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Syllabus

B. Sc. (Part III) Mathematics

MATHEMATICS–I

(SC-127)

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UNIT

1

SEQUENCES

STRUCTURE

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LEARNING OBJECTIVES

After going through this unit you will learn :

- What is meant by sequences ?
- How to classify the convergent, divergent and oscillatory sequences.

• 1.1. INTODUCTION

George Cantor (1845–1918) is known as the **creater** of the set theory. He made a considerable contribution to the development of the theory of real sequence, and found a firm base for most of the fundamental concepts of real analysis in the sequence of rational numbers. Though his lay-outs are not convenient in the initial stages, they are quite advantageous while making advanced investigations. The study of many important and advanced concepts becomes easy if the notion of the sequence is employed.

Set of Numbers

We shalll be using capital latters **N, I, Q** and **R** for the set of numbers as specified below :

$\mathbf{N} = \{n : n = 1, 2, 3, \ldots\}$, the set of natural numbers,

$\mathbf{I} = \{x : x = \ldots -2, -1, 0, 1, 2, \ldots\}$, the set of integers,

$\mathbf{Q} = \{x : x$ is a rational numbers$\}$, the set of rational numbers

and $\mathbf{R} = \{x : x$ is a real numbers$\}$, the set of real numbers.

• 1.1. SEQUENCES

Let **N** *be the set of all natural numbers and S be any set of real numbers. A function whose domain is the set of natural numbers and range is a subset of S, is called a sequence in S.*

Symbolically, if we define a function $f : \mathbf{N} \to S$, then f is a sequence. As in the case of function, we shall denote a sequence in a number of ways :

(i) Usually a sequence is denoted by its images. For a sequence f, the image corresponding to $n \in N$ is denote by f_n or $f\langle n \rangle$ and is called the n^{th} term of the sequence f. For example $\langle 1, 4, 9, \ldots \rangle$ is the sequence whose n^{th} term is n^2.

(ii) Using in order, the first few elements of a sequence, till the rule for writing down different elements becomes clear. For example $\langle 1, 2, 3, \ldots \rangle$ is the sequence whose n^{th} term is n.

(iii) Defining a sequence by a recurrence formula *i.e., by a rule which expresses the n^{th} term by the $(n-1)^{th}$ term.* For example, let

$$a_1 = 1,\ a_{n+1} = 2a_n, \text{ for all } n \geq 1.$$

These above relations define a sequene whose n^{th} term is 2^{n-1}.

Examples :

(i) $\langle \frac{1}{n} \rangle$ is the sequence $\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots, \frac{1}{n} \ldots \rangle$

(ii) $\langle \frac{1}{n^3} \rangle$ is the sequence $\langle 1, \frac{1}{2^3}, \frac{1}{3^3}, \frac{1}{4^3}, \ldots \frac{1}{n^3}, \ldots \rangle$

(iii) $\langle -2n \rangle$ is the sequence $\langle -2, -4, -6, \ldots -2n, \ldots \rangle$

(iv) $\langle \frac{n}{n+1} \rangle$ is the sequence $\langle \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \ldots, \frac{n}{n+1} \ldots \rangle$.

Range of a sequences. The set of all distinct terms of a sequence is known as its range.

Constant sequence. A sequence $\langle s_n \rangle$ defined by $s_n = a$ for all $n \in \mathbf{N}$, is called a constant sequence.

Equality of two sequences. Two sequences $\langle s_n \rangle$ and $\langle t_n \rangle$ are said to be equal, if $s_n = t_n$ $\forall\ n \in \mathbf{N}$.

Operations on sequences. Since the sequences are real valued functions, therefore, the sum, difference, product etc. of two sequences are defined as follows :

(i) If $\langle s_n \rangle$ and $\langle t_n \rangle$ be any two sequences, then the sequences whose n^{th} terms are $s_n + t_n$, $s_n - t_n$ and $s_n . t_n$ are respectively known as the sum, difference and product of the sequences $\langle s_n \rangle$ and $\langle t_n \rangle$ and are denoted by $\langle s_n + t_n \rangle$, $\langle s_n - t_n \rangle$ and $\langle s_n t_n \rangle$ respectively.

(ii) If $s_n \neq 0,\ \forall\ n \in \mathbf{N}$, then the sequence whose n^{th} term is $\frac{1}{s_n}$ is called the reciprocal of the sequence $\langle s_n \rangle$ and is denoted by $\langle \frac{1}{s_n} \rangle$.

(iii) The sequence whose n^{th} term is s_n/t_n $(t_n \neq 0,\ \forall\ n \in \mathbf{N})$ is known as the quotient of the sequence $\langle s_n \rangle$ by the sequence $\langle t_n \rangle$ and is denoted by $\langle \frac{1}{s_n} \rangle$.

(iv) The sequence whose n^{th} term is ks_n, where $k \in \mathbf{R}$ is known as the scalar multiple of the sequence $\langle s_n \rangle$ by k and is denoted by $\langle ks_n \rangle$.

• 1.2. BOUNDED SEQUENCES

(i) Bounded below sequence. A sequence $\langle s_n \rangle$ is said to be bounded below if there exists a real number l such that $s_n \geq l\ \ \forall\ n \in \mathbf{N}$.

The number $\boldsymbol{l}$ is known as the **lower bound** of the sequence $\langle s_n \rangle$.

(ii) Bounded above sequence. A sequence $\langle s_n \rangle$ is said to be bounded above if there exists a real number u such that $s_n \leq u\ \forall\ n \in \mathbf{N}$.

The number $\boldsymbol{u}$ is said to be **upper bound** of the sequence $\langle s_n \rangle$.

(iii) Bounded sequence. A sequence $\langle s_n \rangle$ is said to be bounded if it is bounded above as well as bounded below.

Or

A sequence $\langle s_n \rangle$ is bounded if there exist two real numbers l and u $(l \leq u)$ such that $l \leq s_n \leq u\ \forall\ n \in \mathbf{N}$.

Equivalently, a sequence is bounded iff there exists a real number $k > 0$ such that

$$|s_n| \leq k \quad \forall\ n \in \mathbf{N}.$$

(iv) Unbounded sequence. A sequence $\langle s_n \rangle$ is said to be unbounded if it is **not** bounded.

In sequences, terms with equal values can occur. Therefore, a sequence may have more than one term with the smallest value. In such a case any of those is taken for the smallest value. In fact while talking about the smallest value we are interested in the value of the term rather than the position of the term in the sequence. Similar explanation holds for the greatest value. Note that, like sets of real numbers, a sequence bounded below or above may or may not have a smallest or a greatest member accordingly. Clearly, an unbounded sequence can not have a smallest or a greatest member.

(v) Least upper bound. If a sequence $\langle s_n \rangle$ is bounded above, then there exists a number u_1 such that

$$s_n \leq u_1 \quad \forall\, n \in N. \qquad ...(1)$$

This number u_1 is called an upper bound of the sequence $\langle s_n \rangle$. If $u_1 < u_2$, then from (1) we find that

$$s_n < u_2 \quad \forall\, n \in \mathbf{N}$$

which implies, u_2 is also an upper bound of the sequence $\langle s_n \rangle$. Hence, we can say any number greater than u_1 is an upper bound of $\langle s_n \rangle$.

Hence, a sequence has an infinite number of upper bounds if it is bounded above. Let u is the least of all the upper bounds of the sequence $\langle s_n \rangle$. Then u is defined as the least upper bound (l.u.b.) or **supremum** of the sequence $\langle s_n \rangle$.

(vi) Greatest lower bound. If a sequence $\langle s_n \rangle$ is bound below then there exists a number $l_1 \in R$ such that

$$l_1 \leq s_n \quad \forall\, n \in \mathbf{N}. \qquad ...(ii)$$

This number l_1 is known as the lower bound of $< s_n >$. If $l_2 < l_1$, then from (ii) we have

$$l_2 < s_n \quad \forall\, n \in \mathbf{N}$$

which implies, l_2 is also a lower bound of the sequence $\langle s_n \rangle$. Hence, we can say any number less than l_1 is a lower bound of $\langle s_n \rangle$.

Hence, a sequence has infinite number of lower bounds, if it is bounded below. Let l is the greatest of all the lower bounds of the sequence $\langle s_n \rangle$. Then l is known as greatest lower bound (g.l.b.) or **infimum** of the sequence $\langle s_n \rangle$.

Examples :

(i) The sequence $\langle n^2 \rangle$ is bounded below by 1 but not bounded above.

