is y = m (x d).

- y = mx represent equation of a straight line with slope m and passing through the origin.

Example 2

Find the equation of a straight line whose

(i) Slope is 5 and y intercept is -9

(ii) Inclination is 45° and y intercept is 11

Solution

(i) Given, Slope = 5, y intercept, c = −9

Therefore, equation of a straight line is y = mx +c

y = 5x − 9 gives 5x − y − 9 = 0

(ii) Given, θ = 45° , y intercept, c = 11

Slope m = tan θ = tan 45° = 1

Therefore, equation of a straight line is of the form y = mx +c

Hence we get, y= x + 11 gives x − y + 11 = 0

Example 3

Calculate the slope and y intercept of the straight line 8x − 7y + 6 = 0

Solution

Equation of the given straight line is 8x − 7y + 6 = 0

7y = 8x + 6 (bringing it to the form y = mx +c)

y = 8/7 x + 6/7 ...(1)

Comparing (1) with y = mx +c

Slope m = 8/7 and y intercept c = 6/7

Example 4

The graph relates temperatures y (in Fahrenheit degree) to temperatures x (in Celsius degree) (a) Find the slope and y intercept (b) Write an equation of the line What is the mean temperature of the earth in Fahrenheit degree if its mean temperature is 25° Celsius?

Solution

(a) From the figure, slope =

$$\frac{\text{change in } y \text{ coordinate}}{\text{change is } x \text{ coordinate}} = \frac{68-32}{20-0} = \frac{36}{20} = \frac{9}{5} = 1.8$$

The line crosses the Y axis at (0, 32) So the slope is 9/5 and y intercept is 32.

(b) Use the slope and y intercept to write an equation

The equation is y = 9/5 x + 32

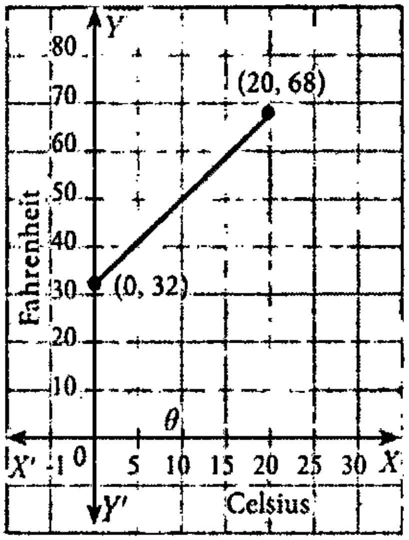


Fig. 5.31

(c) In Celsius, the mean temperature of the earth is 25°. To find the mean temperature in Fahrenheit, we find the value of y when x = 25

y = 9/5 x + 32

y = 9/5 (25)+32

y = 77

Therefore, the mean temperature of the earth is 77° F.

Note

The formula for converting Celsius to Fahrenheit is given by F = 9/5 C + 32. which is the linear equation representing a straight line derived in the example.

5. Point-Slope form

Here we will find the equation of a straight line passing through a given point A(x1 , y1) and having the slope m.

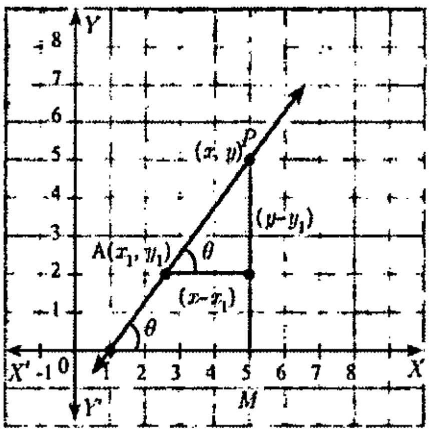


Fig. 5.32

Let P (x, y) be any point other than A on the given line. Slope of the line joining A(x1, y1) and P (x, y) is given by

$$m = \tan\theta = \frac{y - y_1}{x - x_1}$$

Therefore, the equation of the required line is y – y1 = m (x – x1) (Point slope form)

Example 5

Find the equation of a line passing through the point (3, - 4) and having slope -5/7

Solution

Given, (x1 ,y1) = (3, −4) and m = −5/7
The equation of the point-slope form of the straight line is y – y1 = m (x – x1)
we write it as y + 4 = – 5/7 (x-3)
gives us 5x + 7y + 13 = 0

Example 6

Find the equation of a line passing through the point A(1, 4) and perpendicular to the line joining points (2, 5) and (4, 7) .

Solution

Let the given points be A(1, 4) , B(2, 5) and C(4, 7).
Slope of line BC = (7–5)/(4-2) = 2/2 = 1
Let m be the slope of the required line.

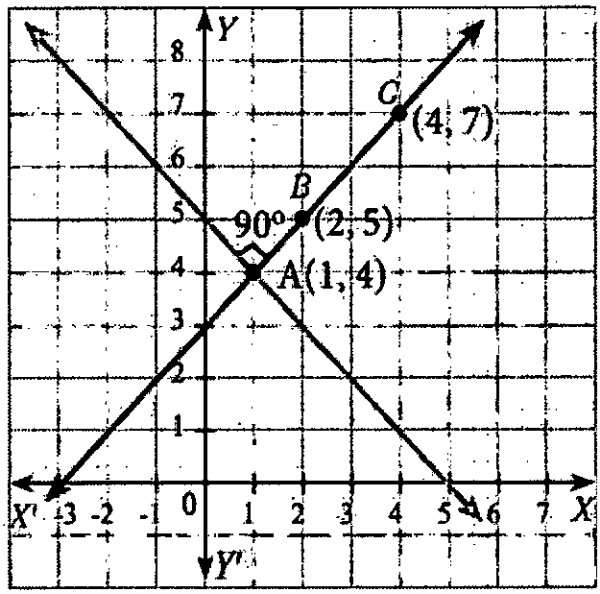


Fig. 5.33

Since the required line is perpendicular to BC,

$m \times 1 = -1$

$m = -1$

The required line also pass through the point A(1,4).

The equation of the required straight line is $y - y1 = m(x - x1)$

$y- 4 = -1(x - 1)$

$y- 4 = -x + 1$

we get, $x + y - 5 = 0$

6. Two Point form

Let A(x1 , y1) and B (x2 , y2) be two given distinct points. Slope of the straight line passing through these points is given by $m = \frac{y_2 - y_1}{x_2 - x_1}, \ (x_2 \neq x_1)$.

From the equation of the straight line in point slope form, we get

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Hence, $$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

(is the equation of the line in two-point form)

Example 7

Find the equation of a straight line passing through (5, - 3) and (7, - 4) .

Solution

The equation of a straight line passing through the two points (x 1 , y1) and (x 2 , y2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Substituting the points we get,

$$\frac{y + 3}{-4 + 3} = \frac{x - 5}{7 - 5}$$

Gives $6y + 2 = - x + 5$

Therefore, $x + 2y + 1 = 0$

Example 8

Two buildings of different heights are located at opposite sides of each other. If a heavy rod is attached joining the terrace of the buildings from (6, 10) to (14, 12) , find the equation of the rod joining the buildings ?

Solution

Let A(6, 10) , B(14, 12) be the points denoting the terrace of the buildings.

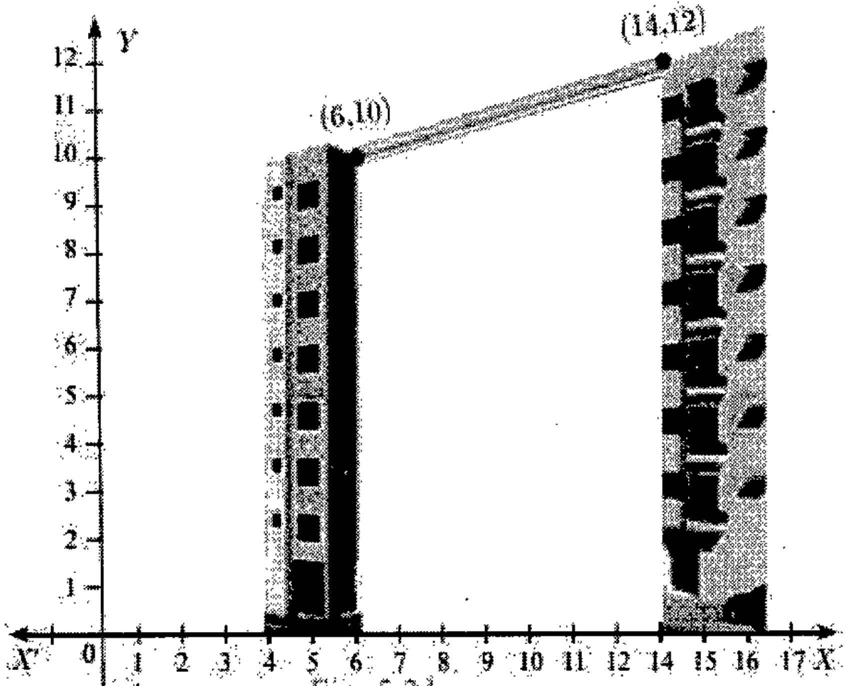


Fig. 5.34

The equation of the rod is the equation of the straight line passing through A(6,10) and B(14,12)

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \quad \text{gives} \quad \frac{y-10}{12-10} = \frac{x-6}{14-6}$$

$$\frac{y-10}{2} = \frac{x-6}{8}$$

Therefore, $x - 4y + 34 = 0$

Hence, equation of the rod is $x - 4y + 34 = 0$

7. Intercept Form

We will find the equation of a line whose intercepts are a and b on the coordinate axes respectively.

Let PQ be a line meeting X axis at A and Y axis at B . Let OA=a, OB=b. Then the coordinates of A and B are (a , 0) and (0,b) respectively. Therefore, the equation of the line joining A and B is

$$\frac{y-0}{b-0} = \frac{x-a}{0-a} \quad \text{we get,} \quad \frac{y}{b} = \frac{x-a}{-a} \quad \text{gives} \quad \frac{y}{b} = \frac{-x}{a} + 1$$

Hence, $\frac{x}{a}+\frac{y}{b}=1$ (Intercept form of a line)

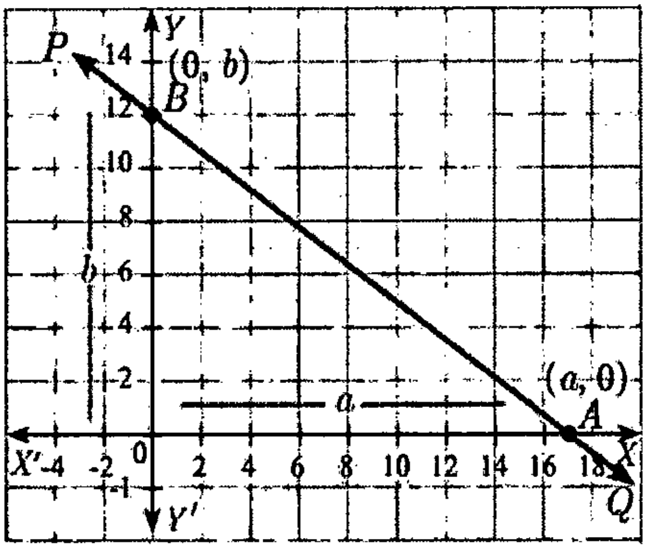


Fig. 5.35

Example 9

Find the equation of a line which passes through (5,7) and makes intercepts on the axes equal in magnitude but opposite in sign.

Solution

Let the x intercept be ‘a’ and y intercept be ‘–a’.

The equation of the line in intercept form is $\frac{x}{a}+\frac{y}{b}=1$

gives $\frac{x}{a}+\frac{y}{-a}=1$ (Here $b=-a$)

Therefore, $x-y=a$ ...(1)

Since (1) passes through (5,7)

Therefore, $5-7=a$ gives $a=-2$

Thus the required equation of the straight line is $x-y=-2$; or $x-y+2=0$

Example 10

Find the intercepts made by the line $4x-9y+36=0$ on the coordinate axes.

Solution

Equation of the given line is $4x-9y+36=0$

we write it as $4x-9y=-36$ (bringing it to the normal form)

$$\frac{x}{-9}+\frac{y}{4}=1$$

Dividing by -36 we get, ...(1)

Comparing (1) with intercept form, we get x intercept a = – 9 ; y intercept b = 4

Example 11

A mobile phone is put to use when the battery power is 100%. The percent of battery power 'y' (in decimal) remaining after using the mobile phone for x hours is assumed as y = –0. 25x + 1

(i) Draw a graph of the equation.

(ii) Find the number of hours elapsed if the battery power is 40%.

(iii) How much time does it take so that the battery has no power?

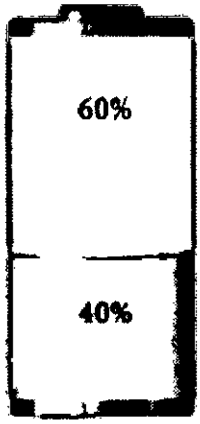
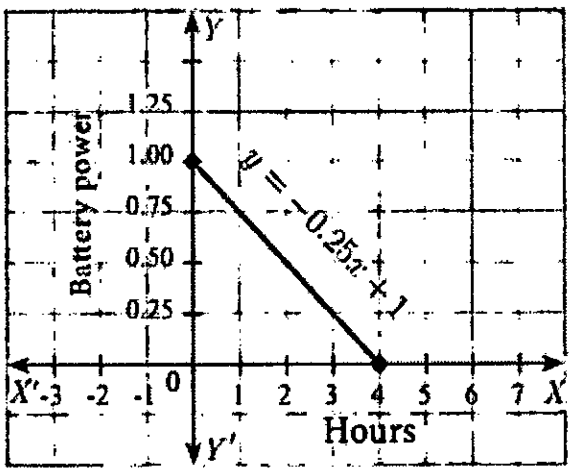


Fig. 5.36

Solution



(i)

(ii) To find the time when the battery power is 40%, we have to take y = 0. 40

0.40 = –0. 25x + 1 gives 0. 25x = 0. 60

we get, x = 0.60/0.25 = 2.4 hours.

(iii) If the battery power is 0 then y = 0

Therefore, 0 = –0.25x + 1 gives 0.25x = 1 hence x = 4 hours.

Thus, after 4 hours, the battery of the mobile phone will have no power.

Example 12

A line makes positive intercepts on coordinate axes whose sum is 7 and it passes through (-3, 8) . Find its equation.

Solution

If a and b are the intercepts then a + b = 7 or b = 7 –a

By intercept form x/a + y/b = 1 ...(1)

We have x/a + y/(7–a) = 1

As this line pass through the point (-3, 8) , we have

–3/a + 8/(7 – a) = 1 gives –3(7–a) + 8a = a(7–a)

– 21 + 3a + 8a = 7a –a2

So, a 2 + 4a – 21 = 0

Solving this equation (a – 3)(a + 7) = 0

a = 3 or a = –7

Since a is positive, we have a = 3 and b = 7–a = 7–3 =4.

Hence x/3 + y/4 = 1

Therefore, 4x + 3y –12 = 0 is the required equation.

Example 13

A circular garden is bounded by East Avenue and Cross Road. Cross Road intersects North Street at D and East Avenue at E. AD is tangential to the circular garden at A(3, 10). Using the figure.

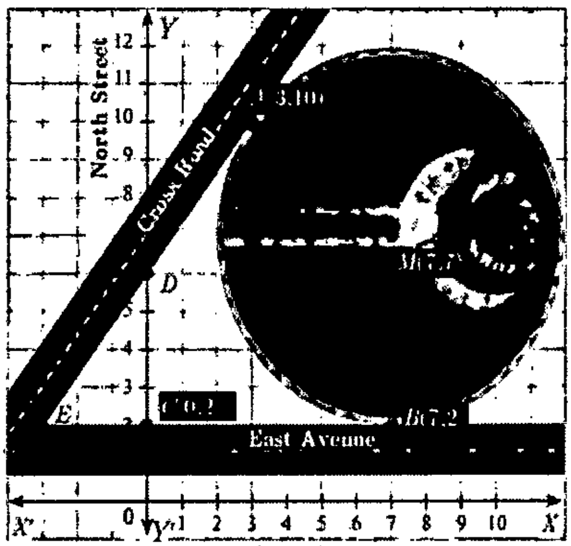


Fig. 5.37

(a) Find the equation of

(i) East Avenue.

(ii) North Street

(iii) Cross Road

(b) Where does the Cross Road intersect the

(i) East Avenue ?

(ii) North Street ?

Solution

(a) (i) East Avenue is the straight line joining C(0, 2) and B(7, 2) . Thus the equation of East Avenue is obtained by using two-point form which is

$$\frac{y-2}{2-2}=\frac{x-0}{7-0}$$

$$\frac{y-2}{0}=\frac{x}{7} \text{ gives } y=2$$

(ii) Since the point D lie vertically above C(0, 2) . The x coordinate of D is 0.

Since any point on North Street has x coordinate value 0.

The equation of North Street is x = 0

(iii) To find equation of Cross Road.

Center of circular garden M is at (7, 7), A is (3, 10) We

first find slope of MA, which we call m1

Thus m1 = (10 −7) / (3-7) = −3/4.

Since the Cross Road is perpendicular to MA, if m2 is the slope of the Cross Road then, m1m2 = −1 gives −3/4 m2 = −1 so m2 = 4/3.

Now, the cross road has slope 4/3 and it passes through the point A(3, 10).

The equation of the Cross Road is y − 10 = 3/4 (x − 3)

3y − 30 = 4x −12

Hence, 4x − 3y + 18 = 0

(b) (i) If D is (0, k) then D is a point on the Cross Road.

Therefore, substituting x = 0 , y = k in the equation of Cross Road,

we get, 0 − 3k + 18 = 0

Value of k = 6

Therefore, D is (0, 6).

(ii) To find E, let E be (q , 2)

Put y = 2 in the equation of the Cross Road,

we get, 4q − 6 + 18 = 0

4q = −12 gives q = −3

Therefore, The point E is (-3, 2)

Thus the Cross Road meets the North Street at D(0, 6) and East Avenue at E (-3, 2) .

3 CONIC SECTIONS

Conic Sections

In addition to the method to determine the curves discussed in Previous Section, geometric description of a conic section is given here. The graph of a circle, an ellipse, a parabola, or a hyperbola can be obtained by the intersection of a plane and a double napped cone. Hence, these figures are referred to as conic sections or simply conics.

1. Geometric description of conic section

A plane perpendicular to the axis of the cone (plane C) intersecting any one nape of the double napped cone yields a circle (Fig. 5.40) . The plane E , tilted so that it is not perpendicular to the axis, intersecting any one nape of the double napped cone yields an ellipse (Fig. 5.40). When the plane is parallel to a side of one napes of the double napped cone, the plane intersecting the cone yields a parabola (Fig. 5.41). When the plane is parallel to the plane containing the axis of the double cone, intersecting the double cone yields a hyperbola (Fig. 5.42).

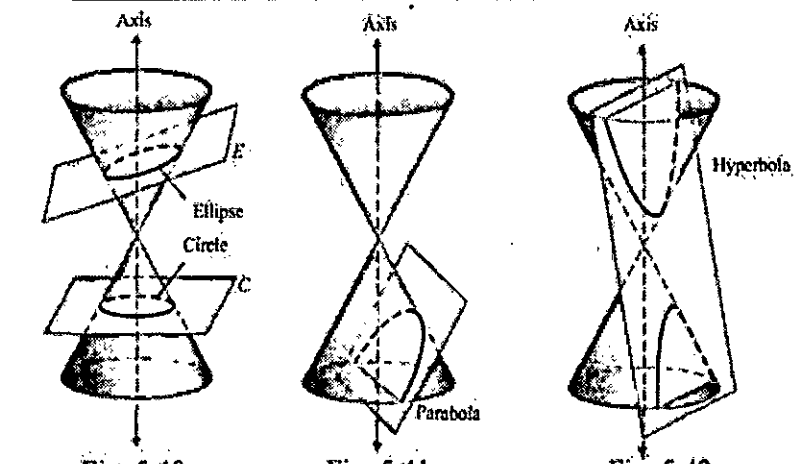


Fig. 5.40 Fig. 5.41 Fig. 5.42

2. Degenerate Forms

Degenerate forms of various conics (Fig. 5.43) are either a point or a line or a pair of straight lines or two intersecting lines or empty set depending on the angle (nature) of intersection of the plane with the double napped cone and passing through the vertex or when the cones degenerate into a cylinder with the plane parallel to the axis of the cylinder.

If the intersecting plane passes through the vertex of the double napped cone and perpendicular to the axis, then we obtain a point or a point circle. If the intersecting plane passes through a generator then we obtain a line or a pair of parallel lines, a degenerate form of a parabola for which A = B = C = 0 in general equation of a conic and if the intersecting plane passes through the axis and passes through the vertex of the double napped cone, then we obtain intersecting lines a degenerate of the hyperbola.

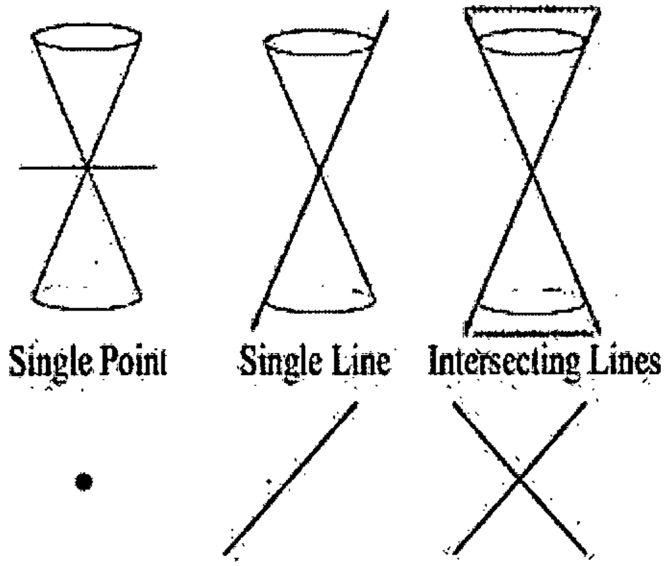


Fig. 5.43

Remark

In the case of an ellipse ($0 < e < 1$) where $e = \sqrt{1-\frac{b^2}{a^2}}$. As $e \to 0$, $b/a \to 1$ i.e., $b \to a$ or the lengths of the minor and major axes are close in size. i.e., the ellipse is close to being a circle. As $e \to 1$, $b/a \to 0$ and the ellipse degenerates into a line segment i.e., the ellipse is flat.

Remark

In the case of a hyperbola ($e > 1$) where $e = \sqrt{1+\frac{b^2}{a^2}}$. As $e \to 1$, $b/a \to 0$ i.e., as $e \to 1$, b is very small related to a and the hyperbola becomes a pointed nose. As $e \to \infty$, b is very large related to a and the hyperbola becomes flat.

3. Identifying the conics from the general equation of the conic Ax2 + Bxy + Cy2 + Dx + Ey + F = 0 .

The graph of the second degree equation is one of a circle, parabola, an ellipse, a hyperbola, a point, an empty set, a single line or a pair of lines. When,

(1) $A = C = 1, B = 0, D = -2h, E = -2k, F = h2 + k\,2 - r2$ the general equation reduces to$(x - h)2 + (y - k)2 = r2$, which is a circle.

(2) $B = 0$ and either A or $C = 0$, the general equation yields a parabola under study, at this level.

(3) $A \neq C$ and A and C are of the same sign, the general equation yields an ellipse.

(4) $A \neq C$ and A and C are of opposite signs, the general equation yields a hyperbola

(5) $A = C$ and $B = D = E = F = 0$, the general equation yields a point $x2 + y2 = 0$.

(6) $A = C = F$ and $B = D = E = 0$, the general equation yields an empty set $x2 + y2 + 1 = 0$, as there is no real solution.

(7) $A \neq 0$ or $C \neq 0$ and others are zeros, the general equation yield coordinate axes.

(8) A=-C and rests are zero, the general equation yields a pair of lines $x2 - y2 = 0$.

Example 1

Identify the type of the conic for the following equations:

(1) $16\,y2 = -4x2 + 64$

(2) $x2 + y2 = -4x - y + 4$

(3) $x2 - 2\,y = x + 3$

(4) $4x2 - 9\,y2 - 16x + 18\,y - 29 = 0$

Solution

Q.no.	Equation	condition	Type of the conic
1	$16y^2 = -4x^2 + 64$	3	Ellipse
2	$x^2 + y^2 = -4x - y + 4$	1	Circle
3	$x^2 - 2y = x + 3$	2	parabola
4	$4x^2 - 9y^2 - 16x + 18y - 29 = 0$	4	Hyperbola

Summary of the Module

Co-ordinate Geometry is a method of analysing geometrical shapes. It is one of the most scoring topics of the mathematics syllabus of IIT JEE and other engineering exams. Besides calculus, this is the only topic that can fetch your maximum marks. It is a vast topic and can further be divided into various parts like: Circle, Parabola, Ellipse, Hyperbola, Straight Lines. All these topics hold great importance from examination point of view but the Straight Line and the Circle are the most important. These topics together fetch maximum questions in the JEE and moreover they are a pre requisite to conic sections as well. Conic section is a vital organ of coordinate geometry. It is easy to gain marks in this section as there are some standard questions asked from this section so they can be easily dealt with. Some topics no doubt are difficult but can be mastered with continuous practice.

Review Questions

Exercise 5.3

1. Find the equation of a straight line passing through the mid-point of a line segment joining the points (1, - 5) , (4,2) and parallel to

(i) X axis

(ii) Y axis

(i) X axis (ii) Y axis

Solution:

i) Mid point of the line segment, the end points are (1, -5), (4, 2)

$$\left(\frac{1+4}{2},\frac{-5+2}{2}\right)=\left(\frac{5}{2},\frac{-3}{2}\right)$$

The required straight line parallel to x-axis, then $\theta = 0$, so the slope $m = \tan 0^{0} = 0$

The required straight line equation is

$$y-y_1 = m\,(x-x_1)$$

$$y+\frac{3}{2} = 0\,(x-\frac{5}{2})$$

$$\frac{2y+3}{2} = 0$$

$$\boxed{2y+3=0}$$

$$2y = -3$$

$$y = \frac{-3}{2}$$

The required straight line equation is $2y+3=0$

ii) y – axis

The required line is parallel to y-axis $\theta = 90^{0}$

$$m = \tan\theta = \tan 90^{0} = \text{undefined} = \frac{1}{0}$$

The required straight line equation

$$y-y_1 = m(x-x_1)$$

$$y-\frac{3}{2} = \frac{1}{0}\left(x-\frac{5}{2}\right)$$

$$0\left(y-\frac{3}{2}\right) = x-\frac{5}{2}$$

$$0 = \frac{2x-5}{2}$$

$$0 = 2x-5$$

The required straight line equation is $2x-5=0$.

2. The equation of a straight line is $2(x-y)+5=0$. Find its slope, inclination and intercept on the Y axis.

Solution:

Given straight line is $2(x-y)+5=0$

$$2x-2y+5 = 0$$

$$2y = 2x+5$$

$$y = \frac{2x}{2}+\frac{5}{2}$$

$$y = x+\frac{5}{2}$$

Which is in the form of $y = mx+c$

here $\boxed{m=1}$

$m = \tan\theta$

$\Rightarrow \tan\theta = 1 = \tan 45^\circ$

$\boxed{\theta = 45^\circ}$

y – intercept $= \frac{5}{2}$

Ans: slope $m = 1$; inclination $= 45^\circ$

intercept on the y-axis $= \frac{5}{2}$

3. Find the equation of a line whose inclination is 30° and making an intercept -3 on the Y axis.

Solution:

Given inclination $\theta = 30^{\circ}$

Slope $m = \tan\theta = \tan 30^{\circ}$

$$\boxed{m = \frac{1}{\sqrt{3}}}$$

Here intercept on the y-axis c = -3

The equation of straight line y = mx+c

$$y = \frac{1}{\sqrt{3}}x + (-3)$$

$$y = \frac{x - 3\sqrt{3}}{\sqrt{3}}$$

$$\sqrt{3}\,y = x - 3\sqrt{3}$$

$$x - 3\sqrt{3} - \sqrt{3}\,y = 0$$

$$\Rightarrow x - \sqrt{3}y - 3\sqrt{3} = 0$$

4. Find the slope and y intercept of $\sqrt{3}x + (1-\sqrt{3})y = 3$.

Solution:

Given straight line is $\sqrt{3}x + (1-\sqrt{3})y = 3$

$(1-\sqrt{3})y = 3 - \sqrt{3}x = -\sqrt{3}x + 3$

$$y = \frac{-\sqrt{3}}{1-\sqrt{3}}x + \frac{3}{1-\sqrt{3}}$$

Rationalization,

$$y = \frac{-\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}x + \frac{3}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{-\sqrt{3}(1+\sqrt{3})}{1-3}x + \frac{3(1+\sqrt{3})}{1-3}$$

$$= \frac{-\sqrt{3}-3}{-2}x + \frac{3+3\sqrt{3}}{-2}$$

$$= \frac{\sqrt{3}+3}{2}x + \left(\frac{3+3\sqrt{3}}{-2}\right)$$

Here $m = \dfrac{\sqrt{3}+3}{2}$; $c = \dfrac{3+3\sqrt{3}}{-2}$

5. Find the value of 'a', if the line through (–2,3) and (8,5) is perpendicular to y = ax + 2

Solution:

Here the two points (–2, 3) & (8, 5)

the slope of these two points is

$$m_1 = \frac{5-3}{8+2} = \frac{2}{10} = \frac{1}{5}$$

Given $y = ax+2$

$m_2 = a$

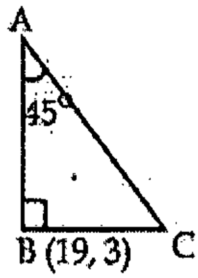
Two straight lines are perpendicular

So, $m_1 \times m_2 = -1$

$$\frac{1}{5} \times a = -1 \Rightarrow \boxed{a = -5}$$

6. The hill in the form of a right triangle has its foot at (19, 3) . The inclination of the hill to the ground is 45°. Find the equation of the hill joining the foot and top.

Solution:



Here $\theta = 45^{\circ}$, slope m = 1

The point is (19, 3)

The required equation of the straight line is,

$$y - y_1 = m(x - x_1)$$

$$y - 3 = (1)(x - 19)$$

$$x - 19 - y + 3 = 0$$

$$x - y - 16 = 0$$

7. Find the equation of a line through the given pair of points
(i) (2,2/3) and (-1/2,-2) (ii) (2, 3) and (- 7, -1)

(i) $\left(2, \frac{2}{3}\right) and \left(\frac{-1}{2}, -2\right)$.

Solution:

Two point form $\Rightarrow \frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

The given two points are $\left(2, \frac{2}{3}\right)$ & $\left(\frac{-1}{2}, -22\right)$

The equation of the straight line

$$\frac{y-\frac{2}{3}}{-2-\frac{2}{3}} = \frac{x-2}{\frac{-1}{2}-2}$$

$$\frac{\frac{3y-2}{3}}{\frac{-6-2}{3}} = \frac{x-2}{\frac{-1-4}{2}}$$

$$\frac{3y-2}{3} \times \frac{3}{(-8)} = \frac{x-2}{1} \times \frac{2}{(-5)}$$

$$\frac{3y-2}{-8} = \frac{2(x-2)}{-5}$$

$$-5(3y-2) = -2\times 8\ (x-2)$$

$$-15y+10 = -16(x-2)$$

$$-15y+10 = -16x+32$$

$$16x - 32 - 15y + 10 = 0$$

$$16x - 15y - 22 = 0$$

the equation of the line is $16x - 15y - 22 = 0$

(ii) (2, 3) and (-7,-1)

Solution:

The equation of the straight line

$$\frac{y-3}{-1-3} = \frac{x-2}{-7-2}$$

$$\frac{y-3}{-4} = \frac{x-2}{-9}$$

$$-9(y-3) = -4(x-2)$$

$$-9y+27 = -4x+8$$

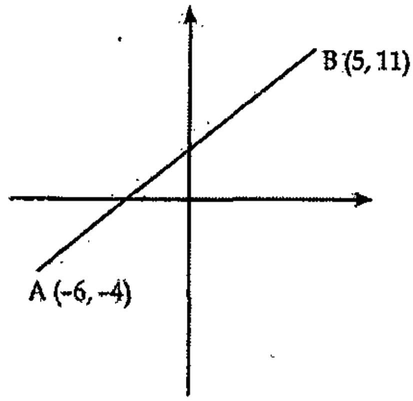
$$4x - 8 - 9y + 27 = 0$$

$$4x - 9y + 19 = 0$$

The equation of a line is $4x-9y+19 = 0$

8. A cat is located at the point(- 6, - 4) in xy plane. A bottle of milk is kept at (5, 11). The cat wish to consume the milk travelling through shortest possible distance. Find the equation of the path it needs to take its milk.

Solution:



Let the two points be A(−6, −4), B(5, 11)

The equation of the straight line

$$\frac{y+4}{11+4} = \frac{x+6}{5+6}$$

$$\frac{y+4}{15} = \frac{x+6}{11}$$

$$11(y+4) = 15(x+6)$$

$$11y+44 = 15x+90$$

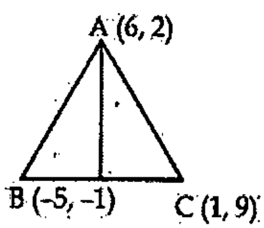
$$15x + 90 - 11y + 44 = 0$$

$$15x-11y+46 = 0$$

The equation of the path cat need to take its milk is $15x-11y+46 = 0$

9. Find the equation of the median and altitude of ΔABC through A where the vertices are A(6, 2) , B(- 5, -1) and C(1, 9)

Solution:



In triangle ΔABC, AD is the median.

Here the midpoint of BC is

$$D = \left(\frac{-5+1}{2}, \frac{-1+9}{2}\right) = \left(\frac{-4}{2}, \frac{8}{2}\right) = (-2, 4)$$

The equation of median AD is

$$\frac{y-2}{4-2} = \frac{x-6}{-2-6}$$

$$\frac{y-2}{2} = \frac{x-6}{-8}$$

$$-8(y-2) = 2(x-6)$$

$$-8y+16 = 2x-12$$

$$2x - 12 + 8y - 16 = 0$$

$$2x+8y-28 = 0$$

$\div$by 2 $x+4y-14 = 0$

The median of the ΔABC is $x+4y-14 = 0$

In Δ ABC, AE is the altitude

Slope of BC, B(–5, –1), C(1, 9)

$$\Rightarrow \frac{9+1}{1+5} = \frac{10}{6} = \frac{5}{3}$$

$m_1 = \frac{5}{3}$, Here BC $\perp^r$ AE

The slope of AE is

$$m_1 \times m_2 = -1$$

$$\frac{5}{3} \times m_2 = -1$$

$$\boxed{m_2 = \frac{-3}{5}}$$

Now, the equation of AE is

slope —, point A(6, 2)

$$y-y_1 = m(x-x_1)$$

$$y-2 = \frac{-3}{5}(x-6)$$

$$5(y-2) = -3(x-6)$$

$$5y-10 = -3x+18$$

$$3x - 18 + 5y - 10 = 0$$

$$3x+5y-28 = 0$$

The altitude of the ΔABC is $3x+5y-28 = 0$

10. Find the equation of a straight line which has slope -5/4 and passing through the point (–1,2).

Solution:

Given slope $m = \frac{5}{-4}$, point (–1, 2)

The equation of straight line is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{5}{-4}(x - 1)$$

$$-4(y - 2) = 5(x + 1)$$

$$-4y + 8 = 5x + 5$$

$$5x + 5 + 4y - 8 = 0$$

$$5x + 4y - 3 = 0$$

The required equation of a straight line:

$5x + 4y - 3 = 0$

11. You are downloading a song. The percent y (in decimal form) of mega bytes remaining to get downloaded in x seconds is given by $y = -0.1x + 1$.

(i) graph the equation

(ii) find the total MB of the song.

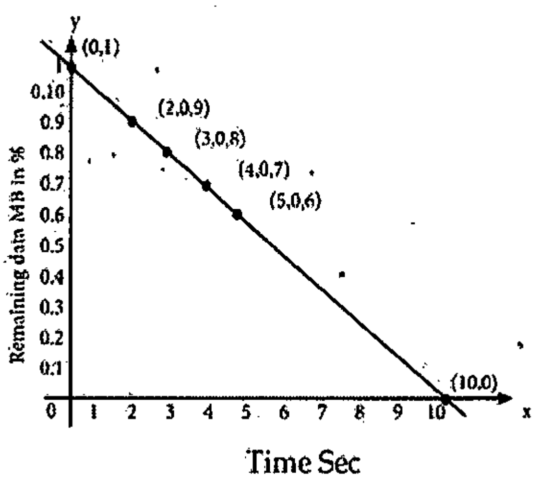
(iii) after how many seconds will 75% of the song gets downloaded?

(iv) after how many seconds the song will be downloaded completely?

Solution:

i) Given equation is $y = -0.1x + 1$ -----(1)

x	0	1	2	3	4	10
y	1	0.9	0.8	0.7	0.6	0



ii) Find the total MB of the song.

The given equation is $y = -0.1x+1$

The downloaded time is zero (i.e) $x = 0$

$\Rightarrow \boxed{y = 1}$

iii) Find after how many seconds is 75% of the song gets downloaded.

Here, Given y = 75% = 0.75 substitute in equation (1)

We get,

$$0.75 = -0.1x+1$$
$$0.1x = 1-0.75$$
$$x = \frac{0.25}{0.1} = 2.5$$
$$\boxed{x = 2.5} \text{ seconds}$$

iv) Find after how many seconds the song will be downloaded completely.

Here y = 0 substitute in equation (1)

We get,

$$0 = -0.1x+1$$
$$0.1x = 1$$
$$x = \frac{1}{0.1}$$
$$\boxed{x = 10} \text{ seconds.}$$

12. Find the equation of a line whose intercepts on the x and y axes are given below.

(i) 4, –6

(ii) -5, 3/4

Solution:

Given x-intercept = a = 4

y- intercept = b = –6

in the intercept form

The eqn of straight line $\frac{x}{a}+\frac{y}{b}=1$

$$\frac{x}{4}+\frac{y}{-6} = 1$$

$$\frac{x}{4}-\frac{y}{6} = 1$$

$$\frac{3x-2y}{12} = 1$$

$$3x-2y = 12$$

The required sraight line eqn. is $3x-2y = 12$

(ii) $-5, \frac{3}{4}$

Solution:

Given x-intercept = a = -5

y- intercept = b = $\frac{3}{4}$ in the intercept form

The equation of straight line $\frac{x}{a}+\frac{y}{b}=1$

$$\frac{x}{-5}+\frac{y}{(3/4)} = 1$$

$$\frac{}{5}\ \frac{}{3} = 1$$

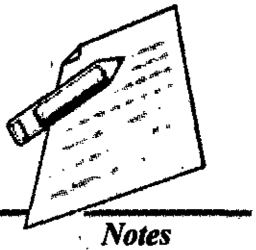
$$\frac{-3x+20y}{15} = 1$$

$$-3x+20y = 15$$

$\times$ by $- \Rightarrow 3x-20y+15 = 0$

Exercise 3.4

1. Find the equation of the following circles having
   (i) the centre (3,5) and radius 5 units
   (ii) the centre (0,0) and radius 2 units
2. Find the centre and radius of the circle
   (i) $x^2+y^2 = 16$
   (ii) $x^2+y^2-22x-4y+25=0$



(iii) $5x^2+5y^2+4x-8y-16=0$

(iv) $(x+2)(x-5)+(y-2)(y-1)=0$

3. Find the equation of the circle whose centre is (–3, –2) and having circumference 16π

4. Find the equation of the circle whose centre is (2,3) and which passes through (1,4)

5. Find the equation of the circle passing through the points (0, 1),(4, 3) and (1, –1).

6. Find the equation of the circle on the line joining the points (1,0), (0,1) and having its centre on the line $x + y = 1$

7. If the lines $x + y = 6$ and $x + 2y = 4$ are diameters of the circle, and the circle passes through the point (2, 6) then find its equation.

8. Find the equation of the circle having (4, 7) and (–2, 5) as the extremities of a diameter.

9. Find the Cartesian equation of the circle whose parametric equations are $x = 3\cos\theta$, $y = 3\sin\theta$, $0 \le \theta \le 2\pi$.

Exercise 3.5

1. Find the equation of the tangent to the circle $x^2+y^2-4x+4y-8=0$ at $(-2,-2)$

2. Determine whether the points $P(1, 0)$, $Q(2, 1)$ and $R(2, 3)$ lie outside the circle, on the circle or inside the circle $x^2+y^2-4x-6y+9=0$

3. Find the length of the tangent from (1, 2) to the circle $x^2+y^2-2x+4y+9=0$

4. Find the value of P if the line $3x+4y-P=0$ is a tangent to the circle $x^2+y^2=16$

EXERCISE

Identify the type of conic section for each of the equations.

1. 2x2 - y2 = 7
2. 3x2 + 3y2 - 4x + 3y +10 = 0
3. 3x2 + 2y2 = 14
4. x2 + y2 + x - y = 0
5. 11x2 - 25y2 - 44x + 50y - 256 = 0
6. y2 + 4x + 3y + 4 = 0

SOLUTION

1) $2x^2 - y^2 = 7$

$$2x^2 - y^2 - 7 = 0 \qquad \text{-----(1)}$$

Compare this equation with the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ -----(2)

$A = 2$, $B = 0$, $C = -1$, $D = 0$, $E = 0$, $F = -7$

$A \neq C$ and A and C are opposite in sign. $\therefore$ The equation represents a hyperbola.

2) $3x^2 + 3y^2 - 4x + 3y + 10 = 0$

$$3x^2 + 3y^2 - 4x + 3y + 10 = 0 \qquad \text{-----(1)}$$

Compare this equation with the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ -----(2)

$A = 3$, $B = 0$, $C = 3$, $D = -4$, $E = 3$, $F = 10$

$A = C$ and $B = 0$. $\therefore$ The given equation represents a circle.

3) $3x^2 + 2y^2 = 14$

$$3x^2 + 2y^2 - 14 = 0 \qquad \text{-----(1)}$$

Compare this equation with the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ -----(2)

$A = 3$, $B = 0$, $C = 2$, $D = 0$, $E = 0$, $F = -14$

$A \neq C$ and A and C are of same sign. $\therefore$ The given equation represents the equation of an ellipse.

4) $x^2 + y^2 + x - y = 0$

$$x^2 + y^2 + x - y = 0 \qquad \text{-----(1)}$$

Compare this equation with the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ -----(2)

$A = 1$, $B = 0$, $C = 1$, $D = 1$, $E = -1$, $F = 0$

Hence $A = C = 1$ and $B = 0$. $\therefore$ The given equation represents a circle.

5) $11x^2 - 25y^2 - 44x + 50y - 256 = 0$

$$11x^2 - 25y^2 - 44x + 50y - 256 = 0 \qquad \text{-----(1)}$$

Compare this equation with the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ -----(2)

$A = 11$, $B = 0$, $C = -25$, $D = -44$, $E = 50$, $F = -256$

$A \neq C$ and A and C are opposite in sign. $\therefore$ The given equation represents a hyperbola.

6) $y^2 + 4x + 3y + 4 = 0$

$$y^2 + 4x + 3y + 4 = 0 \qquad \text{-----(1)}$$

Compare this equation with the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ -----(2)

$A = 0$, $B = 0$, $C = 1$, $D = 4$, $E = 3$, $F = 4$

Hence $B = 0$ and $A = 0$ and $C \neq 0$. $\therefore$ The given equation represents the equation of a parabola.

Answers:

1. hyperbola
2. circle
3. ellipse
4. circle
5. hyperbola
6. parabola

1 MEASURES OF DISPERSION

- Understand the concept of measure of dispersion.
- Understand the concept of range.
- Discuss the concept of standard deviation.
- Discuss the concept of probability.
- Understand the practical application of probability.
- Discuss the concept of events.
- Understand the concept of random experiment.

Objective of the Module:

The basic objective of this chapter is to through some light on the initial concepts of dispersion and probability so that the practical application of measures and probability can be examined in detailed.

The following data provide the runs scored by two batsmen in the last 10 matches.

Batsman A: 25, 20, 45, 93, 8, 14, 32, 87, 72, 4

Batsman B: 33, 50, 47, 38, 45, 40, 36, 48, 37, 26

$$\text{Mean of Batsman A} = \frac{25 + 20 + 45 + 93 + 8 + 14 + 32 + 87 + 72 + 4}{10} = 40$$

$$\text{Mean of Batsman B} = \frac{33 + 50 + 47 + 38 + 45 + 40 + 36 + 48 + 37 + 26}{10} = 40$$

The mean of both datas are same (40), but they differ significantly.

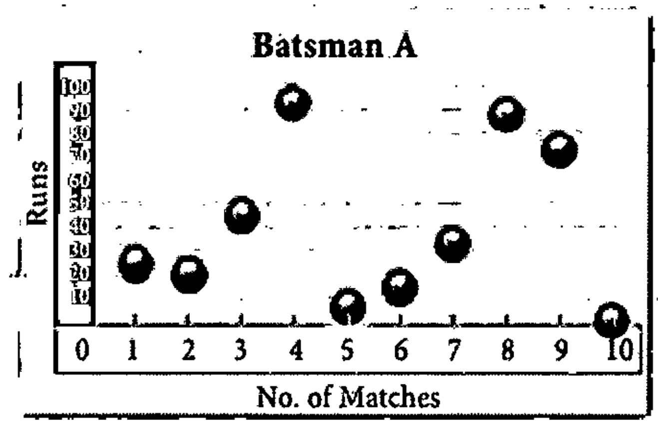


Fig. 8.1(a)

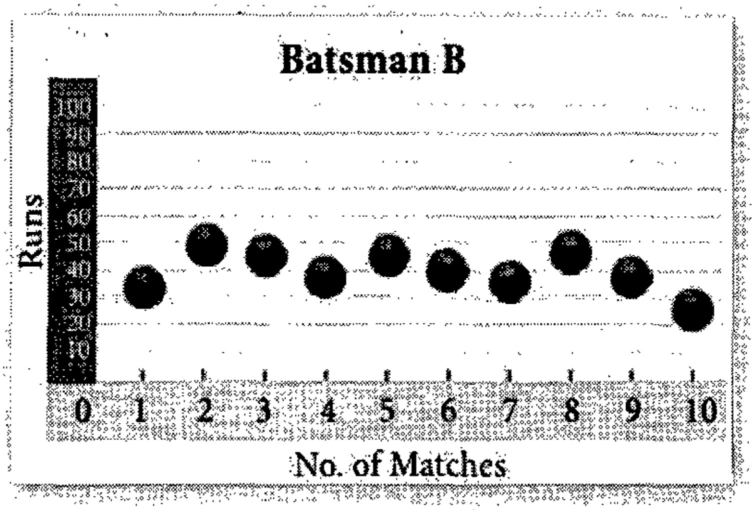


Fig. 8.1(b)

From the above diagrams, we see that runs of batsman B are grouped around the mean. But the runs of batsman A are scattered from 0 to 100, though they both have same mean.

Thus, some additional statistical information may be required to determine how the values are spread in data. For this, we shall discuss Measures of Dispersion.

Dispersion is a measure which gives an idea about the scatteredness of the values.

Measures of Variation (or) Dispersion of a data provide an idea of how observations spread out (or) scattered throughout the data.

Different Measures of Dispersion are

1. Range
2. Mean deviation
3. Quartile deviation
4. Standard deviation
5. Variance
6. Coefficient of Variation

1. Range

The difference between the largest value and the smallest value is called Range.

Range R = L – S

Coefficient of range = (L –S) / (L + S)

where L - Largest value; S - Smallest value

Example 1 Find the range and coefficient of range of the following data: 25, 67, 48, 53, 18, 39, 44.

Solution Largest value L = 67; Smallest value S =18

Range R = L –S = 67 –18 = 49

Coefficient of range = (L –S) / (L + S)

Coefficient of range = (67 – 18) / (67 +18) = 49/85

= 0.576

Example 2 Find the range of the following distribution.

Age (in years)	16-18	18-20	20-22	22-24	24-26	26-28
Number of students	0	4	6	8	2	2

Solution Here Largest value L = 28

Smallest value S = 18

Range R = L –S

R = 28 –18 = 10 Years

Example 3 The range of a set of data is 13.67 and the largest value is 70.08. Find the smallest value.

Solution

Range R = 13.67

Largest value L = 70.08

Range R = L –S

13.67 = 70.08 – S

S = 70.08 –13.67 = 56.41

Therefore, the smallest value is 56.41.

2. Deviations from the mean

For a given data with n observations x 1 , x2,¼xn , the deviations from the mean $\overline{x}$ are

$$x_1 - \overline{x},\ x_2 - \overline{x},\ \ldots,\ x_n - \overline{x}.$$

3. Squares of deviations from the mean

The squares of deviations from the mean $\overline{x}$ of the observations x1, x2, ... , xn are

$$(x_1 - \overline{x})^2, (x_2 - \overline{x})^2, \ldots, (x_n - \overline{x})^2 \text{ or } \sum_{i=1}^{n}(x_i - \overline{x})^2$$

Note

We note that (x i $- \overline{x}$)2 $\geq$ 0 for all observations xi , i = 1,2,3,…,n. If the deviations from the mean (x i $- \overline{x}$) are small, then the squares of the deviations will be very small.

4. Variance

The mean of the squares of the deviations from the mean is called Variance.

It is denoted by σ2 (read as sigma square).

Variance = Mean of squares of deviations

$$\text{Variance} = \text{Mean of squares of deviations}$$

$$= \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\ldots+(x_n-\bar{x})^2}{n}$$

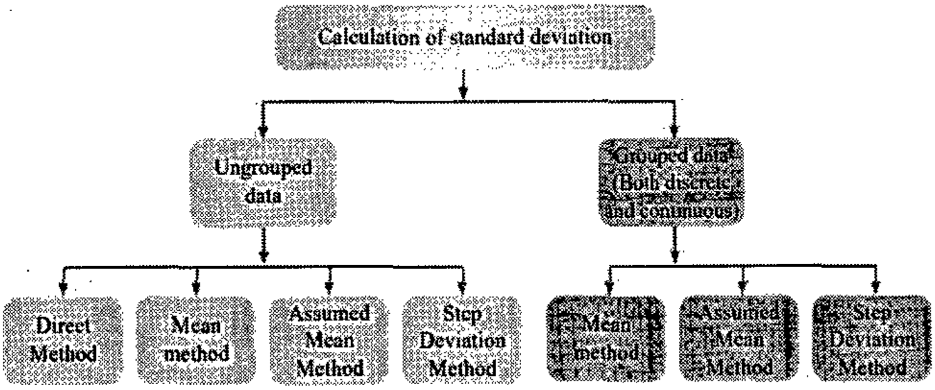
$$\text{Variance } \sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$$

5. Standard Deviation

The positive square root of Variance is called Standard deviation. That is, standard deviation is the positive square root of the mean of the squares of deviations of the given values from their mean. It is denoted by σ.

Standard deviation gives a clear idea about how far the values are spreading or deviating from the mean.

$$\textbf{Standard deviation } \sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}}$$



1. Calculation of Standard Deviation for ungrouped data

(i) Direct Method

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma(x_i-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{\Sigma(x_i^2-2x_i\bar{x}+\bar{x}^2)}{n}}$$

$$= \sqrt{\frac{\Sigma x_i^2}{n}-2\bar{x}\frac{\Sigma x_i}{n}+\frac{\bar{x}^2}{n}\times(1+1+\cdots to\ n \text{ times})}$$

$$= \sqrt{\frac{\Sigma x_i^2}{n}-2\bar{x}\times\bar{x}+\frac{\bar{x}^2}{n}\times n} = \sqrt{\frac{\Sigma x_i^2}{n}-2\bar{x}^2+\bar{x}^2} = \sqrt{\frac{\Sigma x_i^2}{n}-\bar{x}^2}$$

Standard deviation, $\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$

Note

- While computing standard deviation, arranging data in ascending order is not mandatory.
- If the data values are given directly then to find standard deviation we can use the formula Ã = $\sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$
- If the data values are not given directly but the squares of the deviations from the mean of each observation is given then to find standard deviation we can use the formula Ã = $\sqrt{\frac{\Sigma\left(x_i - \bar{x}\right)^2}{n}}$

Example 4 The number of televisions sold in each day of a week are 13, 8, 4, 9, 7, 12, 10.

Find its standard deviation.

Solution

x_i	x_i^2
13	169
8	64
4	16
9	81
7	49
12	144
10	100
$\Sigma x_i = 63$	$\Sigma x_i^2 = 623$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{623}{7} - \left(\frac{63}{7}\right)^2}$$

$$= \sqrt{89 - 81} = \sqrt{8}$$

Hence, $\sigma \simeq 2.83$

(ii) Mean method

Another convenient way of finding standard deviation is to use the following formula.

Standard deviation (by mean method) σ = $\sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}$

If di = xi – $\bar{x}$ are the deviations, then $\sigma = \sqrt{\frac{\Sigma d_i^2}{n}}$

Example 5 The amount of rainfall in a particular season for 6 days are given as 17.8 cm, 19.2 cm, 16.3 cm, 12.5 cm, 12.8 cm and 11.4 cm. Find its standard deviation.

Solution Arranging the numbers in ascending order we get, 11.4, 12.5, 12.8, 16.3, 17.8, 19.2.

Number of observations n = 6

$$\text{Mean} = \frac{11.4 + 12.5 + 12.8 + 16.3 + 17.8 + 19.2}{6} = \frac{90}{6} = 15$$

x_i	$d_i = x_i - \bar{x}$ $= x - 15$	d_i^2
11.4	−3.6	12.96
12.5	−2.5	6.25
12.8	−2.2	4.84
16.3	1.3	1.69
17.8	2.8	7.84
19.2	4.2	17.64
		$\Sigma d_i^2 = 51.22$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma d_i^2}{n}}$$

$$= \sqrt{\frac{51.22}{6}} = \sqrt{8.53}$$

$$\text{Hence, } \sigma \simeq 2.9$$

(iii) Assumed Mean method

When the mean value is not an integer (since calculations are very tedious in decimal form) then it is better to use the assumed mean method to find the standard deviation.

Let x 1 , x2, x 3 , ..., xn be the given data values and let $\bar{x}$ be their mean.

Let di be the deviation of xi from the assumed mean A, which is usually the middle value or near the middle value of the given data.

di= xi − A gives, xi = di + A ...(1)

Σdi= Σ(xi −A)

= Σxi −(A + A + A + . . . to n times)

Σdi = Σxi − A × n

$$\Sigma d_i = \Sigma x_i - A \times n$$

$$\frac{\Sigma d_i}{n} = \frac{\Sigma x_i}{n} - A$$

$$\bar{d} = \bar{x} - A \text{ (or) } \bar{x} = \bar{d} + A \qquad \ldots(2)$$

Now, Standard deviation

$$\sigma = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}} = \sqrt{\frac{\Sigma(d_i + A - \bar{d} - A)^2}{n}} \qquad \text{(using (1) and (2))}$$

$$= \sqrt{\frac{\Sigma(d_i - \bar{d})^2}{n}} = \sqrt{\frac{\Sigma(d_i^2 - 2d_i \times \bar{d} + \bar{d}^2)}{n}}$$

$$= \sqrt{\frac{\Sigma d_i^2}{n} - 2\bar{d}\frac{\Sigma d_i}{n} + \frac{\bar{d}^2}{n}(1+1+1+\cdots \text{ to } n \text{ times})}$$

$$= \sqrt{\frac{\Sigma d_i^2}{n} - 2\bar{d} \times \bar{d} + \frac{\bar{d}^2}{n} \times n} \qquad \text{(since } \bar{d} \text{ is a constant)}$$

$$= \sqrt{\frac{\Sigma d_i^2}{n} - \bar{d}^2}$$

Standard deviation $\sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$

Thinking Corner

For any collection of n values, can you find the value of

(i) $\Sigma(x_i - \bar{x})$ (ii) $(\Sigma x_i) - \bar{x}$

Example 6 The marks scored by 10 students in a class test are 25, 29, 30, 33, 35, 37, 38, 40, 44, 48. Find the standard deviation.

Solution The mean of marks is 35.9 which is not an integer. Hence we take assumed mean, A = 35, n = 10 .

x_i	$d_i = x_i - A$ $d_i = x_i - 35$	d_i^2
25	−10	100
29	−6	36
30	−5	25
33	−2	4
35	0	0
37	2	4
38	3	9
40	5	25
44	9	81
48	13	169
	$\Sigma d_i = 9$	$\Sigma d_i^2 = 453$

Standard deviation

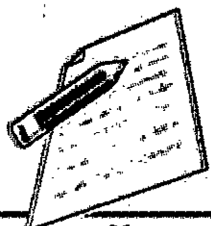
$$\sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

$$= \sqrt{\frac{453}{10} - \left(\frac{9}{10}\right)^2}$$

$$= \sqrt{45.3 - 0.81}$$

$$= \sqrt{44.49}$$

$$\sigma \simeq 6.67$$



(iv) Step deviation method

Let x 1 , x2, x 3 ,...xn be the given data. Let A be the assumed mean.

Let c be the common divisor of x i - A .

$$\text{Let} \quad d_i = \frac{x_i - A}{c}$$

$$\text{Then} \quad x_i = d_i c + A \qquad \ldots(1)$$

$$\Sigma x_i = \Sigma\left(d_i c + A\right) = c\Sigma d_i + A \times n$$

$$\frac{\Sigma x_i}{n} = c\frac{\Sigma d_i}{n} + A$$

$$\bar{x} = c\bar{d} + A \qquad \ldots(2)$$

$$x_i - \bar{x} = cd_i + A - c\bar{d} - A = c(d_i - \bar{d}) \quad \textbf{(using (1) and (2))}$$

$$\sigma = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}} = \sqrt{\frac{\Sigma(c(d_i - \bar{d}))^2}{n}} = \sqrt{\frac{c^2\Sigma(d_i - \bar{d})^2}{n}}$$

$$\sigma = c \times \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

Note

We can use any of the above methods for finding the standard deviation

Example 7 The amount that the children have spent for purchasing some eatables in one day trip of a school are 5, 10, 15, 20, 25, 30, 35, 40. Using step deviation method, find the standard deviation of the amount they have spent.

Solution We note that all the observations are divisible by 5. Hence we can use the step deviation method. Let the Assumed mean A = 20, n = 8.

x_i	$d_i = x_i - A$ $d_i = x_i - 20$	$d_i = \frac{x_i - A}{c}$ $c = 5$	d_i^2
5	−15	−3	9
10	−10	−2	4
15	−5	−1	1
20	0	0	0
25	5	1	1
30	10	2	4
35	15	3	9
40	20	4	16
		$\Sigma d_i = 4$	$\Sigma d_i^2 = 44$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2} \times c$$

$$= \sqrt{\frac{44}{8} - \left(\frac{4}{8}\right)^2} \times 5 = \sqrt{\frac{11}{2} - \frac{1}{4}} \times 5$$

$$= \sqrt{5.5 - 0.25} \times 5 = 2.29 \times 5$$

$$\sigma \simeq 11.45$$

Example 8 Find the standard deviation of the following data 7, 4, 8, 10, 11. Add 3 to all the values then find the standard deviation for the new values.

Solution Arranging the values in ascending order we get, 4, 7, 8, 10, 11 and n = 5

x_i	x_i^2
4	16
7	49
8	64
10	100
11	121
$\Sigma x_i = 40$	$\Sigma x_i^2 = 350$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{350}{5} - \left(\frac{40}{5}\right)^2}$$

$$\sigma = \sqrt{6} \simeq 2.45$$

When we add 3 to all the values, we get the new values as 7,10,11,13,14.

x_i	x_i^2
7	9
10	100
11	121
13	169
14	196
$\Sigma x_i = 55$	$\Sigma x_i^2 = 635$

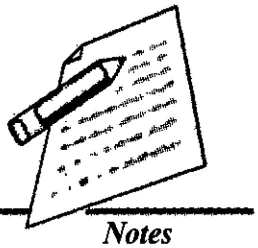
Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{635}{5} - \left(\frac{55}{5}\right)^2}$$

$$\sigma = \sqrt{6} \simeq 2.45$$

From the above, we see that the standard deviation will not change when we add some fixed constant to all the values.



Example 9 Find the standard deviation of the data 2, 3, 5, 7, 8. Multiply each data by 4. Find the standard deviation of the new values.

Solution Given, n = 5

x_i	x_i^2
2	49
3	9
5	25
7	49
8	64
$\Sigma x_i = 25$	$\Sigma x_i^2 = 151$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$\sigma = \sqrt{\frac{151}{5} - \left(\frac{25}{5}\right)^2} = \sqrt{30.2 - 25} = \sqrt{5.2} \simeq 2.28$$

When we multiply each data by 4, we get the new values as 8, 12, 20, 28, 32.

x_i	x_i^2
8	64
12	144
20	400
28	784
32	1024
$\Sigma x_i = 100$	$\Sigma x_i^2 = 2416$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{2416}{5} - \left(\frac{100}{5}\right)^2} = \sqrt{483.2 - 400} = \sqrt{83.2}$$

$$\sigma = \sqrt{16 \times 5.2} = 4\sqrt{5.2} \simeq 9.12$$

From the above, we see that when we multiply each data by 4 the standard deviation also get multiplied by 4.

Example

10 Find the mean and variance of the first n natural numbers.

Solution

$$\text{Mean } \bar{x} = \frac{\text{Sum of all the observations}}{\text{Number of observations}}$$

$$= \frac{\Sigma x_i}{n} = \frac{1+2+3+\ldots+n}{n} = \frac{n(n+1)}{2 \times n}$$

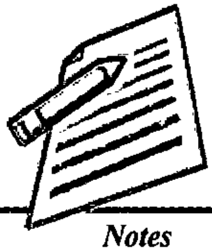
$$\text{Mean } \bar{x} = \frac{n+1}{2}$$

$$\text{Variance } \sigma^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2 \left[\begin{array}{l}\Sigma x_i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 \\ (\Sigma x_i)^2 = (1+2+3+\ldots+n)^2\end{array}\right]$$

$$= \frac{n(n+1)(2n+1)}{6 \times n} - \left[\frac{n(n+1)}{2 \times n}\right]^2$$

$$= \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4}$$

$$\text{Variance } \sigma^2 = \frac{4n^2+6n+2-3n^2-6n-3}{12} = \frac{n^2-1}{12}.$$



Calculation of Standard deviation for grouped data

(i) Mean method

Standard deviation $\sigma = \sqrt{\dfrac{\Sigma f_i(x_i - \bar{x})^2}{N}}$

Let, $d_i = x_i - \bar{x}$

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N}}, \text{ where } N = \sum_{i=1}^{n} f_i$$

(f_i are frequency values of the corresponding data points x_i)

Example 11

48 students were asked to write the total number of hours per week they spent on watching television. With this information find the standard deviation of hours spent for watching television.

x	6	7	8	9	10	11	12
f	3	6	9	13	8	5	4

Solution

x_i	f_i	$x_i f_i$	$d_i = x_i - \bar{x}$	d_i^2	$f_i d_i^2$
6	3	18	–3	9	27
7	6	42	–2	4	24
8	9	72	–1	1	9
9	13	117	0	0	0
10	8	80	1	1	8
11	5	55	2	4	20
12	4	48	3	9	36
	$N = 48$	$\Sigma x_i f_i = 432$	$\Sigma d_i = 0$		$\Sigma f_i d_i^2 = 124$

Mean

$$\bar{x} = \frac{\Sigma x_i f_i}{N} = \frac{432}{48} = 9 \quad (\text{Since } N = \sum f_i)$$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N}} = \sqrt{\frac{124}{48}} = \sqrt{2.58}$$

$\sigma \simeq 1.6$

(ii) Assumed Mean Method

Let x 1 , x2, x 3 , ...xn be the given data with frequencies f1 , f2, f3 , ... fn respectively. Let x be their mean and A be the assumed mean..

$$d_i = x_i - A$$

$$\text{Standard deviation, } \sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

Example 12

The marks scored by the students in a slip test are given below.

x_i	4	6	8	10	12
f	7	3	5	9	5

Find the standard deviation of their marks.