(ii) The sequence $\langle \frac{n}{n+1} \rangle$ is bounded as $\frac{1}{2} \leq \frac{n}{n+1} < 1 \quad \forall\, n \in \mathbf{N}$.

(iii) The sequence $\langle -n^2 \rangle$ is bounded above by -1 but not bounded below.

(iv) The sequence $\langle \frac{1}{n} \rangle$ is bounded since $\left|\frac{1}{n}\right| \leq 1 \quad \forall\, n \in \mathbf{N}$.

(v) The sequence $\langle (-1)^n \rangle$ is bounded since $|(-1)^n| \leq 1 \;\forall\, n \in \mathbf{N}$.

$$(\because \; |(-1)^n| = 1 \quad \forall\, n \in \mathbf{N})$$

(vi) The sequence $\langle s_n \rangle$ defined by $s_n = 1 + (-1)^n$ for all $n \in \mathbf{N}$ is bounded since the range set of the sequence is $\{0, 2\}$, which is a finite set.

(vii) The sequence $\langle (-1)^n/n \rangle$ is bounded since $|(-1)^n/n| \leq 1$ for all $n \in \mathbf{N}$.

(viii) The sequence $\langle 2^n \rangle$ is bounded below and has smallest term as 2. Every member of $]-\infty, 2]$ is a lower bound of the sequence and the sequence is unbounded above.

Theorem 1. *A sequence $\langle s_n \rangle$ is bounded iff there exists a positive integer m and $l \in \mathbf{R}, a < 0$ such that*

$$|s_n - l| < a \quad \forall\, n \geq m.$$

Proof. Let $\langle s_n \rangle$ be a bounded sequence. Then there exist two real numbers c_1 and c_2 such that

$$c_1 < s_n < c_2 \quad \forall\, n \in \mathbf{N}$$

or
$$\left(c_1 - \frac{c_1 + c_2}{2}\right) < \left(s_n - \frac{c_1 + c_2}{2}\right) < \left(c_2 - \frac{c_1 + c_2}{2}\right) \quad \forall\, n \in \mathbf{N}$$

or
$$\frac{c_1 - c_2}{2} < \left(s_n - \frac{c_1 + c_2}{2}\right) < \frac{c_2 - c_1}{2} \quad \forall\, n \in \mathbf{N}$$

or
$$-a < (s_n - l) < a \quad \forall\, n \in \mathbf{N} \text{ where, } a = \frac{c_2 - c_1}{2} \text{ and } l = \frac{c_1 + c_2}{2}$$

or
$$|s_n - l| < a \quad \forall\, n \in \mathbf{N} \text{ where } m = 1 \in N, l \in \mathbf{R} \text{ and } a > 0.$$

Conversly, let there exists $l \in \mathbf{R}$, $a > 0$ and $m \in \mathbf{N}$ such that

$$|s_n - l| < a \quad \forall\, n \geq m$$

This gives $\quad l - a < s_n < l + a \quad \forall\, n \geq m$

Let $\quad k_1 = \min\{s_1, s_2, \ldots, s_{m-1}, l - a\}$

and $\quad k_2 = \max\{s_1, s_2, \ldots s_{m-1}, l + a\}$.

Then $\quad k_1 \leq s_n \leq k_2 \quad \forall\, n \in \mathbf{N}$.

Hence, $\langle s_n \rangle$ is bounded sequence.

Limit point of the sequence. *A real number l is called a limit point of a sequence $\langle s_n \rangle$ if every nbd of l contains infinite number of terms of the sequence.*

Thus $l \in \mathbf{R}$ is a limit point of the sequence $\langle s_n \rangle$ if for given $\varepsilon > 0$ $s_n \in\,]l - \varepsilon, l + \varepsilon[$, for infinitely many points.

The limit points of a sequence may be classified in **two** types :

(i) those for which $l = s_n$ for infinitely many values of $n \in \mathbf{N}$.

(ii) those for which $l = s_n$ for only a finite number of values of $n \in \mathbf{N}$.

But this distinction is not very much needed. As such we do not distinguish the above mentioned two types of limit points of sequences by different titles.

Examples on Limit Points :

(i) The sequence $\langle \frac{1}{n} \rangle$ has one limit point namely 0.

(ii) The sequence $\langle (-1)^n \rangle$ has two limit points 1 and -1.

(iii) The sequence $\langle n \rangle$ has no limit point.

(iv) The sequence $\langle 1 + \frac{(-1)^n}{n} \rangle$ has one limit point *i.e.*, 1.

(v) The sequence $\langle 1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4} \ldots \rangle$ has one limit point *i.e.*, 1.

(vi) The sequence $\langle n + 1 \rangle$ has no limit point.

Sufficient Conditions for number *l* to be or not to be a limit point of the Sequence $\langle s_n \rangle$.

(i) If for every $\varepsilon > 0$, $\exists\, m \in \mathrm{N}$ such that $s_n \in\,]l - \varepsilon, l + \varepsilon[\;\; \forall\, n \geq m$ or equivelently $|s_n - l| < \varepsilon \;\forall\, n \geq m$, then l is the limit point of the sequence $\langle s_n \rangle$.

(ii) If for any $\varepsilon > 0$, $s_n \in\,]l - \varepsilon, l + \varepsilon[$ for only a finite number of values of n, then l is not a limit point of the sequnece $\langle s_n \rangle$. Such a condition is also necessary for a number l not to be a limit point of the sequence $\langle s_n \rangle$.

Theorem 1. (Bolzano-Weierstrass Theorem for sequence).

Every bounded sequence has at least one limit points.

Proof. Let $S = \{s_n : n \in \mathbf{N}\}$ be the range set of the bounded sequence $\langle s_n \rangle$. Then S is bounded set. Now there may be two cases :

Case I. Let S be a finite set. Then $s_n = p$ for infinitely many indices n. Here $p \in \mathbf{R}$. Obviously p is a limit point of $\langle s_n \rangle$.

Case II. Let S be an infinite set. Since S is bounded, then by Bolzano- Weierstrass theorem for sets of real numbers, S has a limits point, say p. Therefore every nbd of p contains infinitely many distinct point of S *i.e.*, infinitely many term of $\langle s_n \rangle$ and hence p is a limit point of the sequence $\langle s_n \rangle$.

• 1.3. LIMIT SUPERIOR AND LIMIT INFERIOR

The greatest limit point of a bounded sequence is called the upper limit or limit superior and is denoted by $\overline{\lim}\, s_n$ and the smallest limit point of a bounded sequence is called the lower limit or limit inferior and is denoted by $\underline{\lim}\, s_n$.

- By definition it is obvious that $\underline{\lim}\, s_n \leq \overline{\lim}\, s_n$.

- A bounded sequence $\langle s_n \rangle$ for which the upper limit and lower limit coincide with real number l is said to converge to l.

Limit of sequence. *A sequence $\langle s_n \rangle$ is said to have a limit l iff for a given $\varepsilon > 0$ $\exists$, a positive integer m such that*

$$|s_n - l| < \varepsilon, \quad \forall\, n \geq m.$$

• 1.4. CONVERGENT SEQUENCES

Definition (1) : *A sequence $\langle s_n \rangle$ is said to converge to a number l, if for a given, $\varepsilon > 0$ there exists a positive integer m such that*

$$|s_n - l| < \varepsilon, \quad \forall\, n \geq m.$$

The number l is called the **limit** of the sequence $\langle s_n \rangle$ and can be written as

$$s_n \to l \text{ as } n \to \infty \quad \text{or} \quad \lim_{n\to\infty} s_n = l \quad \text{or} \quad \lim s_n = l.$$

Definition (2) : *A sequence $\langle s_n \rangle$ is said to be convergent iff it is bounded and has one and only one limit point.*

In such a case the sequence is said to converge to this limit point l.

• 1.5. SUBSEQUENCES

Let $\langle s_n \rangle$ be any sequence. If $\langle n_1, n_2, \ldots, n_k \ldots \rangle$ be a strictly increasing sequence of positive integers *i.e.*, $i > j \Rightarrow n_i > n_j$, then the sequence

$$\langle s_{n_1}, s_{n_2}, \ldots, s_{n_k} \ldots \rangle$$

is called a subsequence of $\langle s_n \rangle$.

SOME IMPORTANT THEOREMS

Theorem 1. *If $\langle s_n \rangle$ is a sequence of non-negative numbers such that* $\lim s_n = l$, *then* $l \geq 0$.