Solution

Let the assumed mean, A = 8

x_i	f_i	$d_i = x_i - A$	$f_i d_i$	$f_i d_i^2$
4	7	−4	−28	112
6	3	−2	−6	12
8	5	0	0	0
10	9	2	18	36
12	5	4	20	80
	$N = 29$		$\Sigma f_i d_i = 4$	$\Sigma f_i d_i^2 = 240$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

$$= \sqrt{\frac{240}{29} - \left(\frac{4}{29}\right)^2} = \sqrt{\frac{240 \times 29 - 16}{29 \times 29}}$$

$$\sigma = \sqrt{\frac{6944}{29 \times 29}}; \quad \sigma \simeq 2.87$$

2. Calculation of Standard deviation for continuous frequency distribution

(i) Mean method

$$\sigma = \sqrt{\frac{\Sigma f_i \left(x_i - \bar{x}\right)^2}{N}}$$

Standard deviation where xi = Middle value of the i th class.

fi = Frequency of the i th class.

(ii) Shortcut method (or) Step deviation method

To make the calculation simple, we provide the following formula. Let A be the assumed mean, xi be the middle value of the ith class and c is the width of the class interval.

Let $d_i = \dfrac{x_i - A}{c}$

$$\sigma = c \times \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

Example 13

Marks of the students in a particular subject of a class are given below.

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Number of students	8	12	17	14	9	7	4

Find its standard deviation.

Solution

Let the assumed mean, A = 35, c = 10

Marks	Midvalue (x_i)	f_i	$d_i = x_i - A$	$d_i = \frac{x_i - A}{c}$	$f_i d_i$	$f_i d_i^2$
0-10	5	8	-30	-3	-24	72
10-20	15	12	-20	-2	-24	48
20-30	25	17	-10	-1	-17	17
30-40	35	14	0	0	0	0
40-50	45	9	10	1	9	9
50-60	55	7	20	2	14	28
60-70	65	4	30	3	12	36
		$N = 71$			$\Sigma f_i d_i = -30$	$\Sigma f_i d_i^2 = 210$

Standard deviation $\sigma = c \times \sqrt{\dfrac{\Sigma f_i d_i^2}{N} - \left(\dfrac{\Sigma f_i d_i}{N}\right)^2}$

$$\sigma = 10 \times \sqrt{\frac{210}{71} - \left(-\frac{30}{71}\right)^2} = 10 \times \sqrt{\frac{210}{71} - \frac{900}{5041}}$$

$$= 10 \times \sqrt{2.779}\ ; \quad \sigma \simeq 16.67$$

Thinking Corner

1. The standard deviation of a data is 2.8, if 5 is added to all the data values then the new standard deviation is ____.
2. If S is the standard deviation of values p, q, r then standard deviation of p-3, q-3, r-3 is ____.

Example 14

The mean and standard deviation of 15 observations are found to be 10 and 5 respectively. On rechecking it was found that one of the observation with value 8 was incorrect. Calculate the correct mean and standard deviation if the correct observation value was 23?

Notes

Solution

$$n = 15,\ \bar{x} = 10,\ \sigma = 5;\quad \bar{x} = \frac{\Sigma x}{n};$$

$$\Sigma x = 15 \times 10 = 150$$

Wrong observation value = 8, Correct observation value = 23.

Correct total = 150 − 8 + 23 = 165

$$\text{Correct mean } \bar{x} = \frac{165}{15} = 11$$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

$$\text{Incorrect value of } \sigma = 5 = \sqrt{\frac{\Sigma x^2}{15} - (10)^2}$$

$$25 = \frac{\Sigma x^2}{15} - 100 \text{ gives, } \frac{\Sigma x^2}{15} = 125$$

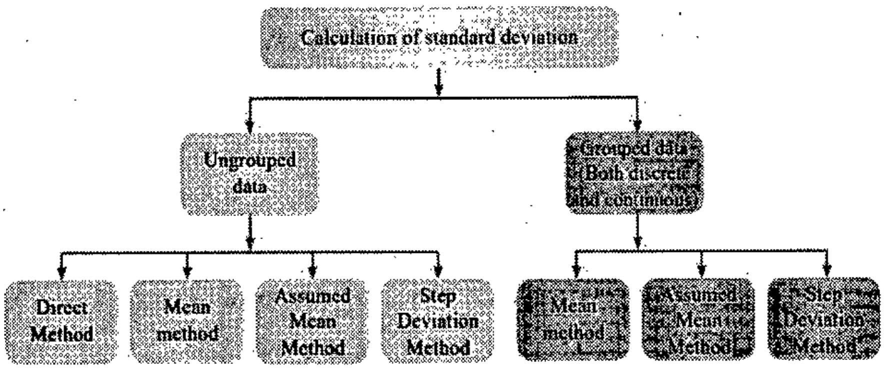
$$\text{Incorrect value of } \Sigma x^2 = 1875$$

$$\text{Correct value of } \Sigma x^2 = 1875 - 8^2 + 23^2 = 2340$$

$$\text{Correct standard deviation } \sigma = \sqrt{\frac{2340}{15} - (11)^2}$$

$$\sigma = \sqrt{156 - 121} = \sqrt{35} \qquad \sigma \simeq 5.9$$

1. Calculation of Standard Deviation for ungrouped data
   (i) Direct Method
   (ii) Mean method
   (iii) Assumed Mean method
   (iv) Step deviation method
2. Calculation of Standard deviation for continuous frequency distribution
   (i) Mean method
   (ii) Assumed Mean method
   (iii) Step deviation method



1. Calculation of Standard Deviation for ungrouped data

(i) Direct Method

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma(x_i-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{\Sigma(x_i^2-2x_i\bar{x}+\bar{x}^2)}{n}}$$

$$= \sqrt{\frac{\Sigma x_i^2}{n} - 2\bar{x}\frac{\Sigma x_i}{n} + \frac{\bar{x}^2}{n}\times\left(1+1+\cdots to\ n\ \text{times}\right)}$$

$$= \sqrt{\frac{\Sigma x_i^2}{n} - 2\bar{x}\times\bar{x} + \frac{\bar{x}^2}{n}\times n} = \sqrt{\frac{\Sigma x_i^2}{n} - 2\bar{x}^2 + \bar{x}^2} = \sqrt{\frac{\Sigma x_i^2}{n} - \bar{x}^2}$$

$$\text{Standard deviation, } \sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

Note

While computing standard deviation, arranging data in ascending order is not mandatory.

If the data values are given directly then to find standard deviation we can use the formula Ã = $\sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$

If the data values are not given directly but the squares of the deviations from the mean of each observation is given then to find standard deviation we can use the formula Ã = $\sqrt{\frac{\Sigma\left(x_i-\bar{x}\right)^2}{n}}$

Example 15 The number of televisions sold in each day of a week are 13, 8, 4, 9, 7, 12, 10.

Find its standard deviation.

Solution

x_i	x_i^2
13	169
8	64
4	16
9	81
7	49
12	144
10	100
$\Sigma x_i = 63$	$\Sigma x_i^2 = 623$

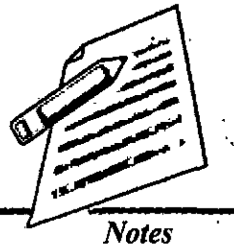
Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{623}{7} - \left(\frac{63}{7}\right)^2}$$

$$= \sqrt{89-81} = \sqrt{8}$$

Hence, $\sigma \simeq 2.83$



(ii) Mean method

Another convenient way of finding standard deviation is to use the following formula.

Standard deviation (by mean method) $\sigma = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}$

If di = xi – $\bar{x}$ are the deviations, then $\sigma = \sqrt{\frac{\Sigma d_i^2}{n}}$

Example 16 The amount of rainfall in a particular season for 6 days are given as 17.8 cm, 19.2 cm, 16.3 cm, 12.5 cm, 12.8 cm and 11.4 cm. Find its standard deviation.

Solution Arranging the numbers in ascending order we get, 11.4, 12.5, 12.8, 16.3, 17.8, 19.2.

Number of observations n = 6

$$\text{Mean} = \frac{11.4 + 12.5 + 12.8 + 16.3 + 17.8 + 19.2}{6} = \frac{90}{6} = 15$$

x_i	$d_i = x_i - \bar{x}$ $= x - 15$	d_i^2
11.4	-3.6	12.96
12.5	-2.5	6.25
12.8	-2.2	4.84
16.3	1.3	1.69
17.8	2.8	7.84
19.2	4.2	17.64
		$\Sigma d_i^2 = 51.22$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma d_i^2}{n}}$$

$$= \sqrt{\frac{51.22}{6}} = \sqrt{8.53}$$

$$\text{Hence, } \sigma \cong 2.9$$

(iii) Assumed Mean method

When the mean value is not an integer (since calculations are very tedious in decimal form) then it is better to use the assumed mean method to find the standard deviation.

Let $x_1, x_2, x_3, \ldots, x_n$ be the given data values and let $\bar{x}$ be their mean.

Let d_i be the deviation of x_i from the assumed mean A, which is usually the middle value or near the middle value of the given data.

$d_i = x_i - A$ gives, $x_i = d_i + A$ ...(1)

$\Sigma d_i = \Sigma(x_i - A)$

$= \Sigma x_i - (A + A + A + \ldots$ to n times)

$\Sigma d_i = \Sigma x_i - A \times n$

$$\Sigma d_i = \Sigma x_i - A \times n$$

$$\frac{\Sigma d_i}{n} = \frac{\Sigma x_i}{n} - A$$

$$\bar{d} = \bar{x} - A \text{ (or) } \bar{x} = \bar{d} + A \qquad \ldots(2)$$

Now, Standard deviation

$$\sigma = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}} = \sqrt{\frac{\Sigma(d_i + A - \bar{d} - A)^2}{n}} \qquad \text{(using (1) and (2))}$$

$$= \sqrt{\frac{\Sigma(d_i - \bar{d})^2}{n}} = \sqrt{\frac{\Sigma(d_i^2 - 2d_i \times \bar{d} + \bar{d}^2)}{n}}$$

$$= \sqrt{\frac{\Sigma d_i^2}{n} - 2\bar{d}\frac{\Sigma d_i}{n} + \frac{\bar{d}^2}{n}(1 + 1 + 1 + \cdots \text{ to } n \text{ times})}$$

$$= \sqrt{\frac{\Sigma d_i^2}{n} - 2\bar{d} \times \bar{d} + \frac{\bar{d}^2}{n} \times n} \qquad \text{(since } \bar{d} \text{ is a constant)}$$

$$= \sqrt{\frac{\Sigma d_i^2}{n} - \bar{d}^2}$$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

Thinking Corner

For any collection of n values, can you find the value of

(i) $\Sigma(x_i - \bar{x})$ (ii) $(\Sigma x_i) - \bar{x}$

Example 17 The marks scored by 10 students in a class test are 25, 29, 30, 33, 35, 37, 38, 40, 44, 48. Find the standard deviation.

Solution The mean of marks is 35.9 which is not an integer. Hence we take assumed mean, A = 35, n = 10 .

x_i	$d_i = x_i - A$ $d_i = x_i - 35$	d_i^2
25	–10	100
29	–6	36
30	–5	25
33	–2	4
35	0	0
37	2	4
38	3	9
40	5	25
44	9	81
48	13	169
	$\Sigma d_i = 9$	$\Sigma d_i^2 = 453$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

$$= \sqrt{\frac{453}{10} - \left(\frac{9}{10}\right)^2}$$

$$= \sqrt{45.3 - 0.81}$$

$$= \sqrt{44.49}$$

$$\sigma \simeq 6.67$$

(iv) Step deviation method

Let x 1 , x2, x 3 ,...xn be the given data. Let A be the assumed mean.

Let c be the common divisor of x i - A .

$$\text{Let} \quad d_i = \frac{x_i - A}{c}$$

$$\text{Then} \quad x_i = d_i c + A \qquad \ldots(1)$$

$$\Sigma x_i = \Sigma\left(d_i c + A\right) = c\Sigma d_i + A \times n$$

$$\frac{\Sigma x_i}{n} = c\frac{\Sigma d_i}{n} + A$$

$$\bar{x} = c\bar{d} + A \qquad \ldots(2)$$

$$x_i - \bar{x} = cd_i + A - c\bar{d} - A = c(d_i - \bar{d}) \quad \text{(using (1) and (2))}$$

$$\sigma = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}} = \sqrt{\frac{\Sigma(c(d_i - \bar{d}))^2}{n}} = \sqrt{\frac{c^2\Sigma(d_i - \bar{d})^2}{n}}$$

$$\sigma = c \times \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

Note

We can use any of the above methods for finding the standard deviation

Example 18 The amount that the children have spent for purchasing some eatables in one day trip of a school are 5, 10, 15, 20, 25, 30, 35, 40. Using step deviation method, find the standard deviation of the amount they have spent.

Solution We note that all the observations are divisible by 5. Hence we can use the step deviation method. Let the Assumed mean A = 20, n = 8.

x_i	$d_i = x_i - A$ $d_i = x_i - 20$	$d_i = \frac{x_i - A}{c}$ $c = 5$	d_i^2
5	−15	−3	9
10	−10	−2	4
15	−5	−1	1
20	0	0	0
25	5	1	1
30	10	2	4
35	15	3	9
40	20	4	16
		$\Sigma d_i = 4$	$\Sigma d_i^2 = 44$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2} \times c$$

$$= \sqrt{\frac{44}{8} - \left(\frac{4}{8}\right)^2} \times 5 = \sqrt{\frac{11}{2} - \frac{1}{4}} \times 5$$

$$= \sqrt{5.5 - 0.25} \times 5 = 2.29 \times 5$$

$$\sigma \simeq 11.45$$

Example 19 Find the standard deviation of the following data 7, 4, 8, 10, 11. Add 3 to all the values then find the standard deviation for the new values.

Solution Arranging the values in ascending order we get, 4, 7, 8, 10, 11 and n = 5

x_i	x_i^2
4	16
7	49
8	64
10	100
11	121
$\Sigma x_i = 40$	$\Sigma x_i^2 = 350$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{350}{5} - \left(\frac{40}{5}\right)^2}$$

$$\sigma = \sqrt{6} \simeq 2.45$$

When we add 3 to all the values, we get the new values as 7,10,11,13,14.

x_i	x_i^2
7	9
10	100
11	121
13	169
14	196
$\Sigma x_i = 55$	$\Sigma x_i^2 = 635$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{635}{5} - \left(\frac{55}{5}\right)^2}$$

$$\sigma = \sqrt{6} \simeq 2.45$$

From the above, we see that the standard deviation will not change when we add some fixed constant to all the values.

Example 20 Find the standard deviation of the data 2, 3, 5, 7, 8. Multiply each data by 4. Find the standard deviation of the new values.

Solution Given, n = 5

x_i	x_i^2
2	49
3	9
5	25
7	49
8	64
$\Sigma x_i = 25$	$\Sigma x_i^2 = 151$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$\sigma = \sqrt{\frac{151}{5} - \left(\frac{25}{5}\right)^2} = \sqrt{30.2 - 25} = \sqrt{5.2} \simeq 2.28$$

When we multiply each data by 4, we get the new values as 8, 12, 20, 28, 32.

x_i	x_i^2
8	64
12	144
20	400
28	784
32	1024
$\Sigma x_i = 100$	$\Sigma x_i^2 = 2416$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{2416}{5} - \left(\frac{100}{5}\right)^2} = \sqrt{483.2 - 400} = \sqrt{83.2}$$

$$\sigma = \sqrt{16 \times 5.2} = 4\sqrt{5.2} \simeq 9.12$$

From the above, we see that when we multiply each data by 4 the standard deviation also get multiplied by 4.

Example 21 Find the mean and variance of the first n natural numbers.

Solution

$$\text{Mean } \bar{x} = \frac{\text{Sum of all the observations}}{\text{Number of observations}}$$

$$= \frac{\Sigma x_i}{n} = \frac{1+2+3+\ldots+n}{n} = \frac{n(n+1)}{2 \times n}$$

$$\text{Mean } \bar{x} = \frac{n+1}{2}$$

$$\text{Variance } \sigma^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2 \left[\begin{array}{l} \Sigma x_i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 \\ (\Sigma x_i)^2 = (1+2+3+\ldots+n)^2 \end{array}\right]$$

$$= \frac{n(n+1)(2n+1)}{6 \times n} - \left[\frac{n(n+1)}{2 \times n}\right]^2$$

$$= \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4}$$

$$\text{Variance } \sigma^2 = \frac{4n^2+6n+2-3n^2-6n-3}{12} = \frac{n^2-1}{12}.$$

Calculation of Standard deviation for grouped data

(i) Mean method

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma f_i (x_i - \bar{x})^2}{N}}$$

Let, $d_i = x_i - \bar{x}$

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N}}, \text{ where } N = \sum_{i=1}^{n} f_i$$

(f_i are frequency values of the corresponding data points x_i)

Example 22

48 students were asked to write the total number of hours per week they spent on watching television. With this information find the standard deviation of hours spent for watching television.

x	6	7	8	9	10	11	12
f	3	6	9	13	8	5	4

Solution

x_i	f_i	$x_i f_i$	$d_i = x_i - \bar{x}$	d_i^2	$f_i d_i^2$
6	3	18	–3	9	27
7	6	42	–2	4	24
8	9	72	–1	1	9
9	13	117	0	0	0
10	8	80	1	1	8
11	5	55	2	4	20
12	4	48	3	9	36
	$N = 48$	$\Sigma x_i f_i = 432$	$\Sigma d_i = 0$		$\Sigma f_i d_i^2 = 124$

Mean

$$\bar{x} = \frac{\Sigma x_i f_i}{N} = \frac{432}{48} = 9 \quad \left(\text{Since } N = \sum f_i\right)$$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N}} = \sqrt{\frac{124}{48}} = \sqrt{2.58}$$

$\sigma \simeq 1.6$

(ii) Assumed Mean Method

Let x 1 , x2, x 3 , ...xn be the given data with frequencies f1 , f2, f3 , ... fn respectively. Let x be their mean and A be the assumed mean..

$$d_i = x_i - A$$

$$\text{Standard deviation, } \sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

Example 23

The marks scored by the students in a slip test are given below.

x	4	6	8	10	12
f	7	3	5	9	5

Find the standard deviation of their marks.

Solution

Let the assumed mean, A = 8

x_i	f_i	$d_i = x_i - A$	$f_i d_i$	$f_i d_i^2$
4	7	-4	-28	112
6	3	-2	-6	12
8	5	0	0	0
10	9	2	18	36
12	5	4	20	80
	$N = 29$		$\Sigma f_i d_i = 4$	$\Sigma f_i d_i^2 = 240$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

$$= \sqrt{\frac{240}{29} - \left(\frac{4}{29}\right)^2} = \sqrt{\frac{240 \times 29 - 16}{29 \times 29}}$$

$$\sigma = \sqrt{\frac{6944}{29 \times 29}}\ ; \quad \sigma \simeq 2.87$$

2. Calculation of Standard deviation for continuous frequency distribution

(i) Mean method

Standard deviation $\sigma = \sqrt{\frac{\Sigma f_i (x_i - \bar{x})^2}{N}}$ where xi = Middle value of the i th class.
fi = Frequency of the i th class.

(ii) Shortcut method (or) Step deviation method

To make the calculation simple, we provide the following formula. Let A be the assumed mean, xi be the middle value of the ith class and c is the width of the class interval.

Let $$d_i = \frac{x_i - A}{c}$$

$$\sigma = c \times \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

CLASS–12

Mathematics

Notes

Example 24

Marks of the students in a particular subject of a class are given below.

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Number of students	8	12	17	14	9	7	4

Find its standard deviation.

Solution

Let the assumed mean, A = 35, c = 10

Marks	Midvalue (x_i)	f_i	$d_i = x_i - A$	$d_i = \frac{x_i - A}{c}$	$f_i d_i$	$f_i d_i^2$
0-10	5	8	-30	–3	–24	72
10-20	15	12	-20	–2	–24	48
20-30	25	17	-10	–1	–17	17
30-40	35	14	0	0	0	0
40-50	45	9	10	1	9	9
50-60	55	7	20	2	14	28
60-70	65	4	30	3	12	36
		$N = 71$			$\Sigma f_i d_i = -30$	$\Sigma f_i d_i^2 = 210$

Standard deviation $\sigma = c \times \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$

$$\sigma = 10 \times \sqrt{\frac{210}{71} - \left(-\frac{30}{71}\right)^2} = 10 \times \sqrt{\frac{210}{71} - \frac{900}{5041}}$$

$$= 10 \times \sqrt{2.779}\ ; \quad \sigma \simeq 16.67$$

Thinking Corner

1. The standard deviation of a data is 2.8, if 5 is added to all the data values then the new standard deviation is ____.
2. If S is the standard deviation of values p, q, r then standard deviation of p-3, q-3, r-3 is ____.

Example 25

The mean and standard deviation of 15 observations are found to be 10 and 5 respectively. On rechecking it was found that one of the observation with value 8 was incorrect. Calculate the correct mean and standard deviation if the correct observation value was 23?

Solution

$$n = 15,\ \bar{x} = 10,\ \sigma = 5;\quad \bar{x} = \frac{\Sigma x}{n};$$

$$\Sigma x = 15 \times 10 = 150$$

Wrong observation value = 8, Correct observation value = 23.

Correct total = 150 − 8 + 23 = 165

$$\text{Correct mean } \bar{x} = \frac{165}{15} = 11$$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

$$\text{Incorrect value of } \sigma = 5 = \sqrt{\frac{\Sigma x^2}{15} - (10)^2}$$

$$25 = \frac{\Sigma x^2}{15} - 100 \text{ gives, } \frac{\Sigma x^2}{15} = 125$$

$$\text{Incorrect value of } \Sigma x^2 = 1875$$

$$\text{Correct value of } \Sigma x^2 = 1875 - 8^2 + 23^2 = 2340$$

$$\text{Correct standard deviation } \sigma = \sqrt{\frac{2340}{15} - (11)^2}$$

$$\sigma = \sqrt{156 - 121} = \sqrt{35} \quad \sigma \simeq 5.9$$

2 PROBABILITY

Introduction

Few centuries ago, gambling and gaming were considered to be fashionable and became widely popular among many men. As the games became more complicated, players were interested in knowing the chances of winning or losing a game from a given situation. In 1654, Chevalier de Mere, a French nobleman with a taste of gambling, wrote a letter to one of the prominent mathematician of the time, Blaise Pascal, seeking his advice about how much dividend he would get for a gambling game played by paying money. Pascal worked this problem mathematically but thought of sharing this problem and see how his good friend and mathematician Pierre de Fermat could solve. Their subsequent correspondences on the issue represented the birth of Probability Theory as a new branch of mathematics.

Random Experiment

A random experiment is an experiment in which

(i) The set of all possible outcomes are known (ii) Exact outcome is not known.

Example : Tossing a coin. 2. Rolling a die. 3. Selecting a card from a pack of 52 cards.

Sample space

The set of all possible outcomes in a random experiment is called a sample space. It is generally denoted by S.

Example : When we roll a die, the possible outcomes are the face numbers 1,2,3,4,5,6 of the die. Therefore the sample space is S = {1,2,3,4,5,6}

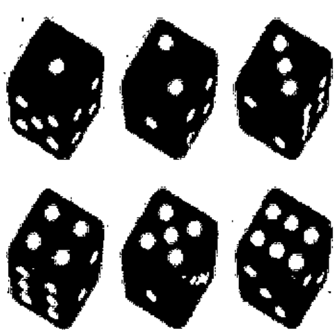


Fig. 8.2

Sample point

Each element of a sample space is called a sample point.

1. Tree diagram

Tree diagram allow us to see visually all possible outcomes of an random experiment. Each branch in a tree diagram represent a possible outcome.

Illustration

(i) When we throw a die, then from the tree diagram (Fig.8.3), the sample space can be written as S = {1,2,3,4,5,6}

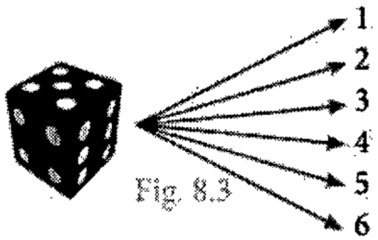


Fig. 8.3

(ii) When we toss two coins, then from the tree diagram (Fig.8.4), the sample space can be written as S={HH,HT,TH,TT}

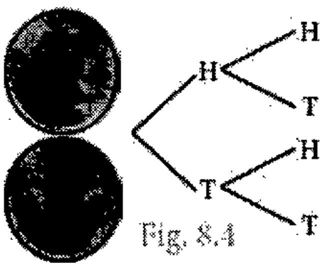
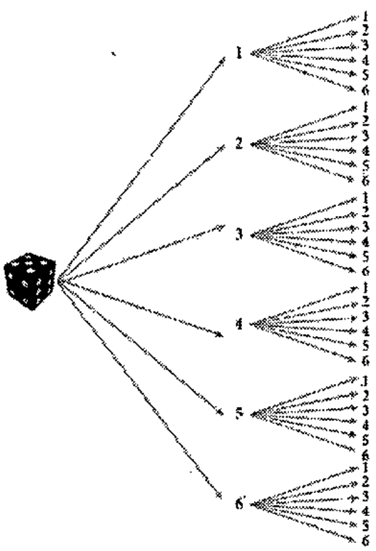


Fig. 8.4

Example: Express the sample space for rolling two dice using tree diagram.

Solution When we roll two dice, since each die contain 6 faces marked with 1,2,3,4,5,6 the tree diagram will look like



Hence, the sample space can be written as

S= {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)

(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)

(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)

(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

Hence, the sample space can be written as

S= {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

Event:

In a random experiment, each possible outcome is called an event. Thus, an event will be a subset of the sample space.

Example:

Getting two heads when we toss two coins is an event.

Trial : Performing an experiment once is called a trial.

Example : When we toss a coin thrice, then each toss of a coin is a trial.

Events	Explanation	Example
Equally likely events	Two or more events are said to be equally likely if each one of them has an equal chance of occurring.	Head and tail are equally likely events in tossing a coin.
Certain events	In an experiment, the event which surely occur is called certain event.	When we roll a die, the event of getting any natural number from one to six is a certain event.
Impossible events	In an experiment if an event has no scope to occur then it is called an impossible event.	When we toss two coins, the event of getting three heads is an impossible event.
Mutually exclusive events	Two or more events are said to be mutually exclusive if they don't have common sample points. i.e., events A, B are said to be mutually exclusive if $A \cap B = \phi$.	When we roll a die the events of getting odd numbers and even numbers are mutually exclusive events.
Exhaustive events	The collection of events whose union is the whole sample space are called exhaustive events.	When we toss a coin twice, the collection of events of getting two heads, exactly one head, no head are exhaustive events.
Complementary events	The complement of an event A is the event representing collection of sample points not in A. It is denoted A' or A' or $\bar{A}$ The event A and its complement A' are mutually exclusive and exhaustive.	When we roll a die, the event 'rolling a 5 or 6' and the event of rolling a 1, 2, 3 or 4 are complementary events.

Note

Elementary event: If an event E consists of only one outcome then it is called an elementary event.

2. Probability of an Event

In a random experiment, let S be the sample space and $E \subset S$. Then if E is an event, the probability of occurrence of E is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to occurence of } E}{\text{Number of all possible outcomes}} = \frac{n(E)}{n(S)}$$

This way of defining the probability is applicable only to finite sample spaces. So in this chapter, we will be dealing problems only with finite sample spaces.

Note

- $P(E) = \frac{n(E)}{n(S)}$
- $P(S) = \frac{n(S)}{n(S)} = 1$. The probability of sure event is 1.
- $P(\phi) = \frac{n(\phi)}{n(s)} = \frac{0}{n(s)} = 0$. The probability of impossible event is 0.
- Since E is a subset of S and ϕ is a subset of any set,

  $\phi \subseteq E \subseteq S$

  $P(\phi) \leq P(E) \leq P(S)$

  $0 \leq P(E) \leq 1$

  Therefore, the probability value always lies from 0 to 1.
- The complement event of E is $\bar{E}$.

  Let $P(E) = \frac{m}{n}$ (where m is the number of favourable outcomes of E and n is the total number of possible outcomes).

  $$P(\bar{E}) = \frac{\text{Number of outcomes unfavourable to occurance of } E}{\text{Number of all possible outcomes}}$$

  $$P(\bar{E}) = \frac{n-m}{n} = 1 - \frac{m}{n}$$

  $$P(\bar{E}) = 1 - P(E)$$

  
- $P(E) + P(\bar{E}) = 1$

Example

A bag contains 5 blue balls and 4 green balls. A ball is drawn at random from the bag. Find the probability that the ball drawn is (i) blue (ii) not blue.

Solution

Total number of possible outcomes n(S) = 5 + 4 = 9

(i) Let A be the event of getting a blue ball.

Number of favourable outcomes for the event A. Therefore, n(A) = 5

Probability that the ball drawn is blue. Therefore, P (A) = n(A)/ n(S) = 5/9

(ii) $\bar{A}$ will be the event of not getting a blue ball. So P ($\bar{A}$) = 1 – P (A) = 1 – 5/9 = 4/9

Example

Two dice are rolled. Find the probability that the sum of outcomes is

(i) equal to 4

(ii) greater than 10

(iii) less than 13

Solution

When we roll two dice, the sample space is given by

S = { (1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)

(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)

(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)

(6,1),(6,2),(6,3),(6,4),(6,5),(6,6) }; n (S) = 36

(i) Let A be the event of getting the sum of outcome values equal to 4.

Then A = {(1,3),(2,2),(3,1)}; n (A) = 3 .

Probability of getting the sum of outcomes equal to 4 is P (A) = n(A)/n(S) = 3/36 =1/12

(ii) Let B be the event of getting the sum of outcome values greater than 10.

Then B = {(5,6),(6,5),(6,6)}; n (B) = 3

Probability of getting the sum of outcomes greater than 10 is P (B) = n(B)/n(S) = 3/36 =1/12

(iii) Let C be the event of getting the sum of outcomes less than 13. Here all the outcomes have the sum value less than 13. Hence C = S.

Therefore, n (C) = n (S) = 36

Probability of getting the total value less than 13 is P (C) = n(C)/n(S) = 36/36 = 1

Example

Two coins are tossed together. What is the probability of getting different faces on the coins?

Solution

When two coins are tossed together, the sample space is

S = { HH, HT, TH, TT } ; n(S) = 4

Let A be the event of getting different faces on the coins.

A = { HT,TH } ; n(A) = 2

Probability of getting different faces on the coins is P(A) = n(A)/n(S) = 2/4 = ½

Example

From a well shuffled pack of 52 cards, one card is drawn at random. Find the probability of getting

(i) red card

(ii) heart card

(iii) red king

(iv) face card

(v) number card

Solution

Suits of playing cards	Spade ♠	Heart ♥	Club or ♣	Diamonds ♦
Cards of each suit	A	A	A	A
	2	2	2	2
	3	3	3	3
	4	4	4	4
	5	5	5	5
	6	6	6	6
	7	7	7	7
	8	8	8	8
	9	9	9	9
	10	10	10	10
	J	J	J	J
	Q	Q	Q	Q
	K	K	K	K
Set of playing card in each suit	13	13	13	13

Fig 8.5

Total number of balls n(S) = 6 + x + 2x = 6 + 3x

It is given that, P (G) = 3 × P(R)

6/ (6+3x) = 3 × x/(6 + 3x)

$$\frac{6}{6+3x} = 3\times\frac{x}{6+3x}$$

3x = 6 gives, x = 2

(i) Number of black balls = 2 × 2 = 4

(ii) Total number of balls = 6 +(3 × 2) = 12

Example

A game of chance consists of spinning an arrow which is equally likely to come to rest pointing to one of the numbers 1, 2, 3, ...12. What is the probability that it will point to

(i) 7

(ii) a prime number

(iii) a composite number?

Solution

Sample space S = {1,2,3,4,5,6,7,8,9,10,11,12}; n(S) = 12

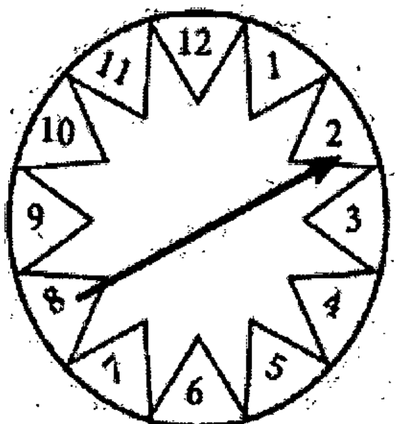


Fig. 8.6

(i) Let A be the event of resting in 7. n(A)=1

P (A) = n(A)/n(S) = 1/12

(ii) Let B be the event that the arrow will come to rest in a prime number.

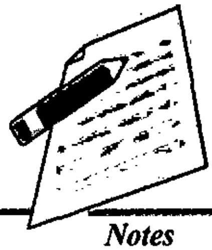
B = {2,3,5,7,11}; n (B) = 5

P (B) = n(B)/n(S) = 5/12

(iii) Let C be the event that arrow will come to rest in a composite number.

C = {4,6,8,9,10,12}; n (C) =6

P (C) = n(C)/n(S) = 6/12 = ½



Summary of the Module

Dispersion is a measure which gives an idea about the scatteredness of the values. Measures of Variation (or) Dispersion of a data provide an idea of how observations spread out (or) scattered throughout the data. Different Measures of Dispersion are Range, Mean deviation, Quartile deviation, Standard deviation, Variance, Coefficient of Variation. The difference between the largest value and the smallest value is called Range. Few centuries ago, gambling and gaming were considered to be fashionable and became widely popular among many men. As the games became more complicated, players were interested in knowing the chances of winning or losing a game from a given situation. In 1654, Chevalier de Mere, a French nobleman with a taste of gambling, wrote a letter to one of the prominent mathematician of the time, Blaise Pascal, seeking his advice about how much dividend he would get for a gambling game played by paying money. Pascal worked this problem mathematically but thought of sharing this problem and see how his good friend and mathematician Pierre de Fermat could solve. Their subsequent correspondences on the issue represented the birth of Probability Theory as a new branch of mathematics. A random experiment is an experiment in which (i) The set of all possible outcomes are known (ii) Exact outcome is not known. The set of all possible outcomes in a random experiment is called a sample space. It is generally denoted by S. Each element of a sample space is called a sample point. In a random experiment, each possible outcome is called an event. Thus, an event will be a subset of the sample space.

Review Questions

Exercise 1

1. Find the range and coefficient of range of the following data.

(i) 63, 89, 98, 125, 79, 108, 117, 68

(ii) 43.5, 13.6, 18.9, 38.4, 61.4, 29.8

(i) 63,89,98,125,79,108,117,68

Solution:

Let us arrange the given data in the ascending order 63,68,79,89,98,108,117,125
Largest value L=125 Smallest value
S=63

Range = L-S

= 125-63=62

$$\text{Coefficient of Range} = \frac{L-S}{L+S}$$

$$= \frac{125 \quad 63}{125 \quad 63} = \frac{62}{188} = 0.33$$

Answer R=62 Coefficient of

Range = 0.33

(ii) 43.5,13.6,18.9,38.4,61.4,29.8

Solution:

Let us arrange in ascending order 13.6,18.9,29.8,38.4,43.5,61.4 Largest value L=61.4 Smallest value S= 13.6

Range = L-S

61.4-13.6 = 47.8

$$\text{Coefficient of Range} = \frac{L-S}{L+S}$$

$$= \frac{61.4-13.6}{61.4+13.6}$$

$$= \frac{47.8}{75} = 0.64$$

Answer: R=47.8 Coefficient of Range = 0.64

2. If the range and the smallest value of a set of data are 36.8 and 13.4 respectively, then find the largest value.

Solution:

Range R = 36.8

Smallest value S = 13.4

R = L-S

36.8= L-13.4

L= 36.8+13.4=50.2

Answer: Largest Value = 50.2

3. Calculate the range of the following data.

Income	400-450	450-500	500-550	550-600	600-650
Number of workers	8	12	30	21	6

Solution:

Largest value L=650
Smallest value S =400
Range R = L-S
=650-450=250

Answer : R=250.

4. A teacher asked the students to complete 60 pages of a record note book. Eight students have completed only 32, 35, 37, 30, 33, 36, 35 and 37 pages. Find the standard deviation of the pages yet to be completed by them.

Solution:

By Assumed Mean Method :

Pages yet to be completed are 28, 25, 23, 30, 27, 24, 25, and 23

Assumed mean A=25 n = 8

x_i	$d_i=x_i-A$ $d_i=x_i-35$	d_i^2
23	-2	4
23	-2	4
24	-1	1
25	0	0
25	0	0
27	2	4
28	3	9
30	5	25
	$\sum d_i=5$	$\sum d_i^2=47$

Standard deviation

$$\sigma=\sqrt{\frac{\sum d_i^2}{n}-\left(\frac{\sum d_i}{n}\right)^2}$$

$$=\sqrt{\frac{47}{8}-\left(\frac{5}{8}\right)^2}$$

$$=\sqrt{\frac{47}{8}-\frac{25}{64}}$$

$$=\sqrt{\frac{351}{64}}$$

$$=\frac{18.735}{8}$$

$$\sigma=2.34$$

Ans: S.D of the pages to be completed = 2.34

5. Find the variance and standard deviation of the wages of 9 workers given below: □310, □290, □320, □280, □300, □290, □320, □310, □280.

Solution Mean Method :

$$\bar{x}=\frac{\Sigma x_i}{n}=\frac{280+280+290+290+300+310+310+232+320}{9}$$

$$=\frac{2700}{9}=300$$

x_i	$d_i = x_i - \overline{x}$ $= x_i - 300$	d_i^2
280	–20	400
280	–20	400
290	–10	100
290	–10	100
300	0	0
310	10	100
310	10	100
320	20	400
320	20	400
	$\Sigma d_i = -5$	$\Sigma d_i^2 = 2000$

Variance

$$\sigma^2 = \frac{\Sigma d_i^2}{n}$$

$$= \frac{2000}{9}$$

$$= 222.22$$

Standard deviation = 14.91

$$\sigma = \sqrt{\text{Variance}}$$

$$= \sqrt{222.22}$$

Answer : Variance = 222.22

S.D = 14.91

6. A wall clock strikes the bell once at 1 o' clock, 2 times at 2 o' clock, 3 times at 3 o' clock and so on. How many times will it strike in a particular day. Find the standard deviation of the number of strikes the bell make a day.

Solution :

The number of strikes the bell make a day.

$= 2\,(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12)$

Number of times strikes in a day.

$$2\left[\frac{n(n+1)}{2}\right]$$

$$2\left(\frac{12\times 13}{2}\right)$$

$$= 2 \times 78 = 156$$

Standard deviation :

S.D of the fist n natural numbers

$$\sigma = \sqrt{\frac{n^2-1}{12}}$$

S.D of number of strikes in a day

$$= 2\sqrt{\frac{n^2-1}{12}}$$

$$= 2\sqrt{\frac{12^2-1}{12}}$$

$$= 2\sqrt{\frac{144-1}{12}}$$

$$= 2\sqrt{\frac{143}{12}} \qquad = 2\sqrt{11.92}$$

$$= 2 \times 3.45$$

$$= 6.90$$

Answer : Standard deviation = 6.9

7. Find the standard deviation of first 21 natural numbers.

Solution :

Standard deviation $\quad \sigma = \sqrt{\frac{n^2-1}{12}}$

$$= \sqrt{\frac{21^2-1}{12}} = \sqrt{\frac{441-1}{12}}$$

$$= \sqrt{\frac{440}{12}} = \sqrt{36.6}$$

$$= 6.05$$

Answer : S.D of 21 natural numbers = 6.05

8. If the standard deviation of a data is 4.5 and if each value of the data is decreased by 5, then find the new standard deviation.

Solution :

$\sigma = 4.5$

each value decreased by 5.

The standard deviation will not change when we subtract some fixed constant to all the values.

$\therefore$ new standard deviation is 4.5

Answer : New S.D $\sigma = 4.5$

9. If the standard deviation of a data is 3.6 and each value of the data is divided by 3, then find the new variance and new standard deviation

Solution :

$\sigma = 3.6$

each value is divided by 3.

When we divide each data by 3 the standard deviation is also get divided by 3.

$\therefore$ new standard deviation $\sigma = \frac{3.6}{3} = 1.2$

and new variance $\sigma^2 = (1.2)^2$

$= 1.44$

Answer : New variance = 1.44

New S.D = 1.2

10. The rainfall recorded in various places of five districts in a week are given below.

Rainfall (in mm)	45	50	55	60	65	70
Number of places	5	13	4	9	5	4

Find its standard deviation.

Solution : Assumed mean method A = 60

x_i	f_i	$d_i = x_i - A$ $= x_i - 60$	$f_i d_i$	$f_i d_i^2$
45	5	–15	–75	1125
50	13	–10	–130	1300
55	4	-5	-20	100
60	9	0	0	0
65	5	5	25	125
70	4	10	40	400
	N = 40		$\Sigma f_i d_i = -160$	$\Sigma f_i d_i^2 = 3050$

Standard deviation

$$\sigma=\sqrt{\frac{\Sigma fidi^2}{N}-\left(\frac{\Sigma fidi}{N}\right)^2}$$

$$=\sqrt{\frac{305\cancel{0}}{4\cancel{0}}-\left(\frac{-16\cancel{0}}{4\cancel{0}}\right)^2}$$

$$=\sqrt{76.25-16}$$

$$=\sqrt{60.25}$$

$$=7.76$$

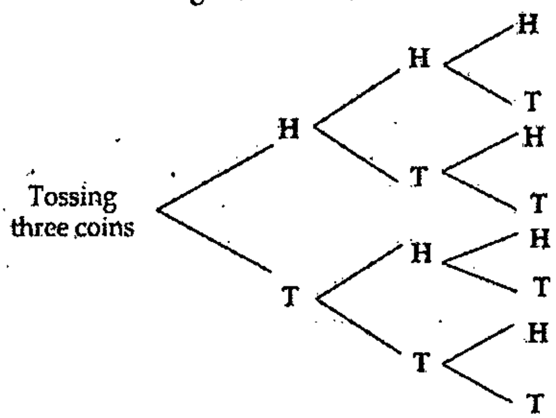
Answer : Standard deviation $\sigma \cong = 7.76$

Exercise 2

1. Write the sample space for tossing three coins using tree diagram.

Solution :

When we tossing a coin the outcomes are Head (H) and Tail (T).

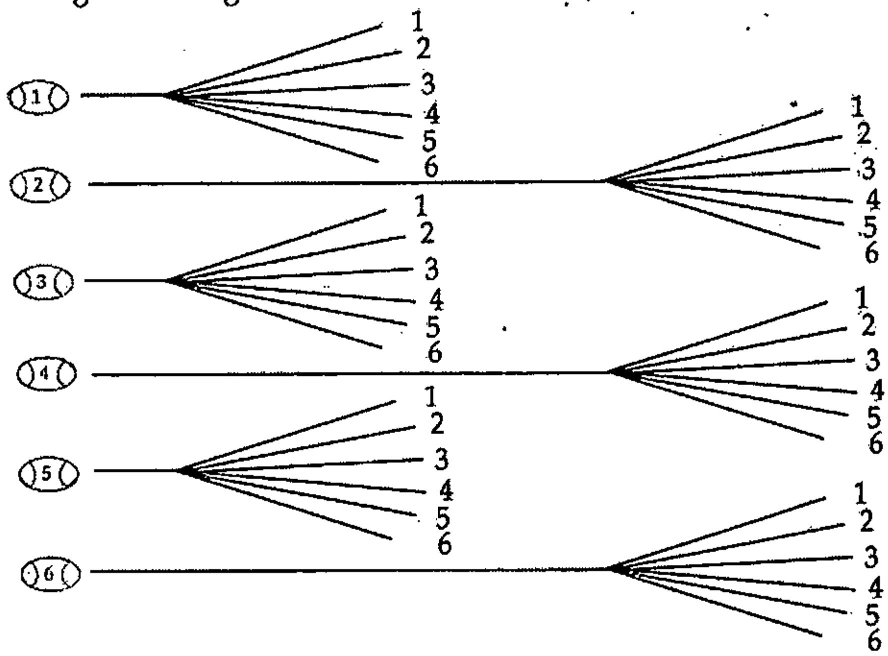


Free tree diagram sample sapce S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

2. Write the sample space for selecting two balls from a bag containing 6 balls numbered 1 to 6 (using tree diagram).

Solution :

A bag containing 6 balls numbered 1 to 6.



sample sapce S = {(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)
(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)
(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)
(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)
(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6)
(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

3. If A is an event of a random experiment such that $P(A) : P(\bar{A}) = 17:15$ and $n(S)=640$ then find (i) $P(\bar{A})$ (ii) $n(A)$.

Solution :

$P(A) : P(\bar{A}) = 17 : 15$ and $n(S) = 640$

$\therefore$ Let $P(A) = 17x$ and $P(\bar{A}) = 15x$.

we have $P(A) + P(\bar{A}) = 1$

$$17x + 15x = 1$$

$$32x = 1$$

$$x = \frac{1}{32}$$

$$\therefore P(\overline{A}) = 15x$$

$$= 15\left(\frac{1}{32}\right)$$

$$p(\overline{A}) = \frac{15}{32}$$

$$\text{and } p(A) = 17x = \frac{17}{32}$$

$$\therefore \frac{n(A)}{n(S)} = \frac{17}{}$$

$$\frac{n(A)}{640} = \frac{17}{32}$$

$$n(A) = \frac{17}{\cancel{32}} \times \cancel{640}^{20} = 340$$

Answer : $p(A) = \frac{15}{32}$ $n(A)=340$

3.B

Solution :

Ayan throws two dice once

sample sapce S = {(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)
(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)
(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)
(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)
(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6)
(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

n (S) = 36

Let A be the event of getting product of the numbers on the dice is 36.

$\therefore$ A = {(6, 6)} n (A) = 1

$$p(A) = \frac{n(A)}{n(S)} = \frac{1}{36}$$

Krishna throws one die

$\therefore S = \{1, 2, 3, 4, 5, 6\}$ $\qquad$ $n(S) = 6$

Let B be the event of getting squares the number on the die is 36.

$\therefore B = \{(6^2)\} = \{36\}$ $\qquad$ $n(B) = 1$

$$p(B) = \frac{n(B)}{n(S)} = \frac{1}{6}$$

Comparing the probability of an event A and B.

Answer : Krishna getting the better change.

4. A coin is tossed thrice. What is the probability of getting two consecutive tails?

Solution :

A coin is tossed thrice.

$S = \{HHH, HTH, HHT, THH, TTH, THT, HTT, TTT\}$

$\therefore n(S) = 8$

Let A be the event of getting two consecutive tails

$A = \{TTH, HTT, TTT\}$

$n(A) = 3$

$$p(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

Answer : P (getting two consecutive tails) $\frac{3}{8}$

5. At a fete, cards bearing numbers 1 to 1000, one number on one card are put in a box. Each player selects one card at random and that card is not replaced. If the selected card has a perfect square number greater than 500, the player wins a prize. What is the probability that (i) the first player wins a prize (ii) the second player wins a prize, if the first has won?

Solution :

$n(S) = 1000$

Let A be the event of getting a perfect square number greater than 500.

(i) $A = \{23^3, 24^2, 25^2, 26^2, 27^2, 28^2, 29^2, 30^2, 31^2\}$

$A = \{529, 576, 625, 676, 729, 784, 841, 900, 961\}$

$\therefore n(A) = 9$

$$\therefore p(A) = \frac{n(A)}{n(S)} = \frac{9}{1000}$$

(ii) The card is not replaced & Let B be the event of winning second player if the first has won.

$\therefore n(S) = 999$ and $n(B) = 8$

$$\therefore p(A) = \frac{n(A)}{n(S)} = \frac{8}{999}$$

6. A bag contains 12 blue balls and x red balls. If one ball is drawn at random (i) what is the probability that it will be a red ball? (ii) If 8 more red balls are put in the bag, and if the probability of drawing a red ball will be twice that of the probability in (i), then find x.

Solution :

A bag has 12 blue balls and x red balls.

$\therefore$ n (S) = 12 + x

(i) Let R be the event of getting red ball.

$\therefore$ n (R) = x

$$p(R) = \frac{n(R)}{n(S)} = \frac{x}{12+x}$$

(ii) 8 more red balls are put in the bag.

$\therefore$ n (S) = 20 + x

$$\therefore p(R_1) = \frac{x+8}{20+x}$$

Given P (R_1) = 2 P (R)

$$\frac{x+8}{20+x} = 2\left(\frac{x}{12+x}\right)$$

$$(12 + x)(x + 8) = 2x(20 + x)$$

$$12x + x^2 + 8x + 96 = 40x + 2x^2$$

$$2x^2 - x^2 + 40x - 20x - 96 = 0$$

$$x^2 + 20x - 96 = 0$$

$$(x + 24)(x - 4) = 0$$

$x + 24 = 0$ (or) $x - 4 = 0$

$x \neq -24$ $\qquad$ $x = 4$

Answer : Value of x = 4

(i) $p(R) = \frac{4}{16} = \frac{1}{4}$

7. Two unbiased dice are rolled once. Find the probability of getting

(i) a doublet (equal numbers on both dice)

(ii) the product as a prime number

(iii) the sum as a prime number

(iv) the sum as 1

Notes

Solution :

Two dice are rolled

sample sapce S = {(1, 1) (1, 2) (1, 3)
(1, 4) (1, 5) (1, 6)
(2, 1) (2, 2) (2, 3)
(2, 4) (2, 5) (2, 6)
(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)
(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)
(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6)
(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

n (S) = 36

(i) Let A be the event of getting a doublet

A = {(1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6)}

n (A) = 6

$$p(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(ii) Let B be the event of getting a product is a prime number

B = {(1, 2) (1, 3) (1, 5) (2, 1) (3, 1) (5, 1)}

n (B) = 6

$$p(B) = \frac{n(B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(iii) Let C be the event of getting a sum is prime number.

C = {(1, 1) (1, 2) (1, 4) (1, 6)
(2, 1) (2, 3) (2, 5) (3, 2)
(3, 4) (4, 1) (4, 3) (5, 2)
(5, 6) (6, 1) (6, 5)}

n (C) = 15

(iv) Let D be the event of getting a sum is 1.

$\because$ D is an impossible event D = { }

$\therefore$ P (D) = 0

8. Three fair coins are tossed together. Find the probability of getting

(i) all heads

(ii) atleast one tail

(iii) atmost one head

(iv) atmost two tails

Solution :

Three fair coins are tossed

$S = \{HHH, \ HHT, \ HTH, \ THH, \ HTT, THT, TTH, TTT\}$

$n(S) = 8$

(i) Let A be the event of getting all heads

$A = \{HHH\} \quad n(H) = 1$

$$p(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$$

(ii) Let B be the event of getting atleast one tail

$B = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$n(B) = 7$

$$p(B) = \frac{n(B)}{n(S)} = \frac{7}{8}$$

(iii) Let C be the event of getting atmost one head.

$C = \{HTT, THT, TTH, TTT\}$

$n(C) = 4$

$$p(C) = \frac{n(C)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

(iv) Let D be the event of getting atmost two tails

$D = \{HHH, HHT, HTH, THH, TTH, THT, HTT\}$

$\therefore n(D) = 7$

$$p(D) = \frac{n(D)}{n(S)} = \frac{7}{8}$$

9. Two dice are numbered 1,2,3,4,5,6 and 1,1,2,2,3,3 respectively. They are rolled and the sum of the numbers on them is noted. Find the probability of getting each sum from 2 to 9 separately.

Solution :

$S = \{(1, 1)\ (1, 1)\ (1, 2)\ (1, 2)\ (1, 3)\ (1, 3)$
$(2, 1)\ (2, 1)\ (2, 2)\ (2, 2)\ (2, 3)\ (2, 3)$
$(3, 1)\ (3, 1)\ (3, 2)\ (3, 2)\ (3, 3)\ (3, 3)$
$(4, 1)\ (4, 1)\ (4, 2)\ (4, 2)\ (4, 3)\ (4, 3)$
$(5, 1)\ (5, 1)\ (5, 2)\ (5, 2)\ (5, 3)\ (5, 3)$
$(6, 1)\ (6, 1)\ (6, 2)\ (6, 2)\ (6, 3)\ (6, 3)\}$

$n(S) = 36$

Let A_1 be the event of getting sum 2.

$A_1 = \{(1, 1)\ (1, 1)\}$ $\quad n(A_1) = 2$

$$p(A_1) = \frac{n(A_1)}{n(S)} = \frac{2}{36} = \frac{1}{18}$$

Let A_2 be the event of getting sum 3.

$A_2 = \{(1, 2)\ (1, 2)\ (2, 1)\ (2, 1)\}$ $\quad n(A_2) = 4$

$$p(A_2) = \frac{n(A_2)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

Let A_3 be the event of getting a sum 4.

$A_3 = \{(2, 2)\ (2, 2)\ (3, 1)\ (3, 1)\ (1, 3)\ (1, 3)\}$ $\quad n(A_3) = 6$

$$p(A_3) = \frac{n(A_3)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

Let A_4 be the event of getting a sum 5.

$A_4 = \{(2, 3)\ (2, 3)\ (3, 2)\ (3, 2)\ (4, 1)\ (4, 1)\}$ $\quad n(A_4) = 6$

$$p(A_4) = \frac{n(A_4)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

Let A_5 be the event of getting a sum 6.

$A_5 = \{(3, 3)\ (3, 3)\ (4, 2)\ (4, 2)\ (5, 1)\ (5, 1)\}$ $\quad n(A_5) = 6$

$$p(A_5) = \frac{n(A_5)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

Let A_6 be the event of getting a sum 7.

$A_6 = \{(4, 3)\ (4, 3)\ (5, 2)\ (5, 2)\ (6, 1)\ (6, 1)\}$ $\quad n(A_6) = 6$

$$p(A_6) = \frac{n(A_6)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

Let A_7 be the event of getting a sum 8.

$A_7 = \{(5, 3)\ (5, 3)\ (6, 2)\ (6, 2)\}$ $\quad n(A_7) = 4$

$$p(A_7) = \frac{n(A_7)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

Let A_8 be the event of getting a sum 9.

$A_8 = \{(6, 3)\ (6, 3)\}$ $\quad n(A_8) = 2$

$$p(A_8) = \frac{n(A_8)}{n(S)} = \frac{2}{36} = \frac{1}{18}$$

10. A bag contains 5 red balls, 6 white balls, 7 green balls, 8 black balls. One ball is drawn at random from the bag. Find the probability that the ball drawn is

(i) White

(ii) black or red

(iii) not white

(iv) neither white nor black

Solution :

A bag contains 5 red balls, 6 white balls, 7 green balls, and 8 black balls.

$\therefore n(S) = 5 + 6 + 7 + 8 = 26$

(i) Let A be the event of getting 'white balls'

$n(A) = 6$

$$p(A) = \frac{n(A)}{n(S)} = \frac{6}{26} = \frac{3}{13}$$

(ii) Let B be the event of getting 'black or red'

$n(B) = 8 + 5 = 13$

$$p(B) = \frac{n(B)}{n(S)} = \frac{13}{26} = \frac{1}{2}$$

(iii) Let C be the event of getting 'nota white ball'

$n(C) = 26 - 6 = 20$

$$p(C) = \frac{n(C)}{n(S)} = \frac{20}{26} = \frac{10}{13}$$

(iv) Let D be the event of getting 'neither white nor black'

$n(D) = 5 + 7 = 12$

$$p(D) = \frac{n(D)}{n(S)} = \frac{12}{36} = \frac{6}{13}$$

1 MATRICES

- Understand the concept of matrices.
- Discuss the types of matrices.
- Understand the determinants.
- Discuss the properties of determinants.
- Discuss the concept of inverse matrices.
- Discuss the classification of matrices.
- Discuss the types of determinants.
- Understand the practical application of matrices.

Objective of the Module:

The basic objective of this chapter is to through some light on the initial concepts of matrices and its determinants so that the practical application of matrices can be examined in detailed.

Definition of Matrices and Classification

A rectangular array of symbols (which could be real or complex numbers) along rows and columns is called a matrix.

Thus, a system m × n symbols arranged in a rectangular formation along m rows and n columns and bounded by the brackets [.] is called an m by n matrix (which is written as m x n matrix)

i.e. $A = \begin{bmatrix} a_{11} & a_{12} \cdots\cdots\cdots & a_{1n} \\ a_{21} & a_{13} \cdots\cdots\cdots & a_{21} \\ -- & ------ & -- \\ -- & ------ & -- \\ a_{m1} & a_{m2} \cdots\cdots\cdots & a_{m3} \end{bmatrix}$ is a matrix of order m × n.

In a compact form the above matrix is represented by A = [aij], 1 < i < m,

1 < j < n, where is, j ε i or simply [aij] m × n.

The numbers a11, a12, ... etc of this rectangular array are called the elements of the matrix. The element aij belongs to the ith row and the jth column and is called the (i, j)th element of the matrix.

Equal Matrices

Two matrices are said to be equal if they have the same order and each element of one is equal to the corresponding element of the other.

Classification of Matrices

Row Matrix

A matrix having a single row is called a row matrix. e.g. [1 3, 5, 7]

Column Matrix

A matrix having a single column is called a column matrix. e.g., $\begin{bmatrix}2\\3\\5\end{bmatrix}$.

Square Matrix

An m x n matrix A is said to be a square matrix if m = n i.e., number of rows = number of columns.

For Example: $A = \begin{bmatrix}1 & 2 & 3\\2 & 3 & 4\\3 & 4 & 5\end{bmatrix}$ is a square matrix of order 3×3.

Note: In a square matrix the diagonal from left hand side upper corner to right hand side lower corner is known as leading diagonal or principal diagonal. In the above example square matrix containing the elements 1, 3, 5 is called the leading or principal diagonal.

Traces of a Matrix

The sum of the elements of a square matrix A lying along the principal diagonal is called the trace of A i.e., tr(A). Thus, if $A = [a_{ij}]_{n\times n}$,

then $tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ... + a_{nn}$.

Diagonal Matrix

A square matrix all of whose elements except those in the leading diagonal are zero is called a diagonal matrix. For a square matrix $A = [a_{ij}]_{n\times n}$ to be a diagonal matrix, $a_{ij} = 0$, whenever $i \neq j$.

For example: $A = \begin{bmatrix}3 & 0 & 0\\0 & 5 & 0\\0 & 0 & -1\end{bmatrix}$ is a diagonal matrix of order 3×3.

Note: Here A can also be represented as diag (3,5 -1).

Scalar Matrix

A diagonal matrix who's all the elements are equal is called a scalar matrix.

For a square matrix $A = [a_{ij}]_{n\times n}$ to be a scalar matrix, $a_{ij} = \begin{cases}0, & i \neq j\\ m, & i = j\end{cases}$, where $m \neq 0$.

For example: $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix.

Unit Matrix or Identity Matrix

A diagonal matrix of order n which has unity for all its elements, is called a unit matrix of order n and is denoted by In.

Thus a square matrix $A = [a_{ij}]_{n \times n}$ is a unit matrix if $a_{ij} = \begin{cases} 1, & i = j \\ 0. & i \neq i. \end{cases}$

For example :

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}.$

Triangular Matrix

A square matrix in which all the elements below the principal diagonal are zero is called Upper Triangular matrix and a square matrix in which all the elements above the principal diagonal are zero is called Lower Triangular matrix.

Given a square matrix $A = [a_{ij}]_{n\times n}$,

for upper triangular matrix, $a_{ij} = 0$, $i > j$

and for lower triangular matrix, $a_{ij} = 0$, $i < j$.

Note:

Diagonal matrix is both upper and lower triangular

A triangular matrix $A = [a_{ij}]_{n\times n}$ is called strictly triangular if $a_{ii} = 0$ for $1 < i < n$.

For Example:

$\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 4 \end{bmatrix}$ are respectively upper and lower triangular matrices.

Null Matrix

If all the elements of a matrix (square or rectangular) are zero, it is called a null or zero matrix.

For $A = [a_{ij}]$ to be null matrix, $a_{ij} = 0 \ \forall \ i, j$.

For Example:

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix.

Transpose of a Matrix

The matrix obtained from any given matrix A, by interchanging its rows and columns, is called the transpose of A and is denoted by A'.

If $A = [a_{ij}]_{m\times n}$ and $A' = [b_{ij}]_{n\times m}$ then $b_{ij} = a_{ij}$, $\forall$ i, j.

For Example:

If $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}_{3\times 2}$, then $A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}_{2\times 3}$

Properties of Transpose

(i) $(A')' = A$

(ii) $(A + B)' = A' + B'$, A and B being conformable matrices

(iii) $(\alpha A)' = \alpha A'$, α being scalar

(iv) $(AB)' = B'A'$, A and B being conformable for multiplication

Conjugate of a Matrix

The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\bar{A}$.

For Example:

$A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$. Then, $\bar{A} = \begin{bmatrix} 1-2i & 2+3i & 3-4i \\ 4+5i & 5-6i & 6+7i \\ 8 & 7-8i & 7 \end{bmatrix}$

Properties of Conjugate

(i) $\overline{(\bar{A})} = A$

(ii) $\overline{(A+B)} = \bar{A} + \bar{B}$

(iii) $\overline{(\alpha A)} = \bar{\alpha}\bar{A}$, α being any number real or complex

(iv) $\overline{(AB)} = \bar{A}\ \ \bar{B}$, A and B being conformable for multiplication

Transpose Conjugate of a Matrix

The transpose of the conjugate of a matrix is called transposed conjugate of A and is denoted by Aq. The conjugate of the transpose of A is the same as the transpose of the conjugate of A i.e.