Proof. Let, if possible $l < 0$ then $-l > 0$. Now $\lim s_n = l$, therefore, for $\varepsilon = -\frac{l}{2} > 0$, there exists a positive integer m such that

$$|s_n - l| < -\frac{l}{2}, \quad \forall\, n \geq m.$$

In particular

$$|s_m - l| < -\frac{l}{2}$$

$$\Rightarrow \qquad l + \frac{l}{2} < s_m < l - \frac{l}{2}$$

$$\Rightarrow \qquad s_m < \frac{l}{2} < 0.$$

which is a contradiction, because $s_m \geq 0$. Therefore our assumption is wrong. Hence, we must have $l \geq 0$.

Theorem 2. *A sequence can not converge to more than one limit point.*

Or

Limit of a sequence is unique.

Proof. Let if possible, a sequence $\langle s_n \rangle$ converges to two distinct numbers l_1 and l_2.

Now $\qquad l_1 \neq l_2 \Rightarrow l_1 - l_2 \neq 0$

$\qquad\qquad \Rightarrow |l_1 - l_2| > 0.$

Let $\varepsilon = \frac{1}{2}|l_1 - l_2|$; then $\varepsilon > 0$.

Since $\langle s_n \rangle$ converges to l_1, there must exists a positive integer m_1 such that

$$|s_n - l_1| < \varepsilon, \quad \forall\, n \geq m_1. \qquad \ldots(1)$$

Similarly $\langle s_n \rangle$ converges to l_2, there must exists a positive integer m_2 such that

$$|s_n - l_2| < \varepsilon \quad \forall\, n \geq m_2. \qquad \ldots(2)$$

Now, let $\qquad m = \max\{m_1, m_2\}$.

Then result (1) and (2) hold for all $n \geq m$. So for all $n \geq m$ we have

$$|l_1 - l_2| = |(s_n - l_1) - (s_n - l_2)|$$
$$\leq |s_n - l_1| + |s_n - l_2|$$

$$< \varepsilon + \varepsilon \qquad \text{[Using (1) and (2)]}$$
$$= 2\varepsilon$$
$$= | l_1 - l_2 |$$
$$\Rightarrow \quad | l_1 - l_2 | < | l_1 - l_2 |$$

which is absurd, hence we must have $l_1 = l_2$ *i.e.*, the limit of the sequence is unique.

Theorem 3. *Every convergent sequence is bounded.*

Proof. Let $\langle s_n \rangle$ be a sequence which converges to l. Take $\varepsilon = 1$. Then there exists a positive integer m such that

$$| s_n - l | < 1, \quad \forall\, n \geq m$$

i.e.,
$$(l - 1) < s_n < (l + 1), \quad \forall\, n \geq m.$$

Let $\quad k_1 = \min \{s_1, s_2, \ldots, s_{m-1}, l - 1\}$

and $\quad k_2 = \max \{s_1, s_2, \ldots, s_{m-1}, l + 1\}$

therefore $\quad k_1 \leq s_n \leq k_2 \quad \forall\, n \in \mathbf{N}.$

Hence the sequence $\langle s_n \rangle$ is bounded.

Note. The converse of the above theorem is **not** necessarily true. *i.e.*, a bounded sequence need not be convergent. For example $\langle (-1)^n \rangle$ is bounded but not convergent.

Theorem 4. *If $\langle s_n \rangle$ converges to l, then any subsequence of $\langle s_n \rangle$ also converges to l.*

Proof. Let $\langle s_{n_k} \rangle$ be any subsequence of $\langle s_n \rangle$. Then by definition of subsequence $n_1, n_2, \ldots, n_k, \ldots$ are positive integers such that

$$n_1 < n_2 < \ldots < n_k < \ldots$$

Now $\quad n_1 \geq 1 \Rightarrow n_k \geq k. \qquad$ (By induction)

Since $\langle s_n \rangle$ converges to l, so given $\varepsilon > 0$, there exists a positive integer m such that

$$| s_k - l | < \varepsilon, \quad \forall\, k \geq m$$

for $k \geq m$, we have

$$n_k \geq k \geq m$$

therefore $\quad | s_{n_k} - l | < \varepsilon$, for all $n_k \geq m$

$\therefore \quad \langle s_{n_k} \rangle$ converges to l.

Theorem 5. *The limit of the sum of two convergent sequences is the sum of their limits.*

Proof. Let $\langle s_n \rangle$ and $\langle t_n \rangle$ be the two given sequences such that

$$\lim s_n = l_1 \qquad \ldots(1)$$

and
$$\lim t_n = l_2. \qquad \ldots(2)$$

Since, $\lim s_n = l_1$, therefore for a given $\varepsilon > 0$, there exists a positive integer m_1 such that

$$| s_n - l_1 | < \varepsilon/2, \quad \forall\, n \geq m.$$

Similarly, $\lim t_n = l_2$, therefore, for a given $\varepsilon > 0$, there must exists a positive integer m_2 such that

$$| t_n - l_2 | < \varepsilon/2, \quad \forall\, n \geq m_2.$$

Let $\quad m = \max \{m_1, m_2\}.$

Therefore $\quad | s_n - l_1 | < \varepsilon/2, \quad \forall\, n \geq m$

and $\quad | t_n - l_2 | < \varepsilon/2, \quad \forall\, n \geq m.$

Now, consider

$$| (s_n + t_n) - (l_1 + l_2) | = | (s_n - l_1) + (t_n - l_2) |, \quad \forall\, n \geq m$$
$$\leq | s_n - l_1 | + | t_n - l_2 |, \quad \forall\, n \geq m$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \ \forall\, n \geq m.$$

Therefore, the sequence $\langle s_n + t_n \rangle$ is convergent and

$$\mathbf{lim}\ (s_n + t_n) = l_1 + l_2 = \mathbf{lim}\ s_n + \mathbf{lim}\ t_n.$$

Theorem 6. *If $\lim s_n = l_1$ and $\lim t_n = l_2$, then $\lim (s_n t_n) = l_1 . l_2$.*

Proof. We have

$$| s_n t_n - l_1 l_2 | = | s_n t_n - l_1 t_n + l_1 t_n - l_1 l_2 |$$
$$= | t_n (s_n - l_1) + l_1 (t_n - l_2) |$$
$$\leq | t_n | | s_n - l_1 | + | l_1 | | t_n - l_2 |. \qquad \ldots(1)$$

The sequence $\langle t_n \rangle$ is convergent, therefore it is bounded, ($\because$ Every convergent sequence is bounded) so there must exists a positive real no c such that

$$|t_n| \le c, \quad \forall\, n \in \mathbf{N}. \qquad \text{...(2)}$$

Since the sequences $\langle s_n \rangle$ and $\langle t_n \rangle$ both are convergent, there must exist, positive integers m_1 and m_2 such that

$$|s_n - l_1| < \varepsilon/2c, \quad \forall\, n \ge m_1 \qquad \text{...(3)}$$

and
$$|t_n - l_2| < \varepsilon/2c, \quad \forall\, n \ge m_2. \qquad \text{...(4)}$$

Let $\qquad m = \max\{m_1, m_2\}$.

From (1), (2), (3) and (4) we have

$$|s_n t_n - l_1 l_2| < c \cdot \frac{\varepsilon}{2c} + |c| \cdot \frac{\varepsilon}{2|c|}, \quad \forall\, n \ge m$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \forall\, n \ge m.$$

Therefore $\qquad \mathbf{lim}\ (s_n t_n) = l_1 l_2$.

Theorem 7. *If* $lim\ s_n = l_1$, $l_1 \ne 0$ *and* $s_n \ne 0$, $\forall\, n \in \mathbf{N}$ *then*

$$lim\left(\frac{1}{s_n}\right) = \frac{1}{l_1}.$$

Proof. Since $l_1 \ne 0$, there exists a positive number c and positive integer m_1 such that

$$|s_n| > c, \quad \forall\, n \ge m_1. \qquad \text{...(1)}$$

Also $\lim s_n = l_1$, therefore, for a given $\varepsilon > 0$, there must exists a positive integer m_2 such that

$$|s_n - l_1| < c\,|l_1|\,\varepsilon, \quad \forall\, n \ge m_2. \qquad \text{...(2)}$$

Let $m = \max\{m_1, m_2\}$. Then

$$\left|\frac{1}{s_n} - \frac{1}{l_1}\right| = \left|\frac{s_n - l_1}{|s_n||l_1|}\right| < \frac{c\,|l_1|}{c\,|l_1|}\varepsilon, \quad \forall\, n \ge m$$
$$= \varepsilon, \quad \forall\, n \ge m.$$

Therefore, $\qquad \mathbf{lim}\ \dfrac{1}{s_n} = \dfrac{1}{l_1}$.