$\overline{(A')} = \overline{(A)}' = A\Theta$.

If $A = [aij]m\times n$, then $A\Theta = [bij]n\times m$ where $bji = \overline{a_{ij}}$

i.e. the (j, i)th element of $A\Theta$ = the conjugate of (i, j)th element of A.

For Example:

If $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$ then $A\Theta = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$.

Properties of Transpose Conjugate

(i) $(A^{\Theta})^{\Theta} = A$

(ii) $(A + B)^{\Theta} = A^{\Theta} + B^{\Theta}$

(iii) $(kA)^{\Theta} = \bar{k}\, A^{\Theta}$, k being any number

(iv) $(AB)^{\Theta} = B^{\Theta}A^{\Theta}$

Addition and Subtraction of Matrices

Any two matrices can be added if they are of the same order and the resulting matrix is of the same order. If two matrices A and B are of the same order, they are said to be conformable for addition.

For example:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} + \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 + c_1 & b_1 + d_1 \\ a_2 + c_2 & b_2 + d_2 \\ a_3 + c_3 & b_3 + d_3 \end{bmatrix}.$$

Similarly, $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} - \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 - c_1 & b_1 - d_1 \\ a_2 - c_2 & b_2 - d_2 \\ a_3 - c_3 & b_3 - d_3 \end{bmatrix}.$$

Note:

Only matrices of the same order can be added or subtracted.

Addition of matrices is commutative as well as associative.

Cancellation laws hold well in case of addition.

The equation A + X = 0 has a unique solution in the set of all m × n matrices.

Scalar Multiplication

The matrix obtained by multiplying every element of a matrix A by a scalar λ is called the multiple of A by λ and its denoted by λA i.e. if $A = [aij]$ then $\lambda A = [\lambda aij]$.

For example:

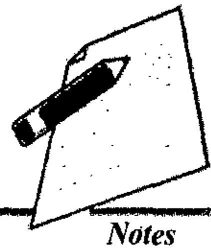
$$A = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 7 & 8 \end{bmatrix}_{2\times 3}.$$

$$\text{Thus, } 2A = \begin{bmatrix} 4 & 6 & 10 \\ 12 & 14 & 16 \end{bmatrix}_{2\times 3}.$$

Note: All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalar.

Multiplication of Matrices

Two matrices can be multiplied only when the number of columns in the first, called the pre factor, is equal to the number of rows in the second, called the post factor. Such matrices are said to be conformable for multiplication.



$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}_{n\times p} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mp} \end{bmatrix}_{m\times p}$$

where $c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj} = \sum_{k=1}^{n} a_{ik} b_{ki}$ $\forall$ $i = 1, 2, 3 \dots\dots, m$ and $j = 1, 2, 3 \dots\dots, p$.

Properties of Multiplication

Matrix multiplication may or may not be commutative. i.e., AB may or may not be equal to BA

If AB = BA, then matrices A and B are called Commutative Matrices.

If AB = BA, then matrices A and B are called Anti-Commutative Matrices.

Matrix multiplication is Associative

Matrix multiplication is Distributive over Matrix Addition.

Cancellation Laws not necessary hold in case of matrix multiplication i.e., if AB = AC => B = C even if A $\neq$ 0.

AB = 0 i.e., Null Matrix, does not necessarily imply that either A or B is a null matrix.

Illustration:

$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$, show that $AB \neq BA$.

Solution:

Here $A.B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2+6+1 & 4+3+3 \\ 1+6+2 & 2+3+6 \end{bmatrix} = \begin{bmatrix} 9 & 10 \\ 9 & 11 \end{bmatrix}$

and $B.A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 2+2 & 3+6 & 1+4 \\ 4+1 & 6+3 & 2+2 \\ 2+3 & 3+9 & 1+6 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 5 \\ 5 & 9 & 5 \\ 5 & 12 & 7 \end{bmatrix}$.

Thus $A.B \neq B.A$.

Illustration:

If $A\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then find $A\alpha A\beta$.

Solution:

We have

$$A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = A_{\alpha+\beta}.$$

Special Matrices

Symmetric and Skew Matrices

A square matrix A = [aij] is said to be symmetric when aij = aij for all i and j. If aij = -aij for all i and j and all the leading diagonal elements are zero, then the matrix is called a skew symmetric matrix.

For example:

$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix and $\begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$ is a skew-symmetric matrix?

Hermitian and Skew - Hermitian Matrices

A square matrix A = [aij] is said to be Hermitian matrix if aij = $\overline{a}$ ij, $\forall$ i, j i.e. Aθ = -A.

For example:

$\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}, \begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$ are Hermitian matrices.

Note:

* If A is a Hermitian matrix then aii = $\overline{a}$ ij is a real $\forall$ i. Thus every diagonal element of a Hermitian Matrix must be real.
* A Hermitian matrix over the set of real numbers is actually a real symmetric matrix.

And a square matrix, A = [aij] is said to be a skew-Hermitian if aij = -$\overline{a}$ ij, $\forall$ i, j i.e. Aθ = -A.

For example:

$\begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$ are skew-Hermitian matrices.

Note:

* If A is a skew-Hermitian matrix then aii = $\overline{a}$ ij => aii + $\overline{a}$ ii = 0, i.e. aii must be purely imaginary or zero.
* A skew-Hermitian Matrix over the set of real numbers is actually a real skew-symmetric matrix.

Singular and Non-singular Matrices

Any square matrix A is said to be non-singular if |A| ≠ 0, and a square matrix A is said to be singular if |A| = 0. Here |A| (or det(A) or simply det A) means corresponding determinants of square matrix A e.g. if

$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ 10 - 12 = -2 => A is a non-singular matrix.

Unitary Matrix

A square matrix is said to be unitary if $\bar{A}'A = I$. Since $|\bar{A}'| = |A|$ and $|\bar{A}'A| = |\bar{A}'|$, we have $|\bar{A}'||A| = 1$.

Thus the determinant of a unitary matrix is of unit modulus. For a matrix to be unitary it must be non-singular.

Hence, $\bar{A}'A = I \Rightarrow A\bar{A}' = I$.

Orthogonal Matrix

Any square matrix A of order n is said to be orthogonal if AA' = A'A = In.

Idempotent Matrix

A square matrix A is called idempotent provided it satisfies the relation A2 = A.

For example:

The matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent as

$$A2 = A.A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} . \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A.$$

Involuntary Matrix

A matrix such that A2 = I is called involuntary matrix.

Nilpotent Matrix

A square matrix A is called a nilpotent matrix if there exists a positive integer m such that Am = O. If m is the least positive integer such that Am = O, then m is called the index of the nilpotent matrix A

Illustration:

Suppose a, b, c are real numbers such that abc = 1. If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ is such that A›A = I, then find the value of a3 + b3 + c3.

Solution:

Note A› = A.

Thus, I = A›A = AA = A2

|A2| = |A2| = |I| = 1

|A| = + 1. But |A| = a3 + b3 + c3 - 3abc.

Thus, a3 + b3 + c3 - 3abc = + 1

=> a3 + b3 + c3 = 4, 2.

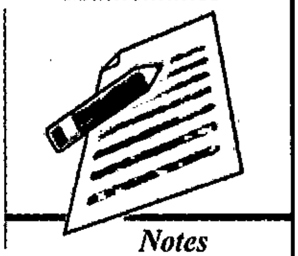


Illustration:

If $\omega \neq 1$ is a cube root of unity, then show that

$A = \begin{bmatrix} 1+2\omega^{100}+\omega^{200} & \omega^2 & 1 \\ 1 & 1+\omega^{100}+2\omega^{200} & \omega \\ \omega & \omega^2 & 2+\omega^{100}+2\omega^{200} \end{bmatrix}$ is singular matrix.

Solution:

$$A = \begin{bmatrix} 1+2\omega^{100}+\omega^{200} & \omega^2 & 1 \\ 1 & 1+\omega^{100}+2\omega^{200} & \omega \\ \omega & \omega^2 & 2+\omega^{100}+2\omega^{200} \end{bmatrix}$$

$$= \begin{bmatrix} \omega & \omega^2 & 1 \\ 1 & 1 & \omega \\ \omega & \omega^2 & -\omega \end{bmatrix} \Rightarrow |A| = \omega \begin{vmatrix} \omega & \omega & 1 \\ 1 & 1 & \omega \\ \omega & \omega & -\omega \end{vmatrix}$$

Hence A is singular matrix.

Illustration:

Show that the matrix = $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of index 3.

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$\Rightarrow A^3 = A^2.A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

=> A3 = 0 i.e. Ak = 0. Here k = 3.

Hence A is nilpotent matrix of index 3.

Adjoint of a Square Matrix

Let A = [aij] be a square matrix of order n and let Cij be cofactor of aij in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix};$$

where Cij denotes the cofactor of aij in A.

For example: $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, C11 = s, C12 = -r, C21 = -q, C22 = p

=> adj $A = \begin{bmatrix} s & -r \\ -q & p \end{bmatrix}^T = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$.

Theorem: Let A be a square matrix of order n. Then A(adj A) = |A| In = (adj A)A.

Proof: Let A = [aij], and let Cij be cofactor of aij in A. Then

(adj A)ij = Cij $\forall$ 1 < i, j , n.

Now, (A(adj A))ij = $\sum$nr=1(A)ir (adj A)rj = $\sum$nr=1 aij Cjr $\begin{cases} |A|, & if\ i = i \\ 0, & if\ i \neq j \end{cases}$

$$\Rightarrow A\,(adj\ A) = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = |A|\ In.$$

Similarly ((adj A)A)ij= $\sum$nr=1(adj A)ir (A)rj= $\sum$nr=1Cri arj= $\begin{cases} |A|, & if\ i = i \\ 0, & if\ i \neq j \end{cases}$

Hence, A(adj A) = |A| In = (adj A)A.

Note: The adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing the signs of off-diagonal (left hand side lower corner to right hand side upper corner) elements.

2 INVERSE OF A MATRIX AND ITS APPLICATION

Introduction

A non-singular square matrix of order n is invertible if there exists a square matrix B of the same order such that AB = In = BA.

In such a case, we say that the inverse of A is B and we write, A-1 = B.

The inverse of A is given by A-1 = 1/|A|. adj A.

Properties of Inverse of a Matrix

(i) Every invertible matrix possesses a unique inverse.

(ii) (Reversal law) If A and B are invertible matrices of the same order, then AB is invertible and (AB)-1 = B-1 A-1.

In general, if A,B,C,...are invertible matrices then (ABC....)-1 =..... C-1 B-1 A-1.

(iii) If A is an invertible square matrix, then AT is also invertible and (AT)-1 = (A-1)T.

(iv) If A is a non-singular square matrix of order n, then |adj A| = |A|n-1.

(v) If A and B are non-singular square matrices of the same order, then adj (AB) = (adj B) (adj A).

(vi) If A is an invertible square matrix, then adj(AT) = (adj A)T.

(vii) If A is a non-singular square matrix, then adj(adjA) = |A|n-1A.

Elementary Operations of a Matrix

The following three operations applied on the rows (columns) of a matrix are called elementary row (column) transformations.

(i) Interchange of any two rows (columns)

If ith row (column) of a matrix is interchanged with the jth row (column), it will be denoted by Ri ↔ Rj (Ci ↔ Cj).

For example: $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, then by applying R1 → R2 we get $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$.

(ii) Multiplying all elements of a row (column) of a matrix by a non-zero scalar. If the elements of ith row (column) are multiplied by non-zero scalar k, it will be denoted by Rl→Ri (k) [Ci→Ci (k)] or Rl→kRi [Ci→kCi].

If $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$, then by applying R2 → 3R2, we obtain $B = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$.

(iii) Adding to the elements of a row (column), the corresponding elements of any other row (column) multiplied by any scalar k.

If k times the elements of jth row (column) are added to the corresponding elements of the ith row (column), it will be denoted by $R_i \to R_i + k R_j$ ($C_i \to C_i + kC_j$).

If $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$, then application of elementary operation $R3 \to R3 + 2R1$ lead to $B = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 4 & 3 & 9 & 3 \end{bmatrix}$.

Illustration:

Using elementary row transformations and find the inverse of the matrix $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$.

Solution:

We write A = IA

$$\text{or } \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad (R_1 \to R_1 - R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} A \quad (R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} A \quad (R_2 \to \tfrac{1}{2} R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5 & 6 & 1 \end{bmatrix} A \quad (R_1 \to R_1 + R_2 \text{ and } R_3 \to R_3 + 2R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A \quad (R_3 \to \tfrac{1}{4} R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A \quad (R_1 \to R_1 + \tfrac{1}{2} R_3 \text{ and } R_2 \to R_2 - \tfrac{1}{2} R_3)$$

$$\text{Hence } A^{-1} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix}.$$

Adjoint of a Square Matrix

We recall the properties of the cofactors of the elements of a square matrix. Let A be a square matrix of by order n whose determinant is denoted | A | or det (A). Let aij be the element sitting at the intersection of the ith row and jth column of A. Deleting the ith row and jth column of A, we obtain a sub-matrix of order (n −1). The determinant of this sub-matrix is called minor of the element aij . It is denoted by Mij .The product of Mij and (−1)i+ j is called cofactor of the element aij . It is denoted by Aij. Thus the cofactor of aij is Aij = (−1)i+j Mij.

An important property connecting the elements of a square matrix and their cofactors is that the sum of the products of the entries (elements) of a row and the corresponding cofactors of the elements of the same row is equal to the determinant of the matrix; and the sum of the products of the entries (elements) of a row and the corresponding cofactors of the elements of any other row is equal to 0. That is,

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} |A| & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

where |A| denotes the determinant of the square matrix A. Here |A| is read as "determinant of A " and not as " modulus of A ". Note that | A| is just a real number and it can also be negative. For instance, we have

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 2(1-2) - 1(1-2) + 1(2-2) = -2 + 1 + 0 = -1.$$

Definition

Let A be a square matrix of order n. Then the matrix of cofactors of A is defined as the matrix obtained by replacing each element aij of A with the corresponding cofactor Aij. The adjoint matrix of A is defined as the transpose of the matrix of cofactors of A. It is denoted by adj A.

Note

adj A is a square matrix of order n and adj A = [Aij]T = [(−1)i + j Mij]T

In particular, adj A of a square matrix of order 3 is given below:

$$\text{adj } A = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}.$$

Theorem

For every square matrix A of order n , A(adj A) = (adj A) A = |A| In .

Proof

For simplicity, we prove the theorem for n = 3 only.

Consider $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then, we get

$$a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}=|A|,\quad a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{23}=0,\quad a_{11}A_{31}+a_{12}A_{32}+a_{13}A_{33}=0;$$
$$a_{21}A_{11}+a_{22}A_{12}+a_{23}A_{13}=0,\quad a_{21}A_{21}+a_{22}A_{22}+a_{23}A_{23}=|A|,\quad a_{21}A_{31}+a_{22}A_{32}+a_{23}A_{33}=0;$$
$$a_{31}A_{11}+a_{32}A_{12}+a_{33}A_{13}=0,\quad a_{31}A_{21}+a_{32}A_{22}+a_{33}A_{23}=0,\quad a_{31}A_{31}+a_{32}A_{32}+a_{33}A_{33}=|A|.$$

By using the above equations, we get

$$A(\text{adj}A)=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}=\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}=|A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=|A|I_3 \qquad \ldots(1)$$

$$(\text{adj}A)A=\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}=|A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=|A|I_3, \qquad \ldots(2)$$

where I3 is the identity matrix of order 3.

So, by equations (1) and (2), we get

A(adj A) = (adj A) A = |A| I3.

Note

If A is a singular matrix of order n , then | A | = 0 and so A(adj A) = (adj A) A = On , where On denotes zero matrix of order n.

Example

If A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, verify that A (adj A) = (adj A) A = | A | I3 .

Solution

We find that $|A|=\begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}=8(21-16)+6(-18+8)+2(24-14)=40-60+20=0.$

By the definition of adjoint, we get

$$\text{adj}\,A=\begin{bmatrix} (21-16) & -(-18+8) & (24-14) \\ -(-18+8) & (24-4) & -(-32+12) \\ (24-14) & -(-32+12) & (56-36) \end{bmatrix}^T=\begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}.$$

So, we get

$$A(\text{adj } A) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 40-60+20 & 80-120+40 & 80-120+40 \\ -30+70-40 & -60+140-80 & -60+140-80 \\ 10-40+30 & 20-80+60 & 20-80+60 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I_3 = |A|I_3.$$

Similarly, we get

$$(\text{adj } A)A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 40-60+20 & -30+70-40 & 10-40+30 \\ 80-120+40 & -60+140-80 & 20-80+60 \\ 80-120+40 & -60+140-80 & 20-80+60 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I_3 = |A|I_3.$$

Hence, $A(\text{adj } A) = (\text{adj } A)A = |A|I_3$.

Definition of inverse matrix of a square matrix

Now, we define the inverse of a square matrix.

Definition

Let A be a square matrix of order n. If there exists a square matrix B of order n such that AB = BA = In , then the matrix B is called an inverse of A.

Theorem

If a square matrix has an inverse, then it is unique.

Proof

Let A be a square matrix order n such that an inverse of A exists. If possible, let there be two inverses B and C of A. Then, by definition, we have AB = BA = In and AC = CA = In

Using these equations, we get

C = CIn = C(AB) = (CA)B = InB = B.

Hence the uniqueness follows.

Notation The inverse of a matrix A is denoted by A−1.

Note

AA−1 = A−1 A = In .

Theorem

Let A be square matrix of order n. Then, A−1 exists if and only if A is non-singular.

Proof

Suppose that A−1 exists. Then AA−1 = A−1 A = In .

By the product rule for determinants, we get

det(AA−1) = det(A) det(A−1) = det(A−1) det(A) = det(In) = 1. So, |A| = det(A) $\neq$ 0.

Hence A is non-singular.

Conversely, suppose that A is non-singular.

Then |A | $\neq$ 0. By Theorem 1.1, we get

A(adj A) = (adj A) A = |A| In.

$$A(\text{adj } A) = (\text{adj } A)A = |A|I_n.$$

$$\text{So, dividing by } |A|, \text{ we get } A\left(\frac{1}{|A|}\text{adj } A\right) = \left(\frac{1}{|A|}\text{adj } A\right)A = I_n.$$

Thus, we are able to find a matrix B = 1/|A| adj A such that AB = BA = In .

Hence, the inverse of A exists and it is given by $A^{-1} = \frac{1}{|A|}\textbf{adj } A.$

Remark

The determinant of a singular matrix is 0 and so a singular matrix has no inverse.

Example

If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular, find A−1

Solution

We first find adj A. By definition, we get

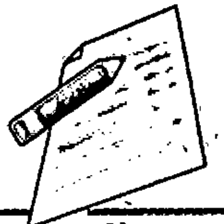
$$\text{adj } A = \begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$\text{Since } A \text{ is non-singular, } |A| = ad - bc \neq 0.$$

$$\text{As } A^{-1} = \frac{1}{|A|}\text{adj } A, \text{ we get } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example

Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}.$



Solution

$$\text{Let } A=\begin{bmatrix}2&-1&3\\-5&3&1\\-3&2&3\end{bmatrix}.\ \text{Then } |A|=\begin{vmatrix}2&-1&3\\-5&3&1\\-3&2&3\end{vmatrix}=2(7)+(-12)+3(-1)=-1\neq 0.$$

Therefore, A^{-1} exists. Now, we get

$$\operatorname{adj}A=\begin{bmatrix}+\begin{vmatrix}3&1\\2&3\end{vmatrix} & -\begin{vmatrix}-5&1\\-3&3\end{vmatrix} & +\begin{vmatrix}-5&3\\-3&2\end{vmatrix}\\ -\begin{vmatrix}-1&3\\2&3\end{vmatrix} & +\begin{vmatrix}2&3\\-3&3\end{vmatrix} & -\begin{vmatrix}2&-1\\-3&2\end{vmatrix}\\ +\begin{vmatrix}-1&3\\3&1\end{vmatrix} & -\begin{vmatrix}2&3\\-5&1\end{vmatrix} & +\begin{vmatrix}2&-1\\-5&3\end{vmatrix}\end{bmatrix}^T=\begin{bmatrix}7&12&-1\\9&15&-1\\-10&-17&1\end{bmatrix}^T=\begin{bmatrix}7&9&-10\\12&15&-17\\-1&-1&1\end{bmatrix}.$$

$$\text{Hence, } A^{-1}=\frac{1}{|A|}(\operatorname{adj}A)=\frac{1}{(-1)}\begin{bmatrix}7&9&-10\\12&15&-17\\-1&-1&1\end{bmatrix}=\begin{bmatrix}-7&-9&10\\-12&-15&17\\1&1&-1\end{bmatrix}.$$

Properties of inverses of matrices

We state and prove some theorems on non-singular matrices.

Theorem

If A is non-singular, then

$$\text{(i) } |A^{-1}|=\frac{1}{|A|}\quad \text{(ii) } \left(A^T\right)^{-1}=\left(A^{-1}\right)^T\quad \text{(iii) } (\lambda A)^{-1}=\frac{1}{\lambda}A^{-1},$$

, where λ is a non-zero scalar.

Proof

Let A be non-singular. Then |A| ≠ 0 and A−1 exists. By definition,

$$AA^{-1}=A^{-1}A=I_n. \qquad \text{...(1)}$$

(i) By (1), we get $|AA^{-1}|=|A^{-1}A|=|I_n|$.

Using the product rule for determinants, we get $|A||A^{-1}|=|I_n|=1$.

Hence, $|A^{-1}|=\dfrac{1}{|A|}$.

(ii) From (1), we get $\left(AA^{-1}\right)^T=\left(A^{-1}A\right)^T=(I_n)^T$.

Using the reversal law of transpose, we get $\left(A^{-1}\right)^T A^T=A^T\left(A^{-1}\right)^T=I_n$. Hence $\left(A^T\right)^{-1}=\left(A^{-1}\right)^T$.

(iii) Since λ is a non-zero scalar, from (1), we get $(\lambda A)\left(\frac{1}{\lambda}A^{-1}\right)=\left(\frac{1}{\lambda}A^{-1}\right)(\lambda A)=I_n$.

So, $(\lambda A)^{-1}=\dfrac{1}{\lambda}A^{-1}$.

Theorem (Left Cancellation Law)

Let A, B, and C be square matrices of order n. If A is non-singular and AB = AC, then B = C.

Proof

Since A is non-singular, A−1 exists and AA−1 = A−1 A = In . Taking AB = AC and pre-multiplying both sides by A−1, we get A−1 (AB) = A−1 (AC). By using the associative property of matrix multiplication and property of inverse matrix, we get B = C.

Theorem (Right Cancellation Law)

Let A, B, and C be square matrices of order n. If A is non-singular and BA = CA, then B = C.

Proof

Since A is non-singular, A−1 exists and AA−1 = A−1 A = In . Taking BA = CA and post-multiplying both sides by A−1, we get (BA) A−1 = (CA) A−1. By using the associative property of matrix multiplication and property of inverse matrix, we get B = C.

Note

If A is singular and AB = AC or BA = CA, then B and C need not be equal. For instance, consider the following matrices:

$$A=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}, B=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} \text{ and } C=\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}.$$

We note that |A| = 0 and AB = AC ; . but B $\neq$ C

Theorem (Reversal Law for Inverses)

If A and B are non-singular matrices of the same order, then the product AB is also non-singular and (AB)−1 = B−1 A−1.

Proof

Assume that A and B are non-singular matrices of same order n. Then,| A | $\neq$ 0, | B | $\neq$ 0, both A−1 and B−1 exist and they are of order n. The products AB and B−1 A−1 can be found and they are also of order n. Using the product rule for determinants, we get |AB| =| A || B |$\neq$ 0. So, AB is non-singular and

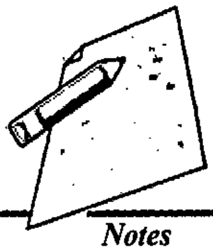
(AB)(B-1 A-1) = (A(BB-1)) A-1 = (AIn) A-1 = AA-1 = In ;

(B-1 A-1)(AB) = (B-1 (A-1 A))B = (B-1In)B = B-1B = In .

Hence (AB)−1 = B−1 A−1.

Theorem (Law of Double Inverse)

If A is non-singular, then A−1 is also non-singular and (A−1)−1 = A.



Notes

Proof

Assume that A is non-singular. Then |A | ≠ 0, and A−1 exists.

Now, |A-1| = 1/|A| ≠ 0

□ A-1 is also non-singular, and AA-1 = A-1A = I

Now, A-1A = I □ (AA-1)-1 = I □ (A-1)-1 A-1 = I.

Post-multiplying by A on both sides of equation (1), we get (A-1)-1 =A.

Theorem

If A is a non-singular square matrix of order n , then

(i) $(\text{adj } A)^{-1} = \text{adj}(A^{-1}) = \frac{1}{|A|}A$ (ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$ (iv) $\text{adj}(\lambda A) = \lambda^{n-1}\text{adj}(A)$, λ is a nonzero scalar

(v) $|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}$ (vi) $(\text{adj } A)^T = \text{adj}(A^T)$

Proof

Since A is a non-singular square matrix, we have |A| ≠ 0 and so, we get

(i) $A^{-1} = \frac{1}{|A|}(\text{adj } A) \Rightarrow \text{adj } A = |A| A^{-1} \Rightarrow (\text{adj } A)^{-1} = (|A|A^{-1})^{-1} = \frac{1}{|A|}(A^{-1})^{-1} = \frac{1}{|A|}A$.

Replacing A by A^{-1} in $\text{adj } A = |A|A^{-1}$, we get $\text{adj}(A^{-1}) = |A^{-1}|(A^{-1})^{-1} = \frac{1}{|A|}A$.

Hence, we get $(\text{adj } A)^{-1} = \text{adj}(A^{-1}) = \frac{1}{|A|}A$.

(ii) $A(\text{adj } A) = (\text{adj } A)A = |A| I_n \Rightarrow \det(A(\text{adj } A)) = \det((\text{adj } A)A) = \det(|A| I_n)$

$\Rightarrow |A||\text{adj } A| = |A|^n \Rightarrow |\text{adj } A| = |A|^{n-1}$.

(iii) For any non-singular matrix B of order n, we have $B(\text{adj } B) = (\text{adj } B)B = |B| I_n$.

Put $B = \text{adj } A$. Then, we get $(\text{adj } A)(\text{adj}(\text{adj } A)) = |\text{adj } A| I_n$.

So, since $|\text{adj } A| = |A|^{n-1}$, we get $(\text{adj } A)(\text{adj}(\text{adj } A)) = |A|^{n-1} I_n$.

Pre-multiplying both sides by A, we get $A((\text{adj } A)(\text{adj } (\text{adj } A))) = A(|A|^{n-1} I_n)$.

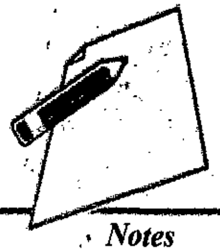
Using the associative property of matrix multiplication, we get

$(A(\text{adj } A))\text{adj}(\text{adj } A) = A(|A|^{n-1} I_n)$.

Hence, we get $(|A| I_n)(\text{adj}(\text{adj } A)) = |A|^{n-1} A$. That is, $\text{adj}(\text{adj } A) = |A|^{n-2} A$.

(iv) Replacing A by λA in $\text{adj}(A) = |A|A^{-1}$ where λ is a non-zero scalar, we get

$\text{adj}(\lambda A) = |\lambda A|(\lambda A)^{-1} = \lambda^n |A| \frac{1}{\lambda} A^{-1} = \lambda^{n-1}|A|A^{-1} = \lambda^{n-1}\text{adj}(A)$.



(v) By (iii), we have $\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} A$. So, by taking determinant on both sides, we get

$$|\operatorname{adj}(\operatorname{adj} A)| = \left||A|^{n-2} A\right| = \left(|A|^{(n-2)}\right)^n |A| = |A|^{n^2-2n+1} = |A|^{(n-1)^2}.$$

(vi) Replacing A by A^T in $A^{-1} = \frac{1}{|A|}\operatorname{adj} A$, we get $\left(A^T\right)^{-1} = \frac{1}{|A^T|}\operatorname{adj}\left(A^T\right)$ and hence, we

get $\operatorname{adj}\left(A^T\right) = |A^T|\left(A^T\right)^{-1} = |A|\left(A^{-1}\right)^T = \left(|A|A^{-1}\right)^T = \left(|A|\frac{1}{|A|}\operatorname{adj} A\right)^T = (\operatorname{adj} A)^T$.

Note

If A is a non-singular matrix of order 3, then, $|A| \neq 0$. By theorem 1.9 (ii), we get $|\text{adjA}| = |A|2$ and so, $|\text{adj A}|$ is positive. Then, we get $|A| = \pm \sqrt{}|\text{adjA}|$.

So, we get $|A| = \pm\sqrt{|\operatorname{adj} A|}$.

Further, by property (iii), we get $A^{-1} = \pm\frac{1}{\sqrt{|\operatorname{adj} A|}}\operatorname{adj} A$.

Hence, if A is a non-singular matrix of order 3, then we get $A = \pm\frac{1}{\sqrt{|\operatorname{adj} A|}}\operatorname{adj}(\operatorname{adj} A)$.

Example

If A is a non-singular matrix of odd order, prove that |adj A| is positive.

Solution

Let A be a non-singular matrix of order 2m + 1, where m = 0,1, 2, ... Then, we get $|A| \neq 0$ and, by theorem 1.9 (ii), we have |adj A| = |A|(2m+1)-1 = |A|2m

Since |A|2m is always positive, we get that |adj A| is positive.

Example

Find a matrix A if adj(A) = $\begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$.

Solution

First, we find $|\operatorname{adj}(A)| = \begin{vmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{vmatrix} = 7(77-35) - 7(-7-77) - 7(-5-121) = 1764 > 0.$

So, we get

$$A = \pm\frac{1}{\sqrt{|\operatorname{adj} A|}}\operatorname{adj}(\operatorname{adj} A) = \pm\frac{1}{\sqrt{1764}}\begin{bmatrix} +(77-35) & -(-7-77) & +(-5-121) \\ -(49+35) & +(49+77) & -(35-77) \\ +(49+77) & -(49-7) & +(77+7) \end{bmatrix}^T$$

$$= \pm\frac{1}{42}\begin{bmatrix} 42 & 84 & -126 \\ -84 & 126 & 42 \\ 126 & -42 & 84 \end{bmatrix}^T = \pm\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}.$$

Example

If adj A = $\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ find A−1.

Solution

$$\text{We compute } |\text{adj } A| = \begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 9.$$

$$\text{So, we get } A^{-1} = \pm \frac{1}{\sqrt{|\text{adj}(A)|}} \text{adj}(A) = \pm \frac{1}{\sqrt{9}} \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \pm \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Example

If A is symmetric, prove that adj A is also symmetric.

Solution

Suppose A is symmetric. Then, AT = A and so, by theorem 1.9 (vi), we get adj (AT) = (adj A) T □ adj A = (adj A)T □ adj A is symmetric

Theorem

If A and B are any two non-singular square matrices of order n , then adj(AB) = (adj B)(adj A).

Proof

Replacing A by AB in adj(A) = |A|A−1 , we get

adj(AB) = |AB| (AB)-1 = (| B | B-1) (| A | A-1) = adj(B) adj(A)

Example

Verify the property (AT)−1 = (A−1)T with A = $\begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$.

Solution

For the given A, we get |A |= (2) (7) - (9)(1) = 14 − 9 = 5 .

$$\text{So, } A^{-1}=\frac{1}{5}\begin{bmatrix}7 & -9\\ -1 & 2\end{bmatrix}=\begin{bmatrix}\frac{7}{5} & -\frac{9}{5}\\ -\frac{1}{5} & \frac{2}{5}\end{bmatrix}$$

$$\text{Then, } \left(A^{-1}\right)^{T}=\begin{bmatrix}\frac{7}{5} & -\frac{1}{5}\\ -\frac{9}{5} & \frac{2}{5}\end{bmatrix}=\frac{1}{5}\begin{bmatrix}7 & -1\\ -9 & 2\end{bmatrix} \quad \ldots(1)$$

For the given A, We get $A^{T}=\begin{bmatrix}2 & 1\\ 9 & 7\end{bmatrix}$. So $\left|A^{T}\right|=(2)(7)-(1)(9)=5$.

$$\text{Then, } \left(A^{T}\right)^{-1}=\frac{1}{5}\begin{bmatrix}7 & -1\\ -9 & 2\end{bmatrix} \quad \ldots(2)$$

From (1) and (2), we get (A-1) = (AT)-1. Thus, we have verified the given property.

Example

Verify (AB)−1 = B−1 A−1 with $A=\begin{bmatrix}0 & -3\\ 1 & 4\end{bmatrix}, B=\begin{bmatrix}-2 & -3\\ 0 & -1\end{bmatrix}$.

Solution

We get $AB=\begin{bmatrix}0 & -3\\ 1 & 4\end{bmatrix}\begin{bmatrix}-2 & -3\\ 0 & -1\end{bmatrix}=\begin{bmatrix}0+0 & 0+3\\ -2+0 & -3-4\end{bmatrix}=\begin{bmatrix}0 & 3\\ -2 & -7\end{bmatrix}$

$$(AB)^{-1}=\frac{1}{(0+6)}\begin{bmatrix}-7 & -3\\ 2 & 0\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-7 & -3\\ 2 & 0\end{bmatrix} \quad \ldots(1)$$

$$A^{-1}=\frac{1}{(0+3)}\begin{bmatrix}4 & 3\\ -1 & 0\end{bmatrix}=\frac{1}{3}\begin{bmatrix}4 & 3\\ -1 & 0\end{bmatrix}$$

$$B^{-1}=\frac{1}{(2-0)}\begin{bmatrix}-1 & 3\\ 0 & -2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1 & 3\\ 0 & -2\end{bmatrix}$$

$$B^{-1}A^{-1}=\frac{1}{2}\begin{bmatrix}-1 & 3\\ 0 & -2\end{bmatrix}\frac{1}{3}\begin{bmatrix}4 & 3\\ -1 & 0\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-7 & -3\\ 2 & 0\end{bmatrix}. \quad \ldots(2)$$

As the matrices in (1) and (2) are same, (AB) −1 = B-1 A-1 is verified.

Example

If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 + xA + yI_2 = O_2$, Hence, find A^{-1}.

Solution

$$\text{Since } A^2 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix},$$

$$A^2 + xA + yI_2 = O_2 \Rightarrow \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + x\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22+4x+y & 27+3x \\ 18+2x & 31+5x+y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, we get $22 + 4x + y = 0$, $31+5x+y=0$, $27+3x=0$ and $18+2x=0$

Hence $x = -9$ and $y = 14$. Then, we get $A^2 - 9A + 14I_2 = O_2$

Post-multiplying this equation by A^{-1}, we get $A - 9I_2 + 14A^{-1} = O_2$. Hence, we get

$$A^{-1} = \frac{1}{14}(9I_2 - A) = \frac{1}{14}\left(9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\right) = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}.$$

Application of matrices to Geometry

There is a special type of non-singular matrices which are widely used in applications of matrices to geometry. For simplicity, we consider two-dimensional analytical geometry.

Let O be the origin, and $x'Ox$ and $y'Oy$ be the x -axis and y -axis. Let P be a point in the plane whose coordinates are (x, y) with respect to the coordinate system. Suppose that we rotate the x -axis and y -axis about the origin, through an angle θ as shown in the figure. Let $X'OX$ and $Y'OY$ be the new X -axis and new Y -axis. Let (X ,Y) be the new set of coordinates of P with respect to the new coordinate system. Referring to Fig.1.1, we get

$x = OL = ON - LN = X\cos\theta - QT = X\cos\theta - Y\sin\theta$,

$y = PL = PT + TL = QN + PT = X\sin\theta + Y\cos\theta$.

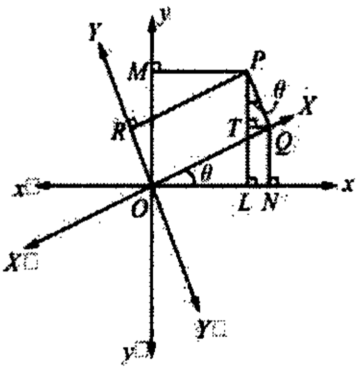


Fig.1.1

These equations provide transformation of one coordinate system into another coordinate system.

The above two equations can be written in the matrix form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}.$$

Let $W = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Then $\begin{bmatrix} x \\ y \end{bmatrix} = W\begin{bmatrix} X \\ Y \end{bmatrix}$ and $|W| = \cos^2\theta + \sin^2\theta = 1$.

So, W has inverse and $W^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. We note that $W^{-1} = W^T$. Then, we get the inverse transformation by the equation

$$\begin{bmatrix} X \\ Y \end{bmatrix} = W^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

Hence, we get the transformation X = x cosθ - y sinθ , Y = x sinθ + y cosθ .
This transformation is used in Computer Graphics and determined by the matrix

$$W = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

We note that the matrix W satisfies a special property W -1 = WT ; that is, WW T = WTW = I .

Definition

A square matrix A is called orthogonal if AAT = ATA = I.

Note

A is orthogonal if and only if A is non-singular and A−1 = AT .

Example

Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Solution

Let $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$. Then, $A^T=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}^T=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$.

So, we get

$$AA^T=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\sin\theta\cos\theta\\ \sin\theta\cos\theta-\cos\theta\sin\theta & \sin^2\theta+\cos^2\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I_2$$

Similarly, we get ATA = I2 . Hence AAT = ATA = I2 □ A is orthogonal.

Example

If A = $\frac{1}{7}\begin{bmatrix}6 & -3 & a\\ b & -2 & 6\\ 2 & c & 3\end{bmatrix}$, is orthogonal, find a, b and c , and hence A−1.

Solution

If A is orthogonal, then AAT = AT A = I3 . So, we have

$$AA^T=I_3\Rightarrow\frac{1}{7}\begin{bmatrix}6 & -3 & a\\ b & -2 & 6\\ 2 & c & 3\end{bmatrix}\frac{1}{7}\begin{bmatrix}6 & b & 2\\ -3 & -2 & c\\ a & 6 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}45+a^2 & 6b+6+6a & 12-3c+3a\\ 6b+6+6a & b^2+40 & 2b-2c+18\\ 12-3c+3a & 2b-2c+18 & c^2+13\end{bmatrix}=49\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow\begin{Bmatrix}45+a^2=49\\ b^2+40=49\\ c^2+13=49\\ 6b+6+6a=0\\ 12-3c+3a=0\\ 2b-2c+18=0\end{Bmatrix}\Rightarrow\begin{Bmatrix}a^2=4, b^2=9, c^2=36,\\ a+b=-1, a-c=-4, b-c=-9\end{Bmatrix}\Rightarrow a=2, b=-3, c=6$$

So, we get $A=\frac{1}{7}\begin{bmatrix}6 & -3 & 2\\ -3 & -2 & 6\\ 2 & 6 & 3\end{bmatrix}$ and hence, $A^{-1}=A^T=\frac{1}{7}\begin{bmatrix}6 & -3 & 2\\ -3 & -2 & 6\\ 2 & 6 & 3\end{bmatrix}$.

Application of matrices to Cryptography

One of the important applications of inverse of a non-singular square matrix is in cryptography. Cryptography is an art of communication between two people by keeping the information not known to others. It is based upon two factors, namely encryption and decryption. Encryption means the process of transformation of an information (plain form) into an unreadable form (coded form). On the other hand, Decryption means the transformation of the coded message back into original form. Encryption and decryption require a secret technique which is known only to the sender and the receiver.

This secret is called a key. One way of generating a key is by using a non-singular matrix to encrypt a message by the sender. The receiver decodes (decrypts) the message to retrieve the original message by using the inverse of the matrix. The matrix used for encryption is called encryption matrix (encoding matrix) and that used for decoding is called decryption matrix (decoding matrix).

We explain the process of encryption and decryption by means of an example.

Suppose that the sender and receiver consider messages in alphabets $A-Z$ only, both assign the numbers 1-26 to the letters $A-Z$ respectively, and the number 0 to a blank space. For simplicity, the sender employs a key as post-multiplication by a non-singular matrix of order 3 of his own choice. The receiver uses post-multiplication by the inverse of the matrix which has been chosen by the sender.

Let the encoding matrix be

$$A=\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}.$$

Let the message to be sent by the sender be "WELCOME".

Since the key is taken as the operation of post-multiplication by a square matrix of order 3, the message is cut into pieces (WEL), (COM), (E), each of length 3, and converted into a sequence of row matrices of numbers:

$[23\ 5\ 12],[3\ 15\ 13],[5\ 0\ 0].$

Note that, we have included two zeros in the last row matrix. The reason is to get a row matrix with 5 as the first entry.

Next, we encode the message by post-multiplying each row matrix as given below:

Uncoded row matrix	Encoding matrix	Coded row matrix

$$[23\ 5\ 12]\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}=[45\ -28\ 23];$$

$$[3\ 15\ 13]\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}=[46\ -18\ 3];$$

$$[5\ 0\ 0]\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}=[5\ -5\ 5].$$

So the encoded message is [45 – 28 –23] [46 -18 3] [5 –5 5] .

The receiver will decode the message by the reverse key, post-multiplying by the inverse of A.

So the decoding matrix is

$$A^{-1} = \frac{1}{|A|}\operatorname{adj} A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}.$$

The receiver decodes the coded message as follows:

Coded row matrix	Decoding matrix	Decoded row matrix

$$\begin{bmatrix} 45 & -28 & 23 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 23 & 5 & 12 \end{bmatrix};$$

$$\begin{bmatrix} 46 & -18 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 15 & 13 \end{bmatrix};$$

$$\begin{bmatrix} 5 & -5 & 5 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}.$$

So, the sequence of decoded row matrices is [23 5 12] , [3 15 13], [5 0 0].

Thus, the receiver reads the message as "WELCOME".

Summary of the module

A rectangular array of symbols (which could be real or complex numbers) along rows and columns is called a matrix. Thus, a system m × n symbols arranged in a rectangular formation along m rows and n columns and bounded by the brackets [.] is called an m by n matrix (which is written as m x n matrix) i.e. A = is a matrix of order m × n. In a compact form the above matrix is represented by A = [aij], 1 < i < m, 1 < j < n, where is, j ε i or simply [aij] m × n. The numbers a11, a12, ... etc of this rectangular array are called the elements of the matrix. The element aij belongs to the ith row and the jth column and is called the (i, j)th element of the matrix.Two matrices are said to be equal if they have the same order and each element of one is equal to the corresponding element of the other. A matrix having a single row is called a row matrix. e.g. [1 3, 5, 7] A matrix having a single column is called a column matrix. e.g., $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$. An m x n matrix A is said to be a square matrix if m = n i.e., number of rows = number of columns. For Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is a square matrix of order 3×3. In a square matrix the diagonal from left hand side upper corner to right hand side lower corner is known as leading diagonal or principal diagonal. In the above example square matrix containing the elements 1, 3, 5 is called the leading or principal diagonal. A non-singular square matrix of order n is invertible if there exists a square matrix B of the same order such that AB = In = BA. In such a case, we say that the inverse of A is B and we write, A-1 = B. The inverse of A is given by A-1 = 1/|A|. adj A.

Review Questions

EXERCISE 1

1. Find the adjoint of the following:

(i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ **(iii)** $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

$\boxed{\textbf{SOLUTION}}$

(i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 2 & -6 \\ -4 & -3 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$$

(ii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$$\text{adj A} = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}^T$$

$$\text{adj A} = \begin{bmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(8-7) & -(6-3) & +(21-12) \\ -(6-7) & +(4-3) & -(14-9) \\ +(3-4) & -(2-3) & +(8-9) \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{adj A} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

(iii) $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Let $\quad A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

$$\text{adj A} = \frac{1}{3}\begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}^T$$

$$= \frac{1}{3}\begin{bmatrix} +\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \frac{1}{3}\begin{bmatrix} +(2+4) & -(-4-2) & +(4-1) \\ -(4+2) & +(4-1) & -(-4-2) \\ +(4-1) & -(4+2) & +(2+4) \end{bmatrix}^T$$

$$= \frac{1}{3}\begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}^T = \frac{1}{3}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

2. Find the inverse (if it exists) of the following:

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

SOLUTION

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

Let $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} = 6 - 4 = 2$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix}^T$$

$$= \frac{1}{2}\begin{bmatrix} -3 & -1 \\ -4 & -2 \end{bmatrix}^T$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

(ii) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Let $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 5 \end{vmatrix}$$

$$= 5(25-1) - 1(5-1) + 1(1-5)$$

$$= 5 \times 24 - 1 \times 4 + 1 \times -4$$

$$= 120 - 4 - 4 = 120 - 8 = 112$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A$$

$$A^{-1} = \frac{1}{112}\begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}^T$$

$$= \frac{1}{112}\begin{bmatrix} +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} \end{bmatrix}^T$$

$$= \frac{1}{112}\begin{bmatrix} +(25-1) & -(5-1) & +(1-5) \\ -(5-1) & +(25-1) & -(5-1) \\ +(1-5) & -(5-1) & +(25-1) \end{bmatrix}^T$$

$$= \frac{1}{112}\begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}^T$$

$$= \frac{1}{112}\begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

$$= \frac{1}{112} \times 4\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

(iii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{vmatrix}$$

$$= 2(8-7) - 3(6-3) + 1(21-12)$$

$$= 2 \times 1 - 3 \times 3 + 9$$

$$= 2 - 9 + 9 \quad = 2$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}^T$$

$$= \frac{1}{2}\begin{bmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \frac{1}{2}\begin{bmatrix} +(8-7) & -(6-3) & +(21-12) \\ -(6-7) & +(4-3) & -(14-9) \\ +(3-4) & -(2-3) & +(8-9) \end{bmatrix}^T$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

3. If $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$, show that $[F(\alpha)]^{-1} = F(-\alpha)$.

SOLUTION

$$F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$|F(\alpha)| = \begin{vmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{vmatrix}$$

$$= \cos\alpha(\cos\alpha - 0) - 0 + \sin\alpha(0 + \sin\alpha)$$

$$= \cos^2\alpha + \sin^2\alpha = 1$$

$$[F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|}\operatorname{adj}(F(\alpha))$$

$$= \frac{1}{1}\begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ -\sin\alpha & \cos\alpha \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & \sin\alpha \\ 0 & \cos\alpha \end{vmatrix} & +\begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix} & -\begin{vmatrix} \cos\alpha & 0 \\ -\sin\alpha & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & \sin\alpha \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} \cos\alpha & \sin\alpha \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(\cos\alpha - 0) & -(0-0) & +(0+\sin\alpha) \\ -(0-0) & +(\cos^2\alpha + \sin^2\alpha) & -(0-0) \\ +(0-\sin\alpha) & -(0-0) & +(\cos\alpha - 0) \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}^T$$

$$[F(\alpha)]^{-1} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad \text{-----(1)}$$

$$F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$$

$$F(-\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad \text{-----(2)}$$

From equations (1) and (2) we get $\quad [F(\alpha)]^{-1} = F(-\alpha)$

4. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = O_2$. Hence find A^{-1}.

SOLUTION

$$A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$A^2 - 3A - 7I_2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 22-15-7 & 9-9-0 \\ -3+3-0 & 1+6-7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= O_2$$

$$A^2 - 3A - 7I_2 = 0$$

Pre multiplying by A^{-1}

$$\therefore \quad A^{-1}(A^2 - 3A - 7I_2) = A^{-1} O_2$$

$$A^{-1}(AA) - 3A^{-1}A - 7A^{-1}I_2 = O_2$$

$$(A^{-1}A)A - 3I_2 - 7A^{-1} = O_2$$

$$I_2A - 3I_2 - 7A^{-1} = O_2$$

$$A - 3I_2 - 7A^{-1} = O_2$$

$$7A^{-1} = A - 3I_2$$

$$7A^{-1} = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$7A^{-1} = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$7A^{-1} = \begin{bmatrix} 5-3 & 3-0 \\ -1-0 & -2-3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

5. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1} = A^T$.

SOLUTION

$$A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$A^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \qquad \text{-----(1)}$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A$$

$$|A| = \left(\frac{1}{9}\right)^3 \begin{vmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{vmatrix}$$

If A is a square matrix of order n, then $\det(kA) = k^n \det A$

$$|A| = \frac{1}{9^3}\left[-8(16+56) - 1(16-7) + 4(-32-4)\right]$$

$$|A| = \frac{1}{9^3}\left[-576 - 9 - 144\right]$$

$$|A| = \frac{1}{9^3} \times -729 = -1$$

$$A^{-1} = \frac{1}{-1} \times \frac{1}{9^2}\begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}^T$$

$$= -\frac{1}{9^2}\begin{bmatrix} +\begin{vmatrix} 4 & 7 \\ -8 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 7 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 4 & 4 \\ 1 & -8 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ -8 & 4 \end{vmatrix} & +\begin{vmatrix} -8 & 4 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} -8 & 1 \\ 1 & -8 \end{vmatrix} \\ +\begin{vmatrix} 1 & 4 \\ 4 & 7 \end{vmatrix} & -\begin{vmatrix} -8 & 4 \\ 4 & 7 \end{vmatrix} & +\begin{vmatrix} -8 & 1 \\ 4 & 4 \end{vmatrix} \end{bmatrix}^T$$

If A is a square matrix of order n, then $\operatorname{adj}(\lambda A) = \lambda^{n-1}\operatorname{adj} A$

$$= -\frac{1}{9^2}\begin{bmatrix} +(16+56) & -(16-7) & +(-32-4) \\ -(4+32) & +(-32-4) & -(64-1) \\ +(7-16) & -(-56-16) & +(-32-4) \end{bmatrix}^T$$

$$= -\frac{1}{9^2}\begin{bmatrix} 72 & -9 & -36 \\ -36 & -36 & -63 \\ -9 & 72 & -36 \end{bmatrix}^T$$

$$= -\frac{1}{9^2}\begin{bmatrix} 72 & -36 & -9 \\ -9 & -36 & 72 \\ -36 & -63 & -36 \end{bmatrix}$$

$$= \frac{1}{9^2}\begin{bmatrix} -72 & 36 & 9 \\ 9 & 36 & -72 \\ 36 & 63 & 36 \end{bmatrix}$$

$$= \frac{1}{9^2} \times 9 \begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \qquad \text{-----(2)}$$

From equations (1) and (2) we get $A^T = A^{-1}$

6. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that

$A(\operatorname{adj} A) = (\operatorname{adj} A)A = |A|\, I_2$.

SOLUTION

$$A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 8 & -4 \\ -5 & 3 \end{vmatrix} = 24 - 20 = 4$$

$$\operatorname{adj} A = \begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 3 & 5 \\ 4 & 8 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$A(\text{adj } A) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$A(\text{adj } A) = \begin{bmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(\text{adj } A)\,A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

$$(\text{adj } A)\,A = \begin{bmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A(\text{adj } A) = (\text{adj } A)\,A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{-----(1)}$$

$$|A| = \begin{vmatrix} 8 & -4 \\ -5 & 3 \end{vmatrix}$$

$$= 24 - 20 = 4$$

Equation (1) $\Rightarrow$

$$A(\text{adj } A) = (\text{adj } A)\,A = |A|\,I_2.$$

7. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

SOLUTION

Given $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15 - 14 = 1$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{1} \begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix}^T$$

$$A^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}^T$$

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$|B| = \begin{vmatrix} -1 & -3 \\ 5 & 2 \end{vmatrix} = -2 + 15 = 13$$

$$B^{-1} = \frac{1}{|B|} \text{adj } B$$

$$= \frac{1}{13} \begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix}^T$$

$$= \frac{1}{13} \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}^T$$

$$B^{-1} = \frac{1}{13} B^{-1} A^{-1}$$

$$= \frac{1}{13} \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$= \frac{1}{13} \begin{bmatrix} 10 - 21 & -4 + 9 \\ -25 + 7 & 10 - 3 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \quad \text{-----(1)}$$

$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} -3+10 & -9+4 \\ -7+25 & -21+10 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 7 & -5 \\ 18 & -11 \end{vmatrix} = -77 + 90 = 13$$

$$(AB)^{-1} = \frac{1}{13}\,\text{adj}(AB)$$

$$= \frac{1}{13}\begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix}^T$$

$$= \frac{1}{13}\begin{bmatrix} -11 & -18 \\ 5 & 7 \end{bmatrix}^T$$

$$(AB)^{-1} = \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \quad \text{-----(2)}$$

From equations (1) and (2) we get $(AB)^{-1} = B^{-1}A^{-1}$

8. If $\text{adj}(A) = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$, find A.

SOLUTION

Given $\text{adj}\,A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

$$A = \pm\frac{1}{\sqrt{|\text{adj}\,A|}}\,\text{adj}(\text{adj}\,A)$$

$$|\text{adj}\,A| = \begin{vmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{vmatrix}$$

$$= 2(24-0) + 4(-6-14) + 2(0+24)$$

$$= 2\times 24 + 4\times -20 + 2\times 24$$

$$= 48 - 80 + 48 = 96 - 80 = 16$$

$$|\text{adj}\,A| = 16$$

$$A = \pm\frac{1}{\sqrt{|\operatorname{adj} A|}}\begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}^T$$

$$A = \pm\frac{1}{\sqrt{16}}\begin{bmatrix} +\begin{vmatrix} 12 & -7 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & -7 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} -3 & 12 \\ -2 & 0 \end{vmatrix} \\ -\begin{vmatrix} -4 & 2 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -2 & 0 \end{vmatrix} \\ +\begin{vmatrix} -4 & 2 \\ 12 & -7 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ -3 & -7 \end{vmatrix} & +\begin{vmatrix} 2 & -4 \\ -3 & 12 \end{vmatrix} \end{bmatrix}^T$$

$$= \pm\frac{1}{4}\begin{bmatrix} +(24-0) & -(-6-14) & +(0+24) \\ -(-8-0) & +(4+4) & -(0-8) \\ +(28-24) & -(-14+6) & +(24-12) \end{bmatrix}^T$$

$$= \pm\frac{1}{4}\begin{bmatrix} 24 & 20 & 24 \\ 8 & 8 & 8 \\ 4 & 8 & 12 \end{bmatrix}^T = \pm\frac{1}{4}\begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$A = \pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

9. If adj (A) = $\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$, find A^{-1}

SOLUTION

Given $\operatorname{adj} A = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$

$$|\operatorname{adj} A| = \begin{vmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{vmatrix}$$

$$|\operatorname{adj} A| = 0(12-0) + 2(36-18) + 0(0+6) = 36$$

$$A^{-1} = \pm\frac{1}{\sqrt{|\operatorname{adj} A|}}(\operatorname{adj} A)$$

$$= \pm\frac{1}{\sqrt{36}}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

$$A^{-1} = \pm\frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

10. Find adj (adj (A)) if $\text{adj}\,A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

SOLUTION

$$\text{adj}\,A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|\text{adj}\,A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 1(2-0) - 0(0-0) + 1(0+2)$$

$$|\text{adj}\,A| = 2 + 2 = 4$$

> If A is a square matrix of order n, then $\text{adj}(\text{adj}(A)) = |A|^{n-2}A$

$$A = \pm \frac{1}{\sqrt{|\text{adj}\,A|}}\ \text{adj}(\text{adj}\,A)$$

$$A = \pm \frac{1}{\sqrt{4}} \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}^T$$

$$= \pm \frac{1}{2} \begin{bmatrix} +\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \pm \frac{1}{2} \begin{bmatrix} (2-0) & -(0-0) & (0+2) \\ -(0-0) & (1+1) & (0-0) \\ +(0-2) & -(0-0) & +(2-0) \end{bmatrix}^T$$

$$= \pm \frac{1}{2} \begin{bmatrix} (2-0) & -(0-0) & (0+2) \\ -(0-0) & (1+1) & (0-0) \\ +(0-2) & -(0-0) & +(2-0) \end{bmatrix}^T$$

$$= \pm \frac{1}{2} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}^T = \pm \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^T$$

$$A = \pm \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$|A| = \pm [1(1-0) - 0(0-0) - 1(0-1)]$$

$$|A| = \pm [1+1] = \pm 2$$

> If A is a square matrix of order n, then $\text{adj}(\text{adj}(A)) = |A|^{n-2}A$

$$\text{adj}(\text{adj}(A)) = (\pm 2)^{3-2} \times \pm \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = (\pm 2) \times \pm \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\text{adj}(\text{adj}(A)) = 2 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

EXERCISE 2

1. Find the adjoint of the following:

$$\text{(i)}\ \begin{bmatrix}-3 & 4\\ 6 & 2\end{bmatrix}\quad \text{(ii)}\begin{bmatrix}2 & 3 & 1\\ 3 & 4 & 1\\ 3 & 7 & 2\end{bmatrix}\quad \text{(iii)}\ \frac{1}{3}\begin{bmatrix}2 & 2 & 1\\ -2 & 1 & 2\\ 1 & -2 & 2\end{bmatrix}$$

2. Find the inverse (if it exists) of the following:

$$\text{(i)}\ \begin{bmatrix}-2 & 4\\ 1 & -3\end{bmatrix}\quad \text{(ii)}\begin{bmatrix}5 & 1 & 1\\ 1 & 5 & 1\\ 1 & 1 & 5\end{bmatrix}\quad \text{(iii)}\begin{bmatrix}2 & 3 & 1\\ 3 & 4 & 1\\ 3 & 7 & 2\end{bmatrix}$$

3. If $F(\alpha)=\begin{bmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ -\sin\alpha & 0 & \cos\alpha\end{bmatrix}$, show that $[F(\alpha)]^{-1}=F(-\alpha)$.

4. If $A=\begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix}$, show that $A^2-3A-7I_2=O_2$. Hence find A^{-1}.

5. If $A=\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$, prove that $A^{-1}=A^T$.

6. If $A=\begin{bmatrix}8 & -4\\ -5 & 3\end{bmatrix}$, verify that $A(\text{adj } A)=(\text{adj } A)A=|A|I_2$.

7. If $A=\begin{bmatrix}3 & 2\\ 7 & 5\end{bmatrix}$ and $B=\begin{bmatrix}-1 & -3\\ 5 & 2\end{bmatrix}$, verify that $(AB)^{-1}=B^{-1}A^{-1}$.

8. If $\text{adj}(A)=\begin{bmatrix}2 & -4 & 2\\ -3 & 12 & -7\\ -2 & 0 & 2\end{bmatrix}$, find A.

9. If $\text{adj}(A)=\begin{bmatrix}0 & -2 & 0\\ 6 & 2 & -6\\ -3 & 0 & 6\end{bmatrix}$, find A^{-1}.

10. Find adj(adj(A)) if $\text{adj } A=\begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ -1 & 0 & 1\end{bmatrix}$.

11. $A=\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A^T A^{-1}=\begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

12. Find the matrix A for which $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}=\begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$.

13. Given $A=\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a matrix X such that $AXB=C$.

14. If $A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1}=\frac{1}{2}\left(A^2-3I\right)$.

15. Decrypt the received encoded message [2 −3] [20 4] with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers 1-26 to the letters A Z – respectively, and the number 0 to a blank space.

Answers:

ANSWERS

Exercise 1.1

1. (i) $\begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$ (iii) $\frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & +2 & 2 \end{bmatrix}$

2. (i) $\frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$ (ii) $\frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$ (iii) $\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$

8. $\pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$ 9. $\pm\frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$ 10. $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$ 13. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ 15. HELP

1 RELATIONS AND FUNCTIONS

- Understand the concept of relations.
- Discuss the types of relations.
- Understand the concepts of functions.
- Discuss the concept of special functions.
- Discuss the types of special functions.
- Discuss the concept of inverse trigonometric function.
- Understand the practical application of relations.

Objective of the Module:

The basic objective of this chapter is to through some light on the initial concepts of relations and functions so that the practical application of relations and functions can be examined in detailed.

Introduction

"Relations and Functions" are the most important topics in algebra. Relations and functions – these are the two different words having different meanings mathematically. You might get confused about their difference. Before we go deeper, let's understand the difference between both with a simple example.

An ordered pair is represented as (INPUT, OUTPUT):

The relation shows the relationship between INPUT and OUTPUT. Whereas, a function is a relation which derives one OUTPUT for each given INPUT.

Note: All functions are relations, but not all relations are functions.

In this section, you will find the basics of the topic – definition of functions and relations, special functions, different types of relations and some of the solved examples.

What is a Function?

A function is a relation which describes that there should be only one output for each input (or) we can say that a special kind of relation (a set of ordered pairs), which follows a rule i.e every X-value should be associated with only one y-value is called a function.

For example:

Domain	Range
-1	-3
1	3
3	9

Let us also look at the definition of Domain and Range of a function.

Domain	It is a collection of the first values in the ordered pair (Set of all input (x) values).
Range	It is a collection of the second values in the ordered pair (Set of all output (y) values).

Example:

In the relation, {(-2, 3), {4, 5), (6, -5), (-2, 3)},

The domain is {-2, 4, 6} and range is {-5, 3, 5}.

Note: Don't consider duplicates while writing the domain and range and also write it in increasing order.

Types of Functions

In terms of relations, we can define the types of functions as:

One to one function or Injective function: A function f: P → Q is said to be one to one if for each element of P there is a distinct element of Q.

Many to one function: A function which maps two or more elements of P to the same element of set Q.

Onto Function or Surjective function: A function for which every element of set Q there is pre-image in set P

One-one correspondence or Bijective function: The function f matches with each element of P with a discrete element of Q and every element of Q has a pre-image in P.

What is the Relation?

It is a subset of the Cartesian product. Or simply, a bunch of points (ordered pairs). In other words, the relation between the two sets is defined as the collection of the ordered pair, in which the ordered pair is formed by the object from each set.

Example: {(-2, 1), (4, 3), (7, -3)}, usually written in set notation form with curly brackets.

Relation Representation

There are other ways too to write the relation, apart from set notation such as through tables, plotting it on XY- axis or through mapping diagram.

Types of Relations

Different types of relations are as follows:

- Empty Relations
- Universal Relations
- Identity Relations
- Inverse Relations
- Reflexive Relations
- Symmetric Relations
- Transitive Relations

Let us discuss all the types one by one.

Empty Relation

When there's no element of set X is related or mapped to any element of X, then the relation R in A is an empty relation, and also called the void relation, i.e R= □.

For example, if there are 100 mangoes in the fruit basket. There's no possibility of finding a relation R of getting any apple in the basket. So, R is Void as it has 100 mangoes and no apples.

Universal relation

R is a relation in a set, let's say A is a universal relation because, in this full relation, every element of A is related to every element of A. i.e $R = A \times A$.

It's a full relation as every element of Set A is in Set B.

Identity Relation

If every element of set A is related to itself only, it is called Identity relation.

I={(A, A), □ a}.