Theorem 8. *If* $lim\ s_n = l_1$ *and* $lim\ t_n = l_2$ $(l_2 \ne 0)$, $t_n \ne 0$, $\forall\, n \in \mathbf{N}$ *then*

$$lim\ \frac{s_n}{t_n} = \frac{l_1}{l_2}.$$

Proof. We have

$$\lim \left|\frac{s_n}{t_n}\right| = \lim\left(s_n \frac{1}{t_n}\right)$$
$$= \lim (s_n) \cdot \lim\left(\frac{1}{t_n}\right)$$

[$\because$ limit of the product of two sequence is equal to the product of the limits]

$$= l_1 \cdot \frac{1}{l_2} \qquad \text{[By previous theorem]}$$

$$\Rightarrow \qquad \lim_{n \to \infty} \frac{s_n}{t_n} = \frac{l_1}{l_2}.$$

• 1.6. DIVERGENT SEQUENCES

Definition (i) *A sequence* $\langle s_n \rangle$ *is said to diverge to* $+\infty$, *if for every real number* $k > 0$, *there exists a positive integer m such that*

$s_n > k$, $\forall\, n \ge m$.

Definition (ii) *A sequence* $\langle s_n \rangle$ *is said to diverge to* $-\infty$, *if for every real number* $k < 0$, *there exists a positive integer m such that*

$s_n < k$, $\forall\, n \ge m$.

Definition (iii) *A sequence is said to be divergent sequence, if it diverges to either* $+\infty$ *or* $-\infty$.

Definition (iv) *A sequence, which is not convergent, is known as divergent sequence.*

Examples :

(i) $\langle 3, 3^2, 3^3, \ldots \rangle$ diverges to $+\infty$.

(ii) $\langle -2, -2^2, -2^3, \ldots \rangle$ diverges to $-\infty$.

(iii) $\langle 2, 4, 6, \ldots, 2n, \ldots \rangle$ diverges to $+\infty$.

(iv) $\langle -2, -4, -6, \ldots, -2n, \ldots \rangle$ diverges to $-\infty$.

• 1.7. OSCILLATORY SEQUENCE

*A sequence $\langle s_n \rangle$ is said to be an **oscillatory** sequence if it is neither convergent nor divergent.*

An oscillatory sequence is said to oscillate finitely or infinitely according as it is bounded or unbounded.

In other words, we can say

(i) A bounded sequence, which is not convergent is said to **oscillate finitely.**

(ii) An unbounded sequene which does not diverge, is said to **oscillate infinitely.**

(iii) A bounded sequence which does not converge and has at least two limit points, is said to be **oscillate finitely.**

Examples :

(i) $\langle 1 + (-1)^n \rangle$ oscillate finitely.

(ii) $\langle (-1)^n \rangle$ oscillate finitely.

(iii) $\langle (-1)^n \left(1 + \frac{1}{n}\right) \rangle$ oscillate finitely.

(iv) $\langle n(-1)^n \rangle$ oscillate infinitely.

SOME IMPORTANT THEOREMS

Theorem 1. *If a sequence $\langle s_n \rangle$ diverges to infinitely, then any subsequence of $\langle s_n \rangle$ also diverges to infinitely.*

Proof. Let $\langle s_{n_k} \rangle$ be any subsequence of the sequence $\langle s_n \rangle$. Then by the definition of subsequence $\langle n_1, n_2, \ldots, n_k, \ldots \rangle$ is a strictly increasing sequence of positive integers

$\Rightarrow$ $$n_1 \geq 1 \Rightarrow n_k \geq k. \qquad \text{(By induction)}$$

Take any positive real number c_1.

Now $\langle s_n \rangle$ diverges to $\infty \Rightarrow$ for $c_1 > 0 \; \exists \; m \in N$ such that $s_n > c_1$ for all $n \geq m$ *i.e.*, $s_k > c_1$, $\forall \; k \geq m$ for $k \geq m$, we have $n_k \geq k \geq m$ *i.e.*, $n_k \geq m$.

$\therefore$ $s_{n_k} > c_1$ for all $c_k \geq m$.

$\Rightarrow$ $\langle s_{n_k} \rangle$ diverges to ∞.

Theorem 2. *If the sequence $\langle s_n \rangle$ diverges to infinity and the sequence $\langle t_n \rangle$ is bounded, then $\langle s_n + t_n \rangle$ diverges to infinity.*

Proof. The sequence $\langle t_n \rangle$ is bounded; therefore for arbitrary positive number k_1 such that

$$|t_n| < k_1.$$

Also, the sequence $\langle s_n \rangle$ diverges to infinity. Therefore for arbitrary positive number k there must exists a positive integer m such that

$$s_n > k + k_1, \quad \forall \; n \geq m.$$

Now, for all $n \geq m$, we have

$$s_n + t_n \geq s_n - |t_n| > k + k_1 - k_1 = k.$$

Thus for $k > 0$, $\exists$ a positive integer m such that

$$s_n + t_n > k, \quad \forall \; n \geq m.$$

$\Rightarrow$ The sequence $\langle s_n + t_n \rangle$ diverges to infinity.

Theorem 3. *If the sequences $\langle s_n \rangle$ and $\langle t_n \rangle$ both diverges to infinity, then the sequences $\langle s_n + t_n \rangle$ and $\langle s_n \cdot t_n \rangle$ diverges to infinity.*

Proof. Since, the sequence $\langle s_n \rangle$ diverges to infinity, therefore for $k_1 > 0$, there must exists a positive, integer m_1 such that $s_n > k_1 \; \forall \; n \geq m_1$. Similarly, the sequence $\langle t_n \rangle$ diverges to infinity, therefore for $k_2 > 0$, there must exists a positive integer m_2 such that

$$t_n > k_2, \quad \forall \; n \geq m_2.$$

Let $m = \max\{m_1, m_2\}$. Then

$$s_n + t_n > k_1 + k_2 = l_1 \text{ (say)}$$

and $$s_n t_n > k_1 \,.\, k_2 = l_2 \text{ (say)}.$$

Therefore, sequences $\langle s_n + t_n \rangle$ and $\langle s_n t_n \rangle$ diverges to infinity.

SOLVED EXAMPLES

Example 1. *Show that the sequence* $\langle \frac{1}{n} \rangle$ *converges to 0.*

Solution. Let $\langle s_n \rangle = \langle \frac{1}{n} \rangle$.

Now $$\lim_{n \to \infty} s_{2n} = \lim_{n \to \infty} \frac{1}{2n} = 0$$

and $$\lim_{n \to \infty} s_{2n+1} = \lim_{n \to \infty} \frac{1}{2n+1} = 0$$

Therefore $$\lim_{n \to \infty} s_{2n} = \lim_{n \to \infty} s_{2n+1} = 0$$

$\Rightarrow$ $$\lim_{n \to \infty} s_n = 0, \ \forall\, n \in \mathbf{N}.$$

Since 0 is a finite quantity. Hence, the sequence $\langle s_n \rangle$ is convergent and converges to 0.

Example 2. *Show that the sequence* $\langle (-1)^n/n \rangle$ *is convergent.*

Solution. Let $\langle s_n \rangle = \langle (-1)^n/n \rangle$.

Here $$\lim_{n \to \infty} s_{2n} = \lim_{n \to \infty} \frac{(-1)^{2n}}{2n} = \lim_{n \to \infty} \frac{1}{2n} = 0$$

and $$\lim_{n \to \infty} s_{2n+1} = \lim_{n \to \infty} \frac{(-1)^{2n+1}}{2n+1} = \lim_{n \to \infty} \frac{-1}{2n+1} = 0$$

which gives, $$\lim_{n \to \infty} s_{2n} = \lim_{n \to \infty} s_{2n+1} = 0$$

$\Rightarrow$ $$\lim_{n \to \infty} s_n = 0, \ \forall\, n \in \mathbf{N}.$$

Since 0 is a finite quantity. Hence, the given sequence $\langle s_n \rangle$ is a convergent sequence.

Example 3. *Discuss the convergence of the sequence* $\langle \frac{1}{3^n} \rangle$.

Solution. Let $\langle s_n \rangle = \langle \frac{1}{3^n} \rangle$.