For Example,

When we throw a dice, the total number of possible outcomes is 36. I.e (1, 1) (1, 2), (1, 3).....(6, 6). From these, if we consider the relation (1, 1), (2, 2), (3, 3) (4, 4) (5, 5) (6, 6), it is an identity relation.

Inverse Relation

If R is a relation from set A to set B i.e R □ A X B. The relation R−1 = {(b,a):(a,b) □ R}.

For Example,

If you throw two dice if R = {(1, 2) (2, 3)}, R−1= {(2, 1) (3, 2)}. Here the domain is the range R−1 and vice versa.

Reflexive Relation

A relation is a reflexive relation iIf every element of set A maps to itself, i.e for every a □ A, (a, a) □ R.

Symmetric Relation

A symmetric relation is a relation R on a set A if (a, b) □ R then (b, a) □ R, for all a & b □ A.

Transitive Relation

If (a, b) □ R, (b, c) □ R, then (a, c) □ R, for all a,b,c □ A and this relation in set A is transitive.

Equivalence Relation

If a relation is reflexive, symmetric and transitive, then the relation is called an equivalence relation.

How to Convert a Relation into a Function?

A special kind of relation (a set of ordered pairs) which follows a rule i.e every X-value should be associated with only one y-value, then the relation is called a function.

Examples

Example 1: Is A = {(1, 5), (1, 5), (3, -8), (3, -8), (3, -8)} a function?

Solution: If there are any duplicates or repetitions in the X-value, the relation is not a function.

But there's a twist here. Look at the following example:

Though X-values are getting repeated here, still it is a function because they are associating with the same values of Y.

The point (1, 5) is repeated here twice and (3, -8) is written thrice. We can rewrite it by writing a single copy of the repeated ordered pairs. So, "A" is a function.

Example 2: Give an example of an Equivalence relation.

Solution:

If we note down all the outcomes of throwing two dice, it would include reflexive, symmetry and transitive relations. Then, throwing two dice is an example of an equivalence relation

2 INVERSE TRIGONOMETRIC FUNCTIONS

Introduction

In everyday life, indirect measurement is used to obtain solutions to problems that are impossible to solve using measurement tools. Trigonometry helps us to find measurements like heights of mountains and tall buildings without using measurement tools. Trigonometric functions and their inverse trigonometric functions are widely used in engineering and in other sciences including physics.

They are useful not only in solving triangles, given the length of two sides of a right triangle, but also they help us in evaluating a certain type of integrals, such as

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx \text{ and } \int \frac{1}{x^2 + a^2}\,dx$$

The symbol sin−1x denoting the inverse trigonometric function arcsine (x) of sine function was introduced by the British mathematician John F.W.Herschel (1792-1871). For his work along with his father, he was presented with the Gold Medal of the Royal Astronomical Society in 1826.

An oscilloscope is an electronic device that converts electrical signals into graphs like that of sine function. By manipulating the controls, we can change the amplitude, the period and the phase shift of sine curves. The oscilloscope has many applications like measuring human heartbeats, where the trigonometric functions play a dominant role.

Let us consider some simple situations where inverse trigonometric functions are often used.

Illustration-1 (Slope problem)

Consider a straight line $y = mx + b$. Let us find the angle θ made by the line with x -axis in terms of slope m. The slope or gradient m is defined as the rate of change of a function, usually calculated by $m = \Delta y/\Delta x = \frac{\Delta y}{\Delta x}$

From right triangle (Fig. 4.1), $\tan\theta = \Delta y/\Delta x = \frac{\Delta y}{\Delta x}$. Thus, $\tan\theta = m$. In order to solve for θ, we need the inverse trigonometric function called "inverse tangent function".

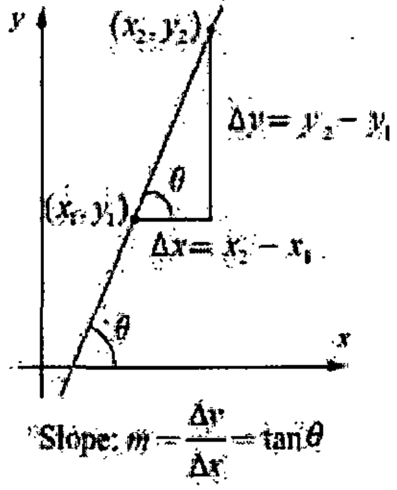


Fig. 4.1

Illustration-2 (Movie Theatre Screens)

Suppose that a movie theatre has a screen of 7 metres tall. When someone sits down, the bottom of the screen is 2 metres above the eye level. The angle formed by drawing a line from the eye to the bottom of the screen and a line from the eye to the top of the screen is called the viewing angle.

In Fig. 4.2, θ is the viewing angle. Suppose that the person sits x metres away from the screen. The viewing angle θ is given by the function

$$\theta(x) = \tan^{-1}\left(\frac{9}{x}\right) - \tan^{-1}\left(\frac{2}{x}\right)$$

Observe that the viewing angle θ is a function of x

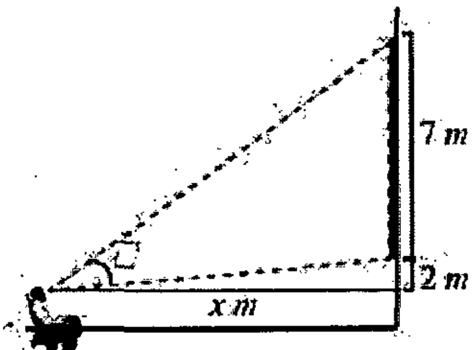


Fig. 4.2

Illustration-3 (Drawbridge)

Assume that there is a double-leaf drawbridge as shown in Fig.4.3. Each leaf of the bridge is 40 metres long. A ship of 33 metres wide needs to pass through the bridge. Inverse trigonometric function helps us to find the minimum angle θ so that each leaf of the bridge should be opened in order to ensure that the ship will pass through the bridge.

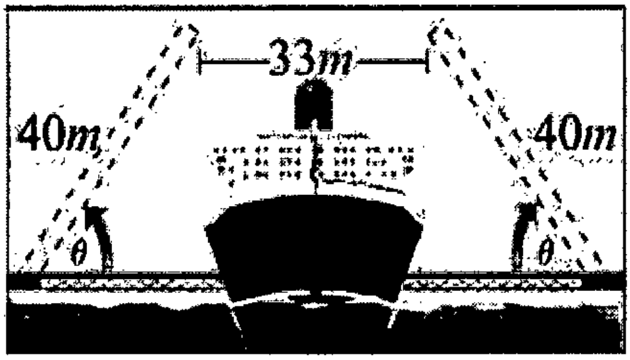


Fig. 4.3

In class XI, we have discussed trigonometric functions of real numbers using unit circle, where the angles are in radian measure. In this chapter, we shall study the inverse trigonometric functions, their graphs and properties. In our discussion, as usual R and Z stand for the set of all real numbers and all integers, respectively. Let us recall the definition of periodicity, domain and range of six trigonometric functions.

Inverse trigonometric functions are simply defined as the inverse functions of the basic trigonometric functions which are sine, cosine, tangent, cotangent, secant, and cosecant functions. They are also termed as arcus functions, antitrigonometric functions or cyclometric functions. These inverse functions in trigonometry are used to get the angle with any of the trigonometry ratios. The inverse trigonometry functions have major applications in the field of engineering, physics, geometry and navigation.

What are Inverse Trigonometric Functions?

Inverse trigonometric functions are also called "Arc Functions" since, for a given value of trigonometric functions, they produce the length of arc needed to obtain that particular value. The inverse trigonometric functions perform the opposite operation of the trigonometric functions such as sine, cosine, tangent, cosecant, secant, and cotangent. We know that trigonometric functions are especially applicable to the right angle triangle. These six important functions are used to find the angle measure in the right triangle when two sides of the triangle measures are known.

Formulas

The basic inverse trigonometric formulas are as follows:

Inverse Trig Functions	Formulas
Arcsine	sin-1(-x) = -sin-1(x), x □ [-1, 1]
Arccosine	cos-1(-x) = π -cos-1(x), x □ [-1, 1]
Arctangent	tan-1(-x) = -tan-1(x), x □ R

Arccotangent	cot-1(-x) = π – cot-1(x), x □ R
Arcsecant	sec-1(-x) = π -sec-1(x), |x| ≥ 1
Arccosecant	cosec-1(-x) = -cosec-1(x), |x| ≥ 1

Inverse Trigonometric Functions Graphs

There are particularly six inverse trig functions for each trigonometry ratio. The inverse of six important trigonometric functions is:

- Arcsine
- Arccosine
- Arctangent
- Arccotangent
- Arcsecant
- Arccosecant

Let us discuss all the six important types of inverse trigonometric functions along with its definition, formulas, graphs, properties and solved examples.

Arcsine Function

Arcsine function is an inverse of the sine function denoted by sin-1x. It is represented in the graph as shown below:

Domain	$-1 \leq x \leq 1$
Range	$-\pi/2 \leq y \leq \pi/2$

Arccosine Function

Arccosine function is the inverse of the cosine function denoted by cos-1x. It is represented in the graph as shown below:

Therefore, the inverse of cos function can be expressed as; y = cos-1x (arccosine x)

Domain & Range of arcsine function:

Domain	$-1 \leq x \leq 1$
Range	$0 \leq y \leq \pi$

Arctangent Function

Arctangent function is the inverse of the tangent function denoted by tan-1x. It is represented in the graph as shown below:

Therefore, the inverse of tangent function can be expressed as; y = tan-1x (arctangent x)

Domain & Range of Arctangent:

Domain	$-\infty < x < \infty$
Range	$-\pi/2 < y < \pi/2$

Arccotangent (Arccot) Function

Arccotangent function is the inverse of the cotangent function denoted by cot-1x. It is represented in the graph as shown below:

Therefore, the inverse of cotangent function can be expressed as; y = cot-1x (arccotangent x)

Domain & Range of Arccotangent:

Domain	$-\infty < x < \infty$
Range	$0 < y < \pi$

Arcsecant Function

What is arcsecant (arcsec)function? Arcsecant function is the inverse of the secant function denoted by sec-1x. It is represented in the graph as shown below:

Therefore, the inverse of secant function can be expressed as; y = sec-1x (arcsecant x)

Domain & Range of Arcsecant:

Domain	$-\infty \le x \le -1$ or $1 \le x \le \infty$
Range	$-\pi/2 < y < \pi/2$; $y \neq 0$

Arccosecant Function

What is arccosecant (arccsc x) function? Arccosecant function is the inverse of the cosecant function denoted by cosec-1x. It is represented in the graph as shown below:

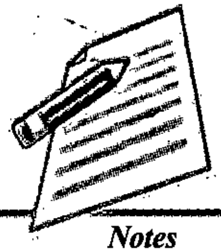
Therefore, the inverse of cosecant function can be expressed as; y = cosec-1x (arccosecant x)

Domain & Range of Arccosecant is:

Domain	$-\infty \le x \le -1$ or $1 \le x \le \infty$
Range	$-\pi/2 < y < \pi/2$; $y \neq 0$

Inverse Trigonometric Functions Table

Let us rewrite here all the inverse trigonometric functions with their notation, definition, domain and range.



Function Name	Notation	Definition	Domain of x	Range
Arcsine or inverse sine	y = sin-1(x)	x=sin y	$-1 \leq x \leq 1$	$-\pi/2 \leq y \leq \pi/2$ $-90° \leq y \leq 90°$
Arccosine or inverse cosine	y=cos-1(x)	x=cos y	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$ $0° \leq y \leq 180°$
Arctangent or Inverse tangent	y=tan-1(x)	x=tan y	For all real numbers	$-\pi/2 < y < \pi/2$ $-90° < y < 90°$
Arccotangent or Inverse Cot	y=cot-1(x)	x=cot y	For all real numbers	$0 < y < \pi$ $0° < y < 180°$
Arcsecant or Inverse Secant	y = sec-1(x)	x=sec y	$x \leq -1$ or $1 \leq x$	$0 \leq y < \pi/2$ or $\pi/2 < y \leq \pi$ $0° \leq y < 90°$ or $90° < y \leq 180°$
Arccosecant	y=csc-1(x)	x=csc y	$x \leq -1$ or $1 \leq x$	$-\pi/2 \leq y < 0$ or $0 < y \leq \pi/2$ $-90° \leq y < 0°$or $0° < y \leq 90°$

Inverse Trigonometric Functions Derivatives

The derivatives of inverse trigonometric functions are first-order derivatives. Let us check here the derivatives of all the six inverse functions.

Inverse Trig Function	dy/dx
y = sin-1(x)	1/√(1-x2)
y = cos-1(x)	-1/√(1-x2)
y = tan-1(x)	1/(1+x2)
y = cot-1(x)	-1/(1+x2)
y = sec-1(x)	1/[|x|√(x2-1)]
y = csc-1(x)	-1/[|x|√(x2-1)]

Inverse Trigonometric Functions Properties

The inverse trigonometric functions are also known as Arc functions. Inverse Trigonometric Functions are defined in a certain interval (under restricted domains).

Trigonometry Basics

Trigonometry basics include the basic trigonometry and trigonometric ratios such as sin x, cos x, tan x, cosec x, sec x and cot x.

Inverse Trigonometric Functions Problems

Example 1: Find the value of x, for sin(x) = 2.

Solution: Given: sin x = 2

x =sin-1(2), which is not possible.

Hence, there is no value of x for which sin x = 2; since the domain of sin-1x is -1 to 1 for the values of x.

Example 2: Find the value of sin-1(sin (π/6)).

Solution:

sin-1(sin (π/6) = π/6 (Using identity sin-1(sin (x)) = x)

Example 3: Find sin (cos-1 3/5).

Solution:

Suppose that, cos-1 3/5 = x

So, cos x = 3/5

We know, sin x = 1–cos2x————√

So, sin x = 1–925——√ = 4/5

This implies, sin x = sin (cos-1 3/5) = 4/5

Example 4: Solve: sin(cot−1x)

Solution:

Let cot−1x=θ□x=cotθ

Now, cosecθ=1+cot2θ————√=1+x2———√

Therefore, sinθ=1cosecθ=11+x2√□θ=sin−111+x2√

Hence sin(cot−1x)=sin(sin−111+x2√)=11+x2√=(1+x2)−1/2

Example 5: sec−1[sec(−30o)]=

Solution:

sec−1[sec(−30o)]=sec−1(sec30o)=30o

Example 6: If sin(sin−115+cos−1x)=1, then what is the value of x?

Solution:

sin−115+cos−1x=π2sin−115=π2−cos−1x=sin−1xx=15

Summary of the Module

"Relations and Functions" are the most important topics in algebra. Relations and functions – these are the two different words having different meanings mathematically. You might get confused about their difference. Before we go deeper, let's understand the difference between both with a simple example. An ordered pair is represented as (INPUT, OUTPUT): The relation shows the relationship between INPUT and

OUTPUT. Whereas, a function is a relation which derives one OUTPUT for each given INPUT. Note: All functions are relations, but not all relations are functions. A function is a relation which describes that there should be only one output for each input (or) we can say that a special kind of relation (a set of ordered pairs), which follows a rule i.e every X-value should be associated with only one y-value is called a function. Inverse trigonometric functions are also called "Arc Functions" since, for a given value of trigonometric functions, they produce the length of arc needed to obtain that particular value. The inverse trigonometric functions perform the opposite operation of the trigonometric functions such as sine, cosine, tangent, cosecant, secant, and cotangent. We know that trigonometric functions are especially applicable to the right angle triangle. These six important functions are used to find the angle measure in the right triangle when two sides of the triangle measures are known.

Review Questions

Exercise

1. If the ordered pairs ($x2 - 3x, y2 + 4y$) and (-2, 5) are equal, then find x and y.

Solution:

Given $x^2 - 3x = -2$

$x^2 - 3x + 2 = 0$

$(x - 1)(x + 2) = 0$

$x = 1$ and $x = 2$

$$\begin{array}{c|c} \multicolumn{2}{c}{+2} \\ \frac{-1}{x} & \frac{-2}{x} \end{array}$$

Given $y^2 + 4y = 5$

$y^2 + 4y - 5 = 0$

$(y - 1)(y + 5) = 0$

$y - 1$ and $y = -5$

$$\begin{array}{c|c} \multicolumn{2}{c}{-5} \\ \frac{-1}{y} & \frac{+5}{y} \end{array}$$

The value of x is 1 and 2

The value of y is 1 and -5

2. The cartesian product $A \times A$ has 9 elements among which (–1, 0) and (0,1) are found. Find the set A and the remaining elements of $A \times A$

Solution:

The set A = {5, 6, 7, 8}

The remaining elements of A × A is

{(-1, -1) (-1, 1) (0, -1) (0, 0) (1, -1) (1, 0) (1, 1)}

3. Given that $f(x) = \begin{cases} \sqrt{x-1} & x \geq 1 \\ 4 & x < 1 \end{cases}$. Find

(i) $f(0)$

(ii) $f(3)$

(iii) $f(a + 1)$ in terms of a.(Given that $a \geq 0$)

Solution:

$$f(x) = \begin{cases} \sqrt{x-1} & \text{if} \quad x = \{1, 2, 3, 4........\} \\ 4 & \text{if} \quad x = \{0, -1, -2..........\} \end{cases}$$

i) $f(0) \quad = 4$

ii) $f(3) \quad = \sqrt{x-1} = \sqrt{3-1} = \sqrt{2}$

iii) $f(a+1) = \sqrt{x-1} = \sqrt{a+1-1} = \sqrt{a}$

4. Let $A = \{9,10,11,12,13,14,15,16,17\}$ and let $f: A \rightarrow N$ be defined by $f(n)$ = the highest prime factor of $n \in A$. Write f as a set of ordered pairs and find the range of f.

Solution:

f(n) = the highest prime factor

f(9) = 3 (factors 1, 3, 9)

f(10) = 5 (factors 1, 2, 5)

f(11) = 11 (factors 1, 11)

f(12) = 3 (factors 1, 2, 3, 4, 6, 12)

f(13) = 13 (factors 1, 13)

f(14) = 7 (factots 1, 2, 7, 14)

f(15) = 5 (factors 1, 3, 5, 15)

f(16) = 2 (factors 1, 2, 4, 8, 16)

f(17) = 17 (factors 1, 17)

Set of ordered pair {(9, 3) (10, 5) (11, 11) (12, 3) (13, 13) (14, 7) (15, 5) (16, 2) (17, 7)}

Range of f = {(2, 3, 5, 11, 13, 17}

5. Find the domain of the function $f(x) = \sqrt{1+\sqrt{1-\sqrt{1-x^2}}}$

Solution:

$$f(x)=\sqrt{1+\sqrt{1+\sqrt{1-\sqrt{1-x^2}}}}$$

Domain = R

Reason : $f(0)=\sqrt{1+\sqrt{1-\sqrt{1-0^2}}}=\sqrt{1}$

$$f(1)=\sqrt{1+\sqrt{1-\sqrt{1-1^2}}}=\sqrt{2}$$

$$f(2)=\sqrt{1+\sqrt{1-\sqrt{1-2^2}}}=\sqrt{3}$$

$$f(-1)=\sqrt{1+\sqrt{1-\sqrt{1-(-1^2)}}}=\sqrt{2}$$

..........................so on.

6. If $f(x) = x2$, $g(x) = 3x$ and $h(x) = x - 2$, Prove that $(f \circ g) \circ h = f \circ (g \circ h)$

Solution:

$$\begin{aligned}
fog(x) &= f(g(x) = f(3x) \\
&= (3x)^2 \\
&= 9x^2 \\
(fog)oh(x) &= fog(h(x) \\
&= fog(x-2) \\
&= 9\,(x-2)^2 \\
&= 9[x^2 - 4x + 4] \\
&= 9x^2 - 36x + 36 \text{ ——— ❶} \\
goh(x) &= g(h(x) = g(x-2) \\
&= 3(x-2) \\
&= 3x - 6 \\
fo(goh)(x) &= fo(3x-6) \\
&= (3x-6)2 \\
&= 9x2 - 36x + 36 \text{ ——— ❷}
\end{aligned}$$

from ❶ and ❷ we get

(fog)oh = fo(goh)

7. Let $A = \{1, 2\}$ and $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$. Verify whether A×C is a subset of B×D?

Solution:

$$A \times C = \{1, 2\} \times \{5, 6\}$$

$$= \{(1, 5)\,(1, 6)\,(2, 5)\,(2, 6)\} \text{ —— } ❶$$

$$B \times D = \{1, 2, 3, 4\} \times \{5, 6, 7, 8\}$$

$$= \left\{ \begin{matrix} (1,5)\,(1,6)\,(1,7)\,(1,8)\,(2,5)\,(2,6) \\ (2,7)\,(2,8)\,(3,5)\,(3,6) \\ (3,7)\,(3,8)\,(4,5)\,(4,6)\,(4,7)\,(4,8) \end{matrix} \right\}$$

—— ❷

from and it is clear

$A \times C \subset B \times D$

8. If $f(x) = [x\text{-}1]/[x\text{+}1]$, x ¹ 1 show that $f(f(x)) = -1/x$ provided $x \neq 0$

Solution:

$$\text{Given } f(x) = \frac{x-1}{x+1}$$

$$f(x) = \left(\frac{x-1}{x+1}\right)$$

$$= \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1}$$

$$= \frac{\frac{x-1-(x+1)}{x+1}}{\frac{x-1+x+1}{x+1}}$$

$$= \frac{-2}{-2x} \Rightarrow = \frac{-1}{x} \text{ proved}$$

9. The functions f and g are defined by $f(x) = 6x + 8$; $g(x) = [x\text{-}2]/3$

(i). Calculate the value of $gg(1/2)$

(ii) Write an expression for $gf(x)$ in its simplest form.

Solution:

Given $f(x) = 6x + 8$

$$g(x) = \frac{x-2}{3}$$

$$g\,g\left(\frac{1}{2}\right) = g\left(\frac{x-2}{3}\right) \text{ where } x = \frac{1}{2}$$

$$= g\left(\frac{\frac{1}{2}-2}{3}\right)$$

$$= g\left(\frac{-1}{2}\right)$$

$$= \frac{x-2}{3} \text{ where } x = -\frac{1}{2}$$

$$= \frac{-\frac{1}{2}-2}{3}$$

$$= \frac{-\frac{5}{2}}{3} \Rightarrow \frac{-5}{2} \times \frac{1}{3} = \frac{-5}{6}$$

ii) Write an expression for g f(x) in its simplest form

Given : $f(x) = 6x + 8$

$$g(x) = \frac{x-2}{3}$$

$$f(x) = g(6x + 8)$$

$$= \frac{x-2}{3} \quad \text{where } x = 6x + 8$$

$$= \frac{6x+8-2}{3}$$

$$= \frac{6x+6}{3} \Rightarrow \frac{6(x+1)}{3}$$

$$= 2(x + 1)$$

10. Write the domain of the following real functions

(i) $f(x)=\dfrac{2x+1}{x-9}$ (ii) $p(x)=\dfrac{-5}{4x^2+1}$

(iii) $g(x)=\sqrt{x-2}$

(iv) $h(x) = x + 6$

Solution:

i) $f(x) = \dfrac{2x+1}{x-9}$

Domain = R – {9}

HINT

If $x = 9$

$f(x) = \dfrac{2(9)+1}{9-9}$

$= \dfrac{18+1}{0}$

= Not defined

ii) $p(x) = \dfrac{-5}{4x^2+1}$

Domain = R

iii) $g(x) = \sqrt{x-2}$

Domain = {2, 3, 4, 5..........}

iv) $h(x) = x + 6$

Domain = R

HINT

If x = 0 and less than 0

$g(0)=\sqrt{0-2}=\sqrt{2} \notin R$

11. Solve: tan (arcsin 12/13)

12. Find the value of x, cos (arccos 1) = cos x

Answers:

1. 1,2 and -5 ,1
2. {-1, 0, 1} , {(- 1, -1),(- 1, 1),(0, -1),(0, 0),(1, -1),(1, 0),(1, 1)}
3. (i) 4 (ii) √2 (iii) √a
4. {(9, 3),(10, 5),(11, 11),(12, 3),(13, 13),(14, 7),(15, 5),(16, 2),(17, 17)} , {2,3,5,11,13,17}
5. {–1,0,1}
9. (i) -5/6 (ii) 2(x + 1)
10. (i) R - {9} (ii) R (iii) [2, ∞) (iv) R

1 LIMITS AND CONTINUITY

- Understand the concept of Calculus.
- Discuss the concept of limit.
- Understand the concepts of continuity.
- Discuss the concept of differentiation.
- Discuss the types of integration.
- Discuss the concept of definite integral.
- Understand the practical application of limit and continuity.

Objective of the Module:

The basic objective of this chapter is to through some light on the initial concepts of calculus so that the practical application of various concepts of calculus like, limits and continuity, differentiations and integration can be examined in detailed.

Introduction

Limits and continuity concept is one of the most crucial topics in calculus. A limit is defined as a number approached by the function as an independent function's variable approaches a particular value. For instance, for a function $f(x) = 4x$, you can say that "The limit of f(x) as x approaches 2 is 8". Symbolically, it is written as;

$\lim x \to 2(4x)=4\times 2=8$

Continuity is another popular topic in calculus. The easy method to test for the continuity of a function is to examine whether a pen can trace the graph of a function without lifting the pen from the paper. When you are doing with pre calculus and calculus, a conceptual definition is almost sufficient, but for higher level, a technical explanation is required. You can learn a better and precise way of defining continuity by using limits.

Continuity Definition

A *function is said to be continuous at a particular point* if the following three conditions are satisfied.

$f(a)$ is defined

$\lim x \to a f(x)$ exists

$\lim x \to a+f(x)=\lim x \to a-f(x)=f(a)$

A function is said to be continuous if you can trace its graph without lifting the pen from the paper. But a function is said to be discontinuous when it has any gap in between.

Types of Discontinuity

There are basically two types of discontinuity:

- Infinite Discontinuity
- Jump Discontinuity
- Infinite Discontinuity

A branch of discontinuity wherein, a vertical asymptote is present at $x = a$ and $f(a)$ is not defined. This is also called as Asymptotic Discontinuity. If a function has values on both sides of an asymptote, then it cannot be connected, so it is discontinuous at the asymptote.

Jump Discontinuity

A branch of discontinuity wherein $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, but both the limits are finite. This is also called simple discontinuity or continuities of first kind.

Positive Discontinuity

A branch of discontinuity wherein a function has a pre-defined two-sided limit at $x=a$, but either $f(x)$ is undefined at a, or its value is not equal to the limit at a.

Limit Definition

A limit of a function is a number that a function reaches as the independent variable of the function reaches a given value. The value (say a) to which the function $f(x)$ gets close arbitrarily as the value of the independent variable x becomes close arbitrarily to a given value a symbolized as $f(x) = A$.

Points to remember:

If $\lim_{x\to a^-} f(x)$ is the expected value of f at $x = a$ given the values of 'f' near x to the left of a. This value is known as the left-hand limit of 'f' at a.

If $\lim_{x\to a^+} f(x)$ is the expected value of f at $x = a$ given the values of 'f' near x to the right of a. This value is known as the right-hand limit of $f(x)$ at a.

If the right-hand and left-hand limits coincide, we say the common value as the limit of $f(x)$ at $x = a$ and denote it by $\lim_{x\to a} f(x)$.

One-Sided Limit

The limit that is based completely on the values of a function taken at x -value that is slightly greater or less than a particular value. A two-sided limit $\lim_{x\to a} f(x)$ takes the values of x into account that are both larger than and smaller than a. A one-sided limit from the left $\lim_{x\to a^-} f(x)$ or from the right $\lim_{x\to a^-} f(x)$ takes only values of x smaller or greater than a respectively.

Properties of Limit

The limit of a function is represented as f(x) reaches L as x tends to limit a, such that; limx→af(x) = L

The limit of the sum of two functions is equal to the sum of their limits, such that: limx→a [f(x) + g(x)] = limx→a f(x) + limx→a g(x)

The limit of any constant function is a constant term, such that, limx→a C = C

The limit of product of the constant and function is equal to the product of constant and the limit of the function, such that: limx→a m f(x) = m limx→a f(x)

Quotient Rule: limx→a[f(x)/g(x)] = limx→af(x)/limx→ag(x); if limx→ag(x) ≠ 0

Examples

1). Compute limx→−2(3x2+5x−9)

Solution:

First, use property 2 to divide the limit into three separate limits. Then use property 1 to bring the constants out of the first two. This gives,

limx→−2(3x2+5x−9)=limx→−2(3x2)+limx→−2(5x)−limx→−2(9)

=3(−2)2+5(−2)−(9)

= 12 – 10 – 9

= -7

2). Find the value of limx→3 [x(x+2)].

Solution:

limx→3 [x(x+2)] = 3(3+2) = 3 x 5 = 15

The calculation of limits

The notion of a limit, which we will discuss extensively in this chapter, plays a central role in calculus and in much of modern mathematics. However, although mathematics dates back over three thousand years, limits were not really understood until the monumental work of the great French mathematician Augustin – Louis Cauchy and Karl Weierstrass in the nineteenth century, the age of rigour in mathematics.

In this section we define limit and show how limits can be calculated.

Illustration 1

We begin by looking at the function y = f (x) = x2 + 3 . Note that f is a function from R→ R .

Let us investigate the behaviour of this function near x = 2. We can use two sets of x values : one set that approaches 2 from the left (values less than 2) and one set that approaches 2 from the right (values greater than 2) as shown in the table.

	x approaches 2 from the left							*x* approaches 2 from the right					
x	1.7	1.9	1.95	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.05	2.1	2.3
f(x)	5.89	6.61	6.8025	6.9601	6.99601	6.99960001	7.	7.0040001	7.004001	7.0401	7.2025	7.41	8.29

It appears from the table that as x gets close to x = 2, f(x) = x2 + 3 gets close to 7. This is not surprising since if we now calculate f(x) at x = 2, we obtain f(2) = 22 + 3 = 7. In order to guess at this limit, we didn't have to evaluate x2 + 3 at x = 2.

That is, as x approaches 2 from either the left (values lower than 2) or right (values higher than

2) the functional values f(x) are approaching 7 from either side; that is, when x is near 2, f(x) is near 7. The above situation is described in a condensed form:

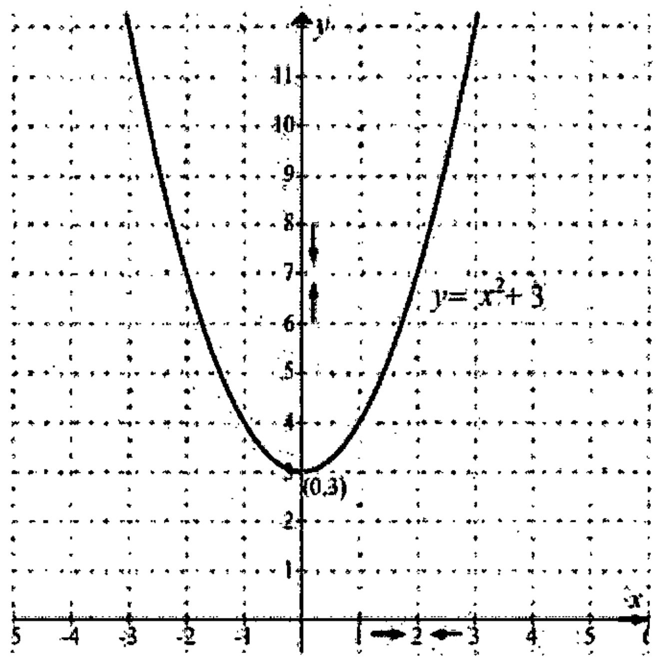


Fig. 9.1

The value 7 is the left limit of f(x) as x approaches 2 from the left as well as 7 is the right limit of f(x) as x approaches 2 from the right and write :

f (x)→7 as x →2- and f (x) → 7 as x→2+

or

$$\lim_{x\to 2^-} f(x) = 7 \text{ and } \lim_{x\to 2^+} f(x) = 7.$$

Note also that lim x→2− f(x) = 7 = lim x→2+ f(x). The common value is written as lim x →2 f(x) = 7.

We also observe that the limit is a definite real number. Here, definiteness means that lim x→2− f(x) and lim x→2+ f(x) are the same and lim x→2 f(x) = lim x→2- f(x) = lim x→2+ f(x) is a unique real number.

The figure in Fig. 9.1 explains the geometrical significance of the above discussion of the behaviour of f(x) = x2 + 3 as x → 2.

Illustration 2

Next, let us look at the rational function $f(x) = \dfrac{16-x^2}{4+x}$.

The domain of this function is R \ {-4}. Although f(- 4) is not defined, nonetheless, $\lim_{x \to -4} f(x)$ can be calculated for any value of x near - 4 because the symbol $\lim_{x \to -4}\left(\dfrac{16-x^2}{4+x}\right)$ says that we consider values of x that are close to - 4 but not equal to - 4. The table below gives the values of f(x) for values of x that approach - 4.

$(x < -4)$ $(x \to -4^-)$	$f(x)$	$(x > -4)$ $(x \to -4^+)$	$f(x)$
-4.1	8.1	-3.9	7.9
-4.01	8.01	-3.99	7.99
-4.001	8.001	-3.999	7.999

For x ≠ - 4, f(x) can be simplified by cancellation :

$$f(x) = \frac{16-x^2}{4+x}$$

$$= \frac{(4+x)(4-x)}{(4+x)} = 4 - x.$$

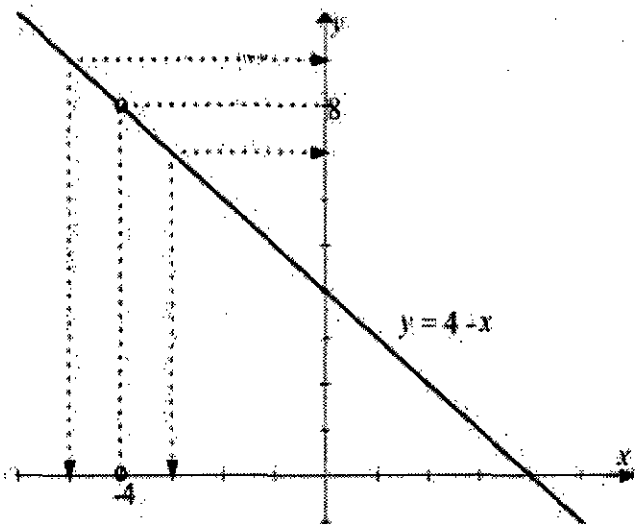


Fig. 9.2

As seen in Fig.9.2, the graph of f(x) is essentially the graph of y = 4 - x with the exception that the graph of f has a hole (puncture) at the point that corresponds to x = - 4. As x gets closer and closer to - 4, represented by the two arrow heads on the x-axis, the two arrow heads on the y-axis simultaneously get closer and closer to the number 8.

Here, note that

$$\lim_{x \to -4^-} f(x) = 8 = \lim_{x \to -4^+} f(x) \text{ and}$$

$$\text{hence } \lim_{x \to -4} f(x) = \lim_{x \to -4} \frac{16 - x^2}{4 + x} = 8.$$

In Illustration 9.2, note that the function is not defined at x = −4 and yet f(x) appears to be approaching a limit as x approaches - 4. This often happens, and it is important to realise that the existence or non-existence of f(x) at x = −4 has no bearing on the existence of the limit of f(x) as x approaches −4 .

2 DIFFERENTIATION

Introduction

In calculus, differentiation is one of the two important concepts apart from integration. Differentiation is a method of finding the derivative of a function. Differentiation is a process, in Maths, where we find the instantaneous rate of change in function based on one of its variables. The most common example is the rate change of displacement with respect to time, called velocity. The opposite of finding a derivative is anti-differentiation.

If x is a variable and y is another variable, then the rate of change of x with respect to y is given by dy/dx. This is the general expression of derivative of a function and is represented as f'(x) = dy/dx, where y = f(x) is any function.

What is Differentiation in Maths

In Mathematics, Differentiation can be defined as a derivative of a function with respect to an independent variable. Differentiation, in calculus, can be applied to measure the function per unit change in the independent variable.

Let y = f(x) be a function of x. Then, the rate of change of "y" per unit change in "x" is given by:

dy / dx

If the function f(x) undergoes an infinitesimal change of 'h' near to any point 'x', then the derivative of the function is defined as

\(\lim\limits_{h \to 0} \frac{f(x+h) – f(x)}{h}\)

Derivative of Function As Limits

If we are given with real valued function (f) and x is a point in its domain of definition, then the derivative of function, f, is given by:

$f'(a) = \lim_{h\to 0}[f(x+h)-f(x)]/h$

provided this limit exists.

Let us see an example here for better understanding.

Example: Find the derivative of f=2x, at x =3.

Solution: By using the above formulas, we can find,

$f'(3) = \lim_{h\to 0}[f(3+h)-f(3]/h = \lim_{h\to 0}[2(3+h)-2(3)]/h$

$f'(3) = \lim_{h\to 0}[6+2h-6]/h$

f'(3) = limh→02h/h

f'(3) = limh→02 = 2

Also, check Continuity And Differentiability to understand the above expression.

Notations

When a function is denoted as y=f(x), the derivative is indicated by the following notations.

D(y) or D[f(x)] is called Euler's notation.

dy/dx is called Leibniz's notation.

F'(x) is called Lagrange's notation.

The meaning of differentiation is the process of determining the derivative of a function at any point.

Linear and Non-Linear Functions

Functions are generally classified in two categories under Calculus, namely:

(i) Linear functions

(ii) Non-linear functions

A linear function varies with a constant rate through its domain. Therefore, the overall rate of change of the function is the same as the rate of change of a function at any point.

However, the rate of change of function varies from point to point in case of non-linear functions. The nature of variation is based on the nature of the function.

The rate of change of a function at a particular point is defined as a derivative of that particular function.

Differentiation Formulas

The important Differentiation formulas are given below in the table. Here, let us consider f(x) is a function and f›(x) is the derivative of the function.

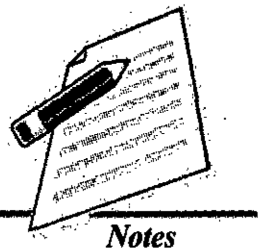
If f(x) = tan (x), then f'(x) = sec2x
If f(x) = cos (x), then f'(x) = -sin x
If f(x) = sin (x), then f'(x) = cos x
If f(x) = ln(x), then f'(x) = 1/x
If f(x) = e^{x}, then f'(x) = e^{x}
If f(x) = x^{n}, where n is any fraction or integer, then f'(x) = nx^{n-1}
If f(x) = k, where k is a constant, then f'(x) = 0

Differentiation Rules

The basic differentiation rules that need to be followed are as follows:

Sum and Difference Rule

Product Rule



Quotient Rule

Chain Rule

Let us discuss here.

Sum or Difference Rule

If the function is sum or difference of two functions, the derivative of the functions is the sum or difference of the individual functions, i.e.,

If $f(x) = u(x) \pm v(x)$

then, $f'(x)=u'(x) \pm v'(x)$

Product Rule

As per the product rule, if the function f(x) is product of two functions u(x) and v(x), the derivative of the function is,

If $f(x) = u(x) \times v(x)$

then, $\mathbf{ f'(x) = u'(x) \times v(x) + u(x) \times v'(x)}$

Quotient rule

If the function f(x) is in the form of two functions [u(x)]/[v(x)], the derivative of the function is

If, $f(x) = \frac{u(x)}{v(x)}$

then, $\large \mathbf{ f'(x) = \frac{u'(x) \times v(x) - u(x) \times v'(x)}{(v(x))^{2}}}$

Chain Rule

If a function y = f(x) = g(u) and if u = h(x), then the chain rule for differentiation is defined as,

$\large \mathbf{\frac{\mathrm{d} y}{\mathrm{d} x} = \frac{\mathrm{d} y}{\mathrm{d} u} \times \frac{\mathrm{d}u }{\mathrm{d} x}}$

This plays a major role in the method of substitution that helps to perform differentiation of composite functions.

Real-Life Applications of Differentiation

With the help of differentiation, we are able to find the rate of change of one quantity with respect to another. Some of the examples are:

Acceleration: Rate of change of velocity with respect to time

To calculate the highest and lowest point of the curve in a graph or to know its turning point, the derivative function is used

To find tangent and normal to a curve

Solved Examples

Q.1: Differentiate f(x) = 6x3-9x+4 with respect to x.

Solution: Given: f(x) = 6x3-9x+4

On differentiating both the sides w.r.t x, we get;

f'(x) = (3)(6)x2 – 9

f'(x) = 18x2 – 9

This is the final answer.

Q.2: Differentiate y = x(3x2 – 9)

Solution: Given, y = x(3x2 – 9)

y = 3x3 – 9x

On differentiating both the sides we get,

dy/dx = 9x2 – 9

This is the final answer.

3 INTEGRATION

Introduction

Integration is the calculation of an integral. Integrals in maths are used to find many useful quantities such as areas, volumes, displacement, etc. When we speak about integrals, it is related to usually definite integrals. The indefinite integrals are used for anti derivatives. Integration is one of the two major calculus topics in Mathematics, apart from differentiation (which measure the rate of change of any function with respect to its variables). Here, you will learn the definition of integrals in Maths, formulas of integration along with examples.

Integration Definition

The integration denotes the summation of discrete data. The integral is calculated to find the functions which will describe the area, displacement, volume, that occurs due to a collection of small data, which cannot be measured singularly. In a broad sense, in calculus, the idea of limit is used where algebra and geometry are implemented. Limits help us in the study of the result of points on a graph such as how they get closer to each other until their distance is almost zero. We know that there are two major types of calculus –

- Differential Calculus
- Integral Calculus

The concept of integration has developed to solve the following types of problems:

- To find the problem function, when its derivatives are given.
- To find the area bounded by the graph of a function under certain constraints.

These two problems lead to the development of the concept called the "Integral Calculus", which consist of definite and indefinite integral. In calculus, the concept of differentiating a function and integrating a function is linked using the theorem called the Fundamental Theorem of Calculus.

Maths Integration

In Maths, integration is a method of adding or summing up the parts to find the whole. It is a reverse process of differentiation, where we reduce the functions into parts. This method is used to find the summation under a vast scale. Calculation of small addition problems is an easy task which we can do manually or by using calculators as well. But for big addition problems, where the limits could reach to even infinity, integration methods are used. Integration and differentiation both are important parts of calculus. The concept level of these topics is very high. Hence, it is introduced to us at higher secondary classes and then in engineering or higher education.

Integral Calculus

According to Mathematician Bernhard Riemann,

"Integral is based on a limiting procedure which approximates the area of a curvilinear region by breaking the region into thin vertical slabs."

Let us now try to understand what does that mean:

Take an example of a slope of a line in a graph to see what differential calculus is:

In general, we can find the slope by using the slope formula. But what if we are given to find an area of a curve? For a curve, the slope of the points varies, and it is then we need differential calculus to find the slope of a curve.

You must be familiar with finding out the derivative of a function using the rules of the derivative. Wasn't it interesting? Now you are going to learn the other way round to find the original function using the rules in Integrating.

Integration – Inverse Process of Differentiation

We know that differentiation is the process of finding the derivative of the functions and integration is the process of finding the anti derivative of a function. So, these processes are inverse of each other. So we can say that integration is the inverse process of differentiation or vice versa. The integration is also called the anti-differentiation. In this process, we are provided with the derivative of a function and asked to find out the function (i.e., primitive).

We know that the differentiation of sin x is cos x.

It is mathematically written as:

$(d/dx) \sin x = \cos x$ ...(1)

Here, cos x is the derivative of sin x. So, sin x is the anti derivative of the function cos x. Also, any real number "C" is considered as a constant function and the derivative of the constant function is zero.

So, equation (1) can be written as

$(d/dx) (\sin x + C)= \cos x +0$

$(d/dx) (\sin x + C)= \cos x$

Where "C" is the arbitrary constant or constant of integration.

Generally, we can write the function as follow:

$(d/dx) [F(x)+C] = f(x)$, where x belongs to the interval I.

To represent the anti derivative of "f", the integral symbol "∫" symbol is introduced. The anti derivative of the function is represented as $\int f(x)\, dx$. This can also be read as the indefinite integral of the function "f" with respect to x.

Therefore, the symbolic representation of the anti derivative of a function (Integration) is:

y = ∫ f(x) dx

∫ f(x) dx = F(x) + C.

Integrals in Maths

You have learned until now the concept of integration. You will come across, two types of integrals in maths:

- Definite Integral
- Indefinite Integral

Definite Integral

An integral that contains the upper and lower limits then it is a definite integral. On a real line, x is restricted to lie. Riemann Integral is the other name of the Definite Integral.

A definite Integral is represented as:

\(\int_{a}^{b} f(x)dx\)

Indefinite Integral

Indefinite integrals are defined without upper and lower limits. It is represented as:

∫f(x)dx = F(x) + C

Where C is any constant and the function f(x) is called the integrand.

Integration Examples

Solve some problems based on integration concept and formulas here.

Example 1: Find the integral of the function: ∫x2 dx

Solution:

Given ∫x2 dx

= (x3/3) + C.

Example 2:

Integrate ∫ (x2-1)(4+3x)dx.

Solution:

Given: ∫ (x2-1)(4+3x)dx.

Multiply the terms, we get

∫ (x2-1)(4+3x)dx = ∫ 4x2+3x3-3x-4 dx

Now, integrate it, we get

∫ (x2-1)(4+3x)dx = 4(x3/3) + 3(x4/4)- 3(x2/2) – 4x + C

The anti derivative of the given function ∫ (x2-1)(4+3x)dx is 4(x3/3) + 3(x4/4)- 3(x2/2) – 4x + C.

4 DIFFERENTIAL EQUATIONS

Introduction

In Mathematics, a differential equation is an equation that contains one or more functions with its derivatives. The derivatives of the function define the rate of change of a function at a point. It is mainly used in fields such as physics, engineering, biology and so on. The primary purpose of the differential equation is the study of solutions that satisfy the equations and the properties of the solutions. Learn how to solve differential equation here.

One of the easiest ways to solve the differential equation is by using explicit formulas. In this chapter, let us discuss the definition, types, methods to solve the differential equation, order and degree of the differential equation, ordinary differential equations with real-word example and a solved problem.

Differential Equation Definition

A differential equation is an equation which contains one or more terms and the derivatives of one variable (i.e., dependent variable) with respect to the other variable (i.e., independent variable)

$dy/dx = f(x)$

Here "x" is an independent variable and "y" is a dependent variable

For example, $dy/dx = 5x$

A differential equation contains derivatives which are either partial derivatives or ordinary derivatives. The derivative represents a rate of change, and the differential equation describes a relationship between the quantity that is continuously varying with respect to the change in another quantity. There are a lot of differential equations formulas to find the solution of the derivatives.

Types of Differential Equations

Differential equations can be divided into several types namely

- Ordinary Differential Equations
- Partial Differential Equations
- Linear Differential Equations
- Non-linear differential equations
- Homogeneous Differential Equations

- Non-homogenous Differential Equations
- Differential Equations Solutions

There exist two methods to find the solution of the differential equation.

- Separation of variables
- Integrating factor

Differential Equation

Separation of the variable is done when the differential equation can be written in the form of $dy/dx = f(y)g(x)$ where f is the function of y only and g is the function of x only. Taking an initial condition, rewrite this problem as $1/f(y)dy= g(x)dx$ and then integrate on both sides.

Also, check: Solve Separable Differential Equations

Integrating factor technique is used when the differential equation is of the form $dy/dx + p(x)y = q(x)$ where p and q are both the functions of x only.

First-order differential equation is of the form $y' + P(x)y = Q(x)$. where P and Q are both functions of x and the first derivative of y. The higher-order differential equation is an equation that contains derivatives of an unknown function which can be either a partial or ordinary derivative. It can be represented in any order.

We also provide differential equation solver to find the solutions for related problems.

Order of Differential Equation

The order of the differential equation is the order of the highest order derivative present in the equation. Here some examples for different orders of the differential equation are given.

$dy/dx = 3x + 2$, The order of the equation is 1

$(d2y/dx2)+ 2 (dy/dx)+y = 0$. The order is 2

$(dy/dt)+y = kt$. The order is 1

First Order Differential Equation

You can see in the first example, it is a first-order differential equation which has degree equal to 1. All the linear equations in the form of derivatives are in the first order. It has only the first derivative such as dy/dx, where x and y are the two variables and is represented as:

$dy/dx = f(x, y) = y'$

Second-Order Differential Equation

The equation which includes second-order derivative is the second-order differential equation. It is represented as;

$d/dx(dy/dx) = d2y/dx2 = f''(x) = y''$

Degree of Differential Equation

The degree of the differential equation is the power of the highest order derivative, where the original equation is represented in the form of a polynomial equation in derivatives such as y',y", y"', and so on.

Suppose (d2y/dx2)+ 2 (dy/dx)+y = 0 is a differential equation, so the degree of this equation here is 1. See some more examples here:

dy/dx + 1 = 0, degree is 1

(y"')3 + 3y" + 6y' – 12 = 0, degree is 3

Ordinary Differential Equation

An ordinary differential equation involves function and its derivatives. It contains only one independent variable and one or more of its derivative with respect to the variable.

The order of ordinary differential equations is defined as the order of the highest derivative that occurs in the equation. The general form of n-th order ODE is given as

F(x, y, y',...., yn) = 0

Applications

Let us see some differential equation applications in real-time.

1) Differential equations describe various exponential growths and decays.

2) They are also used to describe the change in return on investment over time.

3) They are used in the field of medical science for modelling cancer growth or the spread of disease in the body.

4) Movement of electricity can also be described with the help of it.

5) They help economists in finding optimum investment strategies.

6) The motion of waves or a pendulum can also be described using these equations.

The various other applications in engineering are: heat conduction analysis, in physics it can be used to understand the motion of waves. The ordinary differential equation can be utilized as an application in the engineering field for finding the relationship between various parts of the bridge.

Linear Differential Equations Real World Example

To understand Differential equations, let us consider this simple example. Have you ever thought why a hot cup of coffee cools down when kept under normal conditions? According to Newton, cooling of a hot body is proportional to the temperature difference between its temperature T and the temperature T0 of its surrounding. This statement in terms of mathematics can be written as:

dT/dt □ (T – T0)............(1)

This is the form of a linear differential equation.

Introducing a proportionality constant k, the above equation can be written as:

dT/dt = k(T – T0) ....(2)

Here, T is the temperature of the body and t is the time,

T0 is the temperature of the surrounding,

dT/dt is the rate of cooling of the body

Eg: dy/dx = 3x

Here, the differential equation contains a derivative that involves a variable (dependent variable, y) w.r.t another variable (independent variable, x). The types of differential equations are :

1. An ordinary differential equation contains one independent variable and its derivatives. It is frequently called ODE. The general definition of the ordinary differential equation is of the form: Given an F, a function os x and y and derivative of y, we have

F(x, y, y' .....y^(n1)) = y (n) is an explicit ordinary differential equation of order n.

2. Partial differential equation that contains one or more independent variable.

Solved problem

Question:

Verify that the function y = e-3x is a solution to the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

Solution:

The function given is $y = e^{-3x}$. We differentiate both the sides of the equation with respect to x,

$\frac{dy}{dx} = -3e^{-3x}$

Now we again differentiate the above equation with respect to x,

$\frac{d^2y}{dx^2} = 9e^{-3x}$

We substitute the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and y in the differential equation given in the question,

On left hand side we get, LHS = 9e-3x + (-3e-3x) – 6e-3x

= 9e-3x – 9e-3x = 0 (which is equal to RHS)

Therefore, the given function is a solution to the given differential equation.

Summary of the Module

Limits and continuity concept is one of the most crucial topics in calculus. A limit is defined as a number approached by the function as an independent function's variable approaches a particular value. For instance, for a function f(x) = 4x, you can say that "The limit of f(x) as x approaches 2 is 8". Symbolically, it is written as; limx→2(4x)=4×2=8.Continuity is another popular topic in calculus. The easy method to test for the continuity of a function is to examine whether a pen can trace the graph of a function without lifting the pen from the paper. When you are doing with pre calculus and calculus, a conceptual definition is almost sufficient, but for higher level, a technical explanation is required. You can learn a better and precise way of

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defining continuity by using limits. A function is said to be continuous at a particular point if the following three conditions are satisfied. f(a) is defined, limx→af(x) exists, limx→a+f(x)=limx→a−f(x)=f(a). A function is said to be continuous if you can trace its graph without lifting the pen from the paper. But a function is said to be discontinuous when it has any gap in between. In Mathematics, a differential equation is an equation that contains one or more functions with its derivatives. The derivatives of the function define the rate of change of a function at a point. It is mainly used in fields such as physics, engineering, biology and so on. The primary purpose of the differential equation is the study of solutions that satisfy the equations and the properties of the solutions. Learn how to solve differential equation here. One of the easiest ways to solve the differential equation is by using explicit formulas. In this chapter, let us discuss the definition, types, methods to solve the differential equation, order and degree of the differential equation, ordinary differential equations with real-word example and a solved problem.

Review Questions

EXERCISE 9.1

In problems 1-6, complete the table using calculator and use the result to estimate the limit.

(1) $\lim_{x \to 2} \dfrac{x-2}{x^2-x-2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

(2) $\lim_{x \to 2} \dfrac{x-2}{x^2-4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

(3) $\lim_{x \to 0} \dfrac{\sqrt{x+3}-\sqrt{3}}{x}$

x	− 0.1	− 0.01	− 0.001	0.001	0.01	0.1
$f(x)$						

(4) $\lim_{x \to -3} \dfrac{\sqrt{1-x}-2}{x+3}$

x	− 3.1	− 3.01	− 3.00	− 2.999	− 2.99	− 2.9
$f(x)$						

(5) $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

(6) $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x}$

x	-0.1	-0.01	-0.001	0.0001	0.01	0.1
$f(x)$						

In exercise problems 7 - 15, use the graph to find the limits (if it exists). If the limit does not exist, explain why?

(7) $\lim\limits_{x \to 3} (4 - x)$.

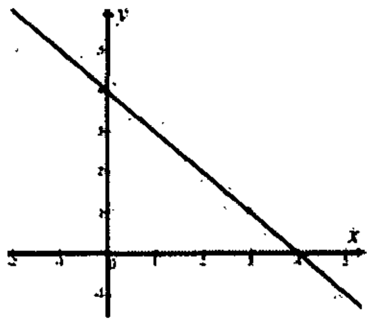

Fig. 9.13

(8) $\lim\limits_{x \to 1} (x^2 + 2)$.

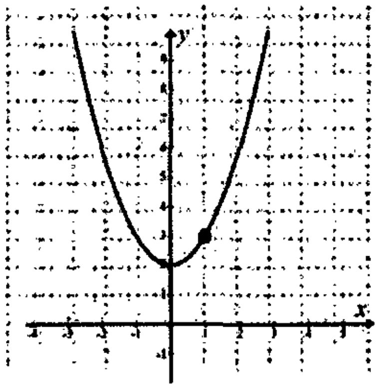

Fig. 9.14

(9) $\lim\limits_{x \to 2} f(x)$

where $f(x) = \begin{cases} 4 - x, & x \neq 2 \\ 0, & x = 2 \end{cases}$.

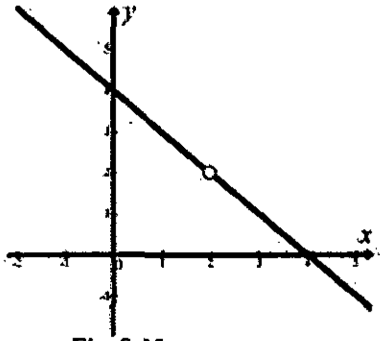

Fig. 9.15

(10) $\lim\limits_{x \to 1} f(x)$

where $f(x) = \begin{cases} x^2 + 2, & x \neq 1 \\ 1, & x = 1 \end{cases}$.

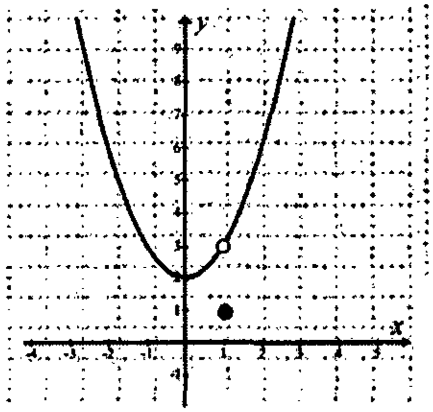

Fig. 9.16

CLASS–12

Mathematics

Notes

(11) $\lim\limits_{x\to 3}\dfrac{1}{x-3}$

(12) $\lim\limits_{x\to 5}\dfrac{|x-5|}{x-5}$

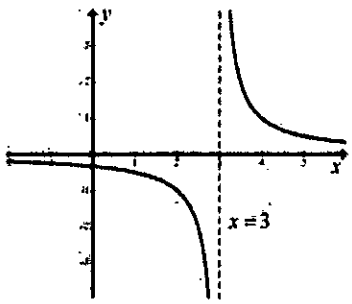

Fig. 9.17

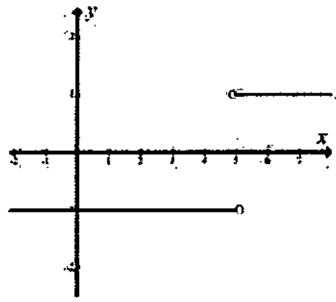

Fig. 9.18

(13) $\lim\limits_{x\to 1}\sin \pi x$

(14) $\lim\limits_{x\to 0}\sec x$

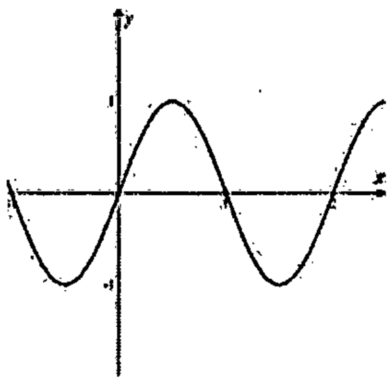

Fig. 9.19

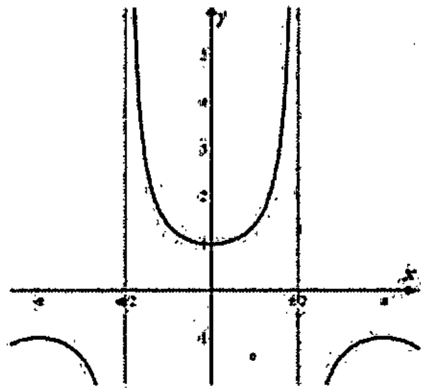

Fig. 9.20

(15) $\lim\limits_{x\to \frac{\pi}{2}}\tan x$

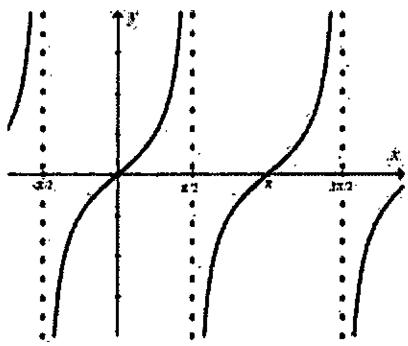

Fig. 9.21

Sketch the graph of f, then identify the values of x0 for which lim x →x0 f (x) exists.

$$(16)\quad f(x)=\begin{cases} x^2, & x\le 2\\ 8-2x, & 2<x<4\\ 4, & x\ge 4\end{cases}$$

$$(17)\quad f(x)=\begin{cases} \sin x, & x<0\\ 1-\cos x, & 0\le x\le \pi\\ \cos x, & x>\pi\end{cases}$$

(18) Sketch the graph of a function f that satisfies the given values :

(i) $f(0)$ is undefined

$\lim_{x\to 0} f(x) = 4$

$f(2) = 6$

$\lim_{x\to 2} f(x) = 3$

(ii) $f(-2) = 0$

$f(2) = 0$

$\lim_{x\to -2} f(x) = 0$

$\lim_{x\to 2} f(x)$ does not exist.

(19) Write a brief description of the meaning of the notation lim x→8f (x) = 25. .

(20) If f(2) = 4, can you conclude anything about the limit of f(x) as x approaches 2?

(21) If the limit of f(x) as x approaches 2 is 4, can you conclude anything about f(2)? Explain reasoning.

(22) Evaluate : $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$ if it exists by finding f (3-) and f (3+) .

(23) Verify the existence of $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} \dfrac{|x-1|}{x-1}, & \text{for } x \neq 1 \\ 0, & \text{for } x = 1 \end{cases}$.

ANSWERS

Exercise 9.1

(1) $0.\bar{3}$	(2) 0.25	(3) 0.288	(4) −0.25
(5) 1	(6) 0	(7) 1	(8) 3
(9) 2	(10) 3	(11) does not exist	(12) does not exist
(13) 0	(14) 1	(15) does not exist	
(16) except at $x_0 = 4$	(17) except at $x_0 = \pi$	(19) $f(8^-) = f(8^+) = 25$	
(20) No	(21) $f(2)$ cannot be concluded	(22) 6, 6	(23) does not exist

EXERCISE 9.2

Evaluate the following limits :

(1) $\lim_{x\to 2} \dfrac{x^4-16}{x-2}$

(2) $\lim_{x\to 1} \dfrac{x^m-1}{x^n-1}$, m and n are integers.

(3) $\lim_{\sqrt{x}\to 3} \dfrac{x^2-81}{\sqrt{x}-3}$

(4) $\lim_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}, x > 0$

(5) $\lim_{x\to 5} \dfrac{\sqrt{x+4}-3}{x-5}$

(6) $\lim_{x\to 2} \dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$

(7) $\lim_{x\to 1} \dfrac{\sqrt{x}-x^2}{1-\sqrt{x}}$

(8) $\lim_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$

(9) $\lim_{x\to 0} \dfrac{\sqrt{1+x}-1}{x}$

(10) $\lim_{x\to 1} \dfrac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1}$

(11) $\lim_{x\to 2} \dfrac{2-\sqrt{x+2}}{\sqrt[3]{2}-\sqrt[3]{4-x}}$

(12) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x^2}-1}{x}$ (13) $\lim\limits_{x \to 0} \dfrac{\sqrt{1-x}-1}{x^2}$ (14) $\lim\limits_{x \to 5} \dfrac{\sqrt{x-1}-2}{x-5}$

(15) $\lim\limits_{x \to a} \dfrac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2} (a > b)$

ANSWERS

Exercise 9.2

(1) 32 (2) $\dfrac{m}{n}$ (3) 108 (4) $\dfrac{1}{2\sqrt{x}}$

(5) $\dfrac{1}{6}$ (6) $-\dfrac{1}{4}$ (7) 3 (8) 4

(9) $\dfrac{1}{2}$ (10) $-\dfrac{1}{4}$ (11) $-\dfrac{3}{4}\sqrt[3]{4}$ (12) 0

(13) $f(x) \to -\infty$ as $x \to 0$ (limit does not exist) (14) $\dfrac{1}{4}$ (15) $\dfrac{1}{4a\sqrt{a-b}}$

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