Then $$\lim_{n \to \infty} s_{2n} = \lim_{n \to \infty} \frac{1}{3^{2n}} = 0$$

and $$\lim_{n \to \infty} s_{2n+1} = \lim_{n \to \infty} \frac{1}{3^{2n+1}} = 0$$

which implies $$\lim_{n \to \infty} s_{2n} = \lim_{n \to \infty} s_{2n+1} = 0$$

Therefore, $$\lim_{n \to \infty} s_n = 0, \ \forall\, n \in \mathbf{N}.$$

Since 0 is a finite quantity, hence, the given sequence $\langle s_n \rangle$ is a convergent sequence.

Example 4. *Show that the sequence* $\langle s_n \rangle$ *defined by*

$$s_n = \langle (\sqrt{n+1} - \sqrt{n}) \rangle, \ \forall\, n \in \mathbf{N}$$

is convergent.

Solution. We have

$$s_n = \sqrt{n+1} - \sqrt{n}$$

For any $\varepsilon > 0$, $|s_n - 0| = \sqrt{n+1} - \sqrt{n} < \varepsilon$

$\Rightarrow$ $$\sqrt{n+1} < (\varepsilon + \sqrt{n}).$$

$\Rightarrow$ $$n + 1 < \varepsilon^2 + 2\varepsilon\sqrt{n} + n.$$

$\Rightarrow$ $$1 < \varepsilon^2 + 2\varepsilon\sqrt{n}$$

i.e. if $$\frac{1}{4\varepsilon^2} < n$$

Thus, for any given $\varepsilon > 0$, $\exists\, m \left(> \frac{1}{4\varepsilon^2} \right) \in \mathbf{N}$ such that

$$|s_n - 0| < \varepsilon, \quad \forall\, n \geq m.$$

Therefore, $\lim s_n = 0.$

Since, 0 is a finite quantity. Hence, the given sequence $\langle s_n \rangle$ is convergent.

Example 5. *Show that the sequence $\langle s_n \rangle$ defined by $s_n = r^n$ converges to 0 if $|r| < 1$.*

Solution. If $|r| < 1$. Then

$$|r| = \frac{1}{1+h}, \text{ where } h > 0.$$

Since,
$$(1+h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \ldots + h^n$$
$$> 1 + nh \quad \forall\, n.$$

Now
$$|s_n - 0| = |r^n|$$
$$= |r|^n = \frac{1}{(1+h)^n}$$
$$< \frac{1}{1+nh} \quad \forall\, n.$$

Let $\varepsilon > 0$. Then

$$|s_n - 0| < \varepsilon \text{ if } \frac{1}{1+nh} < \varepsilon \text{ or } n > \left(\frac{1}{\varepsilon} - 1\right) \Big/ h.$$

Now, if we take a positive integer m such that $m > \left(\frac{1}{\varepsilon} - 1\right) \Big/ h$, then, for all $n \geq m$

$|s_n - 0| < \varepsilon.$

Hence, the sequence $\langle s_n \rangle$ converges to 0.

Example 6. *Show that the sequence $\langle s_n \rangle = \frac{3n}{n + 5n^{1/2}}$ has the limit 3.*

Solution. Let ε be any positive number.

Consider,
$$\left|\frac{3n}{n+5n^{1/2}} - 3\right| = \frac{15n^{1/2}}{n+5n^{1/2}} < \frac{15}{n^{1/2}}.$$

Therefore,
$$\left|\frac{3n}{n+5n^{1/2}} - 3\right| = \varepsilon \text{ if } \frac{15}{n^{1/2}} < \varepsilon \text{ or } n > \frac{225}{\varepsilon^2}.$$

If we choose a positive integer $m > \frac{225}{\varepsilon^2}$, then, we get

$$|s_n - 3| < \varepsilon, \quad \forall\, n \geq m.$$

Hence
$$\lim_{n \to \infty} s_n = 3.$$

Example 7. *Show that $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$.*

Solution. Let $\sqrt[n]{n} = 1 + h$, where $h \geq 0$

$$\Rightarrow \quad n = (1+h)^n$$
$$= 1 + nh + \frac{n(n-1)}{2!}h^2 + \ldots + h^n$$
$$\Rightarrow \quad n > \frac{n(n-1)}{2}h^2, \quad \forall\, n \qquad (\because\ h \geq 0)$$
$$\Rightarrow \quad h^2 < \frac{2}{n-1}, \quad \text{for } n \geq 2$$
$$\Rightarrow \quad |h| < \sqrt{\left(\frac{2}{n-1}\right)}, \text{ for } n \geq 2.$$

Let $\varepsilon > 0$ (any positive number, however small) then

$$|h| < \sqrt{\left(\frac{2}{n-1}\right)} < \varepsilon \text{ provided, } \frac{2}{n-1} < \varepsilon^2 \text{ or } n > \frac{2}{\varepsilon^2} + 1.$$

If we take $m \in \mathbf{N}$ such that $m > \frac{2}{\varepsilon^2} + 1$

then
$$|h| < \varepsilon \quad \forall\, n \geq m$$

or $$|\sqrt[n]{n}-1|<\varepsilon \quad \forall\, n \geq m \Rightarrow \lim_{n\to\infty} \sqrt[n]{n}=1.$$

Example 8. *If $\langle s_n \rangle$ be a sequence such that $s_n \neq 0$ for any $n \in \mathbf{N}$, and $\frac{s_{n+1}}{s_n} = l$. Then prove that if $|l|<1$, then $s_n \to 0$.*

Solution. Since $|l|<1$. Hence there exist $\varepsilon_1 > 0$ such that

$$|l|+\varepsilon_1 = h < 1.$$

Now $\frac{s_{n+1}}{s_n} \to l \Rightarrow$ there exists a positive integer m such that

$$\left|\frac{s_{n+1}}{s_n} - l\right| < \varepsilon, \quad \forall\, n \geq m.$$

We have

$$\left|\frac{s_{n+1}}{s_n}\right| = \left|\left(\frac{s_{n+1}}{s_n} - l\right) + l\right| \leq \left|\frac{s_{n+1}}{s_n} - l\right| + |l|$$

$$< \varepsilon_1 + |l|, \quad \forall\, n \geq m$$

i.e., $$\left|\frac{s_{n+1}}{s_n}\right| < h, \quad \forall\, n \geq m.$$

Replacing n by, $m, m+1, \ldots, n-1$ successively in the above equation and multiplying the corresponding sides of the resulting $(n-m)$ inequalities, we get

$$\left|\frac{s_{m+1}}{s_m}\right| \cdot \left|\frac{s_{m+2}}{s_{m+1}}\right| \cdots \left|\frac{s_n}{s_{n-1}}\right| < h^{n-m},$$

$$\Rightarrow \quad \left|\frac{s_{m+1}}{s_m} \cdot \frac{s_{m+2}}{s_{m+1}} \cdots \frac{s_n}{s_{n-1}}\right| < h^{n-m},$$

$$\Rightarrow \quad |s_n| < h^n\left(\frac{|s_m|}{h^m}\right), \quad \text{for all } n > m. \qquad \ldots(1)$$

Since, $0 < h < 1$, therefore $h^n \to 0$ and hence, given $\varepsilon > 0$, there exists a positive integer m_1 such that

$$|h^n| < \frac{h^m \varepsilon}{|s_m|}, \quad \forall\, n \geq m_1. \qquad \ldots(2)$$

Now, let us choose a positive integer p such that

$$p > \max\{m_1, m_2\}.$$

From (1) and (2), we get

$$|s_n| < \varepsilon \quad \forall\, n \geq p.$$

Hence $s_n \to 0$.

• 1.8. CAUCHY SEQUENCES

A sequence $\langle s_n \rangle$ is said to be Cauchy sequence if, given $\varepsilon > 0$ there exist $m \in \mathbf{N}$ such that

$$|s_n - s_m| < \varepsilon, \quad \forall\, n \geq m$$

or $$|s_p - s_q| < \varepsilon, \quad \forall\, p, q \geq m$$

or $$|s_{n+p} - s_n| < \varepsilon, \quad \forall\, n \geq m \text{ and } p > 0.$$

Examples :

(i) The sequence $\langle \frac{1}{2^n} \rangle$ is a Cauchy sequence.

(ii) The sequence $\langle \frac{1}{n} \rangle$ is a Cauchy sequence.

(iii) The sequence $\langle \frac{1}{n^2} \rangle$ is not a Cauchy sequence.

(iv) The sequence $\langle (-1)^n \rangle$ is not a Cauchy sequence.

SOME IMPORTANT THEOREMS

Theorem 1. *Every Cauchy sequence is bounded.*

Proof. Let $\langle s_n \rangle$ be a Cauchy sequence.

Taking $\varepsilon = 1$, there exists a positive integer m such that

$$|s_n - s_m| < 1, \quad \forall\, n \geq m$$

$$\Rightarrow \quad (s_m - 1) < s_n < (s_m + 1) \quad \forall\, n \geq m.$$

Let $\quad k = \min\{s_m - 1, s_1, s_2, \ldots, s_{m-1}\}$

and $\quad K = \max\{s_m + 1, s_1, s_2, \ldots, s_{m-1}\}$.

Then $\quad k \leq s_n \leq K, \quad \forall\, n.$

$\Rightarrow$ The sequence $\langle s_n \rangle$ is bounded.

Note. Converse of the above theorem is **not** necessarily true, *i.e.*, a bounded sequence need not be a Cauchy sequence, **for example**, the sequence $\langle (-1)^n \rangle$ is bounded, but is not a Cauchy sequence.

Theorem 2. (Cauchy's General Principle of Convergence). *A sequence is convergent if and only if it is a Cauchy sequence.*

Proof. Let us first suppose $\langle s_n \rangle$ be a convergent sequence. Let, this sequence converges to l.

$\therefore$ for a given $\varepsilon > 0$ these exists a positive integer m such that

$$|s_n - l| < \varepsilon/2, \quad \forall\, n \geq m. \qquad \ldots(1)$$

In particular, for $n = m$

$$|s_m - l| < \varepsilon/2. \qquad \ldots(2)$$

Now, consider

$$\begin{aligned} |s_n - s_m| &= |s_n - l + l - s_m| \\ &\leq |s_n - l| + |s_m - l| \\ &< \varepsilon/2 + \varepsilon/2, \quad \forall\, n \geq m \\ &= \varepsilon, \quad \forall\, n \geq m \end{aligned}$$

i.e., $\quad |s_n - s_m| < \varepsilon, \quad \forall\, n \geq m$

$\Rightarrow \quad \langle s_n \rangle$ is a Cauchy sequence.

Conversely, let $\langle s_n \rangle$ be a Cauchy sequence.

$\Rightarrow \langle s_n \rangle$ is a bounded sequence [By Theorem 1]

$\Rightarrow$ By Bolzano-Weierstress theorem $\langle s_n \rangle$ has at least one limit point, say l. We shall show that the sequence $\langle s_n \rangle$ converges to l.

Let $\varepsilon > 0$ be given.

Since, $\langle s_n \rangle$ is a Cauchy sequence

$\therefore \quad \exists$ *a positive integer* m *such that*

$$|s_n - s_m| < \varepsilon/3, \quad \forall\, n \geq m. \qquad \ldots(3)$$

Since, l is the limit point of $\langle s_n \rangle$.

$\therefore \quad$ for above choice of ε and m, $\exists$ a positive integer $k > m$ such that

$$|s_k - l| < \varepsilon/3. \qquad \ldots(4)$$

Since, $k > m$, therefore from (3)

$$|s_k - s_m| < \varepsilon/3. \qquad \ldots(5)$$

Now, consider

$$\begin{aligned} |s_n - l| &= |s_n - s_m + s_m - s_k + s_k - l| \\ &\leq |s_n - s_m| + |s_m - s_k| + |s_k - l| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon \end{aligned}$$

i.e., $\quad |s_n - l| < \varepsilon, \quad \forall\, n \geq m.$

$\Rightarrow \quad \langle s_n \rangle$ is convergent.

SOLVED EXAMPLES

Example 1. *If* $\langle s_n \rangle$ *is a sequence in* **R**, *where*

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

evaluate, $\lim_{n \to \infty} |a_{n+1} - a_n|$. *Verify, if this sequence satisfy the Cauchy criterion.*

Solution. Here $\quad s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

$$\Rightarrow \quad s_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \frac{1}{n+1}.$$

$$\therefore \quad s_{n+1} - s_n = \frac{1}{n+1}$$

$$\Rightarrow \quad \lim_{n \to \infty} |s_{n+1} - s_n| = 0.$$

Also, here we have

$$s_{2n} - s_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$$

$$\geq n\left(\frac{1}{2n}\right). \qquad \left(\because \frac{1}{n+1} > \frac{1}{2n} \text{ etc.}\right)$$

$$\Rightarrow \quad |s_{2n} - s_n| > \frac{1}{2} \quad \forall\, n \in \mathbf{N}.$$

$\Rightarrow$ there exists a positive integer k such that $|s_n - s_k| \geq \frac{1}{2}$ whenever $n \geq k$

$\Rightarrow$ Cauchy criterion is not satisfied.

Example 2. *Show by applying Cauchy's convergent criterion that the sequence* $\langle s_n \rangle$ *given by*

$$s_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \textit{ diverges.}$$

Solution. Here, we have

$$s_{n+1} = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} + \frac{1}{2(n+1)-1}$$

$$= 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} + \frac{1}{2n+1}$$

$$\therefore \quad s_{n+1} - s_n = \left[1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} + \frac{1}{2n+1}\right] - \left[1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right]$$

$$= \frac{1}{2n+1} > 0, \quad \forall\, n \in \mathbf{N}.$$

$$\therefore \quad s_{n+1} > s_n, \quad \forall\, n \in \mathbf{N}.$$

$\Rightarrow$ The sequence $\langle s_n \rangle$ is increasing sequence.

Also, we have

$$s_{2n} = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} + \frac{1}{2n+1} + \ldots + \frac{1}{4n-1}$$

$$\therefore \quad s_{2n} - s_n = \left[1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} + \frac{1}{2n+1} + \ldots + \frac{1}{4n-1}\right] - \left[1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right]$$

$$= \frac{1}{2n+1} + \frac{1}{2n+3} + \ldots + \frac{1}{4n-1}$$

$$\Rightarrow \quad s_{2n} - s_n > n\left(\frac{1}{4n}\right) \qquad \left(\because \frac{1}{2n-1} > \frac{1}{4n} \text{ etc. and there are } n \text{ terms}\right)$$

$$\Rightarrow \quad |s_{2n} - s_n| > \frac{1}{4}, \quad \forall\, n \in \mathbf{N}$$

$\Rightarrow$ there exists a positive integer k such that $|s_n - s_k| > \frac{1}{4}$ whenever $n \geq k$

$\Rightarrow$ Cauchy criterion is not satisfied.

$\Rightarrow$ The sequence $\langle s_n \rangle$ can not converge

$\Rightarrow$ The sequence $\langle s_n \rangle$ diverges to $+\infty$.

SOME IMPORTANT THEOREMS

Theorem 1. (Squeeze Principle). *If $\langle s_n \rangle$, $\langle t_n \rangle$ and $\langle u_n \rangle$ are three sequences such that*

(i) $s_n \le t_n \le u_n \ \forall n$

and (ii) $\langle s_n \rangle$ converges to l and $\langle u_n \rangle$ also converges to l, then $\langle t_n \rangle$ also converges to l.

Proof. Let $\varepsilon > 0$ be given. Since the sequences $\langle s_n \rangle$ and $\langle u_n \rangle$ converges to l, there must exist positive integers m_1 and m_2 such that

$$|s_n - l| < \varepsilon, \ \forall n \ge m_1 \qquad \text{...(1)}$$

$$|u_n - l| < \varepsilon, \ \forall n \ge m_2. \qquad \text{...(2)}$$

Let $m = \max\{m_1, m_2\}$. Then for $n > m$, we have

$$l - \varepsilon < s_n \le t_n \le u_n < l + \varepsilon$$

or $$l - \varepsilon < t_n < l + \varepsilon$$

or $$|t_n - l| < \varepsilon, \ \forall n \ge m$$

$\Rightarrow$ Hence $\lim t_n = l$

$\Rightarrow$ $\langle t_n \rangle$ converges to l.

Theorem 2. (Cauchy's first theorem on limits). *If $\lim\limits_{n \to \infty} s_n = l$, then*

$$\lim_{n \to \infty} \frac{s_1 + s_2 + \dots + s_n}{n} = l.$$

Proof. Let us define a sequence $\langle t_n \rangle$ in such a way that

$$t_n = s_n - l$$

then $$\lim t_n = \lim (s_n - l) = \lim s_n - l = l - l = 0$$

and $$\frac{s_1 + s_2 + \dots + s_n}{n} = l + \frac{t_1 + t_2 + \dots + t_n}{n}.$$

In order to prove this theorem, we have to show that

$$\lim \frac{t_1 + t_2 + \dots + t_n}{n} = 0.$$

Now, sequence $\langle t_n \rangle$ is convergent ($\because$ $\langle s_n \rangle$ is convergent), therefore it is bounded and hence there must exists a positive number k such that

$$|t_n| < k, \quad \forall n \in \mathbf{N}.$$

Also, $\langle t_n \rangle$ converges to zero. Therefore for a given $\varepsilon > 0$ there must exists a positive integer m such that

$$|t_n| < \varepsilon/2, \ \forall n \ge m.$$

Now, consider

$$\left|\frac{t_1 + t_2 + \dots + t_n}{n}\right| = \left|\frac{t_1 + t_2 + \dots + t_m}{n} + \frac{t_{m+1} + \dots + t_n}{n}\right|$$

$$\le \frac{|t_1||t_2| + \dots + |t_m|}{n} + \frac{|t_{m+1}| + \dots |t_n|}{n}$$

$$< \frac{mk}{n} + \frac{\varepsilon}{2}(n - m), \ \forall n \ge m.$$

Keeping m fixed, we have

$$\frac{mk}{n} < \varepsilon/2 \text{ if } n > \frac{2mk}{\varepsilon}.$$

Let, μ be any positive integer $> \frac{2mk}{\varepsilon}$, so that $n \ge \mu$ we have

$$\frac{mk}{n} \le \frac{\varepsilon}{2}.$$

Let $$\lambda = \max\{m, \mu\}.$$

Therefore, for each $n \ge \lambda$, we have

$$\left|\frac{t_1 + t_2 + \dots + t_n}{n}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This gives

$$\lim_{n \to \infty} \frac{t_1 + t_2 + \dots + t_n}{n} = 0.$$

Hence, we have

$$\lim_{n \to \infty} \frac{s_1 + s_2 + \ldots + s_n}{n} = l.$$

Theorem 3. (Cauchy's second theorem on limits). *If $\langle s_n \rangle$ is a sequence of positive terms and $\lim_{n \to \infty} s_n = l$, then*

$$lim\ (s_1, s_2, \ldots, s_n)^{1/n} = l.$$

Proof. Let $\langle t_n \rangle$ be a sequence, such that

$$t_n = \log s_n, \quad \forall\ n \in \mathbf{N}.$$

Now $\quad \lim s_n = l \Rightarrow \lim t_n = \lim \log s_n = \log l$

$$(\because \quad \lim s_n = l \Leftrightarrow \lim \log s_n = \log l \text{ provided } s_n > 0,\ \ \forall\ n \text{ and } l > 0)$$

Then, by Cauchy first theorem on limits, we have

$$\lim_{n \to \infty} \frac{t_1 + t_2 + \ldots + t_n}{n} = \lim t_n = \log l$$

$$\Rightarrow \quad \lim_{n \to \infty} \frac{\log s_1 + \log s_2 + \ldots + \log s_n}{n} = \log l$$

$$\Rightarrow \quad \lim_{n \to \infty} \frac{1}{n} \log (s_1, s_2, \ldots, s_n) = \log l$$

$$\Rightarrow \quad \lim \log (s_1, s_2, \ldots, s_n)^{1/n} = \log l$$

$$\Rightarrow \quad \mathbf{lim}\ (s_1, s_2 \ldots, s_n)^{1/n} = l.$$

Theorem 4. *If $\langle s_n \rangle$ is a sequence such that*

$$lim_{n \to \infty} \frac{s_{n+1}}{s_n} = l \text{ where } |\, l \,| < 1$$

then

$$lim_{n \to \infty} s_n = 0.$$

Proof. Since $|\, l \,| < 1$, let us choose a positive small number ε such that

$$|\, l \,| + \varepsilon < 1.$$

Now, $\lim \frac{s_{n+1}}{s_n} = l$, therefore for $\varepsilon > 0$ there must exists a positive ineger m such that, for all $n \geq m$

$$\left| \frac{s_{n+1}}{s_n} - l \right| < \varepsilon$$

$$\Rightarrow \quad \left| \frac{s_{n+1}}{s_n} \right| - |\, l \,| \leq \left| \frac{s_{n+1}}{s_n} - l \right| < \varepsilon$$

$$\Rightarrow \quad \left| \frac{s_{n+1}}{s_n} \right| < |\, l \,| + \varepsilon = k \text{ (say)}.$$

Now, putting $n = m, m + 1, \ldots, n - 1$ in the above inequality and multiplying them, we get

$$\left| \frac{s_n}{s_m} \right| < k^{n-m}$$

or

$$|\, s_n \,| < \frac{|\, s_m \,|}{k^m} . k^n.$$

But $k < 1 \Rightarrow k^n \to 0$ as $n \to \infty$, which gives $\lim s_n = 0$.

Theorem 5. *If $\langle s_n \rangle$ is a sequence such that $s_n > 0$ and $lim \frac{s_{n+1}}{s_n} = l$, then $lim \sqrt[n]{s_n} = l$.*

Proof. Let us define a sequence $\langle t_n \rangle$ such that

$$t_1 = s_1,\ t_2 = \frac{s_2}{s_1}, \ldots, t_n = \frac{s_n}{s_{n-1}}.$$

Then $t_1 . t_2 \ldots t_n = s_n$.

Also $\quad \lim \frac{s_{n+1}}{s_n} = l \Rightarrow \lim \frac{s_n}{s_{n-1}} = l \Rightarrow \lim t_n = l$

$$= s_n > 0 \Rightarrow t_n = 0, \quad \forall\ n \in N.$$

Hence we have, the sequence $\langle t_n \rangle$ of positive terms and $\lim t_n = l$.

Now, Cauchy's second theorem on limits we have

$$\lim (t_1, t_2, \ldots, t_n)^{1/n} = l$$

or

$$\lim (s_n)^{1/n} = l.$$

Theorem 6. (Cesaro's Theorem). *If* $\lim s_n = l_1$ *and* $\lim t_n = l_2$. *Then*

$$\lim \frac{s_1 t_n + s_2 t_{n-1} + \ldots + s_n t_1}{n} = l_1 l_2.$$

Proof. Let us define $s_n = l_1 + u_n$ and $|u_n| = U_n$.

Then $\lim u_n = 0$ and therefore $\lim U_n = 0$.

Now, by Cauchy's first theorem on limits, we have

$$\lim \frac{1}{n} [U_1 + U_2 + \ldots + U_n] = 0. \qquad \ldots(1)$$

Consider,

$$\frac{1}{n} [s_1 t_n + s_2 t_{n-1} + \ldots + s_n t_1] = \frac{1}{n} [t_1 + t_2 + \ldots + t_n] + \frac{1}{n} [u_1 t_n + u_2 t_{n-1} + \ldots + u_n t_1]. \qquad \ldots(2)$$

Since, the sequence $\langle t_n \rangle$ is convergent. Therefore, it is bounded. Hence, there must exists a positive real number k such that

$$|t_n| < k, \ \forall\, n \in \mathbf{N}.$$

Therefore,

$$\left| \frac{1}{n} (u_1 t_n + u_2 t_{n-1} + \ldots + u_n t_1 \right| \geq 0$$

$$\frac{1}{n} [|u_1||t_n| + |u_2||t_{n-1}| + \ldots + |u_n||t_1|] \geq 0$$

$$\frac{k}{n} [|u_1| + |u_2| + \ldots + |u_n|] > 0$$

$$\frac{k}{n} [u_1 + u_2 + \ldots + u_n] > 0.$$

$\Rightarrow$ $$\frac{k}{n} [u_1 + u_2 + \ldots + u_n] \to 0 \text{ as } n \to \infty \qquad \text{[By using (1)]}$$

Thus $$\lim \frac{1}{n} [u_1 t_n + u_2 t_{n-1} + \ldots + u_n t_1] = 0.$$

Since, $\lim t_n = l_2$, therefore

$$\lim \frac{t_1 + t_2 + \ldots + t_n}{n} = l_2.$$

Now, from (2), we have

$$\lim \frac{1}{n} (s_1 t_n + s_2 t_{n-1} + \ldots + s_n t_1) = l_1 l_2.$$

SOLVED EXAMPLES

Example 1. *Prove that* $\lim\limits_{n \to \infty} s_n = 1$, *where* $s_n = n^{1/n}$.

Solution. For $n = 1, \quad s_n = 1$

For $n \geq 2, \quad s_n > 1.$

Let $s_n = 1 + t_n,\ t_n > 0, \ \forall\, n \geq 2$

$$n = s_n^{\,n} = (1 + t_n)^n$$

$$= 1 + n t_n + \frac{n(n-1)}{2!} t_n^2 + \ldots + t_n^n \qquad \text{[By Binomial Theorem]}$$

$$\geq \frac{n(n-1)}{2!} t_n^2$$

$\Rightarrow$ $$0 \leq t_n^2 \leq \frac{2}{n-1}$$

$\Rightarrow$ $$0 < t_n \leq \sqrt{\frac{2}{n-1}}$$

Since $\sqrt{\dfrac{2}{n-1}} \to 0$ as $n \to \infty$

$t_n \to 0$ as $n \to \infty$. [By Sandwitch Theorem]

∴ Hence $s_n \to 1$ as $n \to \infty$.

Example 2. *If*

$$s_n = \left[\left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n\right]^{1/n}$$

then $s_n \to e$. Hence show that

$$\lim_{n\to\infty} \left[\frac{n^n}{n!}\right]^{1/n} = e.$$

Solution. Let $t_n = \left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n$

so that $s_n = t_n^{1/n}$.

Also, $\dfrac{t_{n+1}}{t_n} = \left(\dfrac{n+2}{n+1}\right)^{n+1} = \left(1 + \dfrac{1}{n+1}\right)^{n+1}$

Now $\lim\limits_{n\to\infty} \dfrac{t_{n+1}}{t_n} = e$.

Hence, by Cauchy's second theorem on limits, we have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n^{1/n} = e.$$

Also
$$s_n = \left[2 \cdot \left(\frac{3}{2}\right)^2 \cdot \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n\right]^{1/n}$$

$$= \left[\frac{(n+1)^n}{n!}\right]^{1/n} = \left[\frac{(n+1)^n}{n^n} \cdot \frac{n^n}{n!}\right]^{1/n}$$

$$= \frac{n+1}{n}\left(\frac{n^n}{n!}\right)^{1/n}$$

∴
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)\left(\frac{n^n}{n!}\right)^{1/n}\right]$$

$$= \lim_{n\to\infty} \frac{n+1}{n} \lim_{n\to\infty} \left[\frac{n^n}{n!}\right]^{1/n}$$

$$e = 1 \cdot \lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{1/n}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{1/n} = e.$$

Example 3. *Show that the sequence $\langle s_n \rangle$ where*

$$s_n = \left\{\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \ldots + \frac{1}{\sqrt{n^2+1}}\right\}$$

converges to 1.

Solution. Here, we have

$$\frac{n}{\sqrt{n^2+n}} \le s_n \le \frac{n}{\sqrt{n^2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+(1/n)}} \le s_n \le 1.$$

Now the sequence $\langle t_n \rangle, \langle u_n \rangle$ are such that

(i) $t_n \le s_n \le u_n$,

and **(ii)** $\lim t_n = \lim u_n = 1$

where, $t_n = \dfrac{1}{\sqrt{1+\left(\dfrac{1}{n}\right)}}$ and $u_n = 1$.

From **(i)** and **(ii)**, we have

$$\lim s_n = 1. \qquad \text{[By Sandwitch theorem]}$$

Example 4. *Prove that*

$$\lim_{n\to\infty}\left[\frac{(n+1)(n+2)(n+3)\ldots(n+n)}{n^n}\right] = \frac{4}{e}.$$

Solution. Let $s_n = \dfrac{(n+1)(n+2)\ldots(n+n)}{n^n} = \dfrac{(2n)\,!}{n^n\,(n\,!)}$.

Then $s_{n+1} = \dfrac{(2n+2)\,!}{(n+1)^{n+1}\,(n+1)\,!}$.

Therefore,
$$\frac{s_{n+1}}{s_n} = \frac{(2n+2)\,!\,n^n\,(n\,!)}{(n+1)^{n+1}\,(n+1)\,(2n)\,!} = \frac{(2n+2)(2n+1)}{(n+1)^{n+2}}$$
$$= \frac{(2n+2)(2n+1)\,n^n}{(n+1)^{n+2}} = \frac{2\,(2n+1)\,n^n}{(n+1)^{n+1}}$$
$$= \frac{2\times 2n\left[1+\dfrac{1}{2n}\right]n^n}{(n+1)(n+1)^n} = \frac{4n\left[1+\dfrac{1}{2n}\right]n^n}{n\left[1+\dfrac{1}{n}\right](n+1)^n}$$
$$= \frac{4\left(1+\dfrac{1}{2n}\right)}{\left(1+\dfrac{1}{n}\right)}\cdot\left[\frac{n}{n+1}\right]^n$$
$$= \frac{4n\left[1+\dfrac{1}{2n}\right]}{\left[1+\dfrac{1}{n}\right]}\cdot\frac{1}{\left[1+\dfrac{1}{n}\right]^n}.$$

Now, taking $\lim n\to\infty$, we have

$$\lim_{n\to\infty}\frac{s_{n+1}}{s_n} = \lim_{n\to\infty}\left[\frac{4\left[1+\dfrac{1}{2n}\right]}{1+\dfrac{1}{n}}\cdot\frac{1}{\left[1+\dfrac{1}{n}\right]^n}\right] = \frac{4}{e}.$$

Now By Cauchy's second theorem on limits, we have

$$\lim_{n\to\infty}(s_n)^{1/n} = \lim_{n\to\infty}\left(\frac{s_{n+1}}{s_n}\right) = \frac{4}{e}$$

$$\Rightarrow \qquad \lim_{n\to\infty}\left[\frac{(n+1)(n+2)\ldots(n+n)}{n^n}\right] = \frac{4}{e}.$$

Example 5. *Prove that*

$$\lim \frac{1}{n}[1+2^{1/2}+3^{1/3}+\ldots+n^{1/n}] = 1.$$

Solution. Let $s_n = n^{1/n}$

$$\lim s_n = \lim n^{1/n} = 1.$$

Then, by Cauchy's first theorem on limits, we have

$$\lim \frac{1}{n}(s_1 + s_2 + \ldots + s_n) = 1$$

$$\Rightarrow \quad \lim \frac{1}{n}[1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n}] = 1.$$

• 1.10. MONOTONIC SEQUENCES

(i) A sequence $\langle s_n \rangle$ is said to be monotonically increasing (or non-decreasing) if

$$s_n \le s_{n+1}, \ \forall\, n$$

or $$s_n \le s_m, \ \forall\, n > m.$$

(ii) A sequence $\langle s_n \rangle$ is said to be strictly increasing if

$$s_n < s_{n+1}, \ \forall\, n \in \mathbf{N}.$$

(iii) A sequence $\langle s_n \rangle$ is said to be monotonically decreasing (or non- increasing) if

$$s_n \ge s_{n+1}, \ \forall\, n \ .$$

or $$s_n \ge s_m, \ \forall\, n < m.$$

(iv) A sequence $\langle s_n \rangle$ is said to be strictly decreasing if

$$s_n > s_{n+1}, \ \forall\, n \in \mathbf{N}.$$

(v) A sequence $\langle s_n \rangle$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

Examples :

(i) $\langle 2, 2, 4, 4, 6, \ldots \rangle$ is monotonically increasing.

(ii) $\langle 1, 2, 3, \ldots n \rangle$ is strictly increasing.

(iii) $\langle 1, 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{5} \ldots \rangle$ is monotonically decreasing.

(iv) $\langle -2, -4, -6, -8, \ldots \rangle$ is strictly decreasing.

(v) $\langle 0, 1, 0, 1, \ldots \rangle$ is not monotonic.

Theorem 1. (Monotone Convergence Theorem). *Every bounded monotonically increasing sequence converges.*

Proof. Let us suppose $\langle s_n \rangle$ be a bounded monotonically increasing sequence. Let

$$S = \{s_n : n \in \mathbf{N}\}$$

denotes its range. Then, obviously S is a non-empty set, which is bounded above. Therefore there exists a number l, which is the supremum of S. We shall show that the sequence $\langle s_n \rangle$ converges to l.

Let $\varepsilon > 0$ be a given number. Since $l - \varepsilon < l$, therefore $l - \varepsilon$ is not an upper bound of S. Hence, there exists a positive integer m such that $s_n > l - \varepsilon$.

Now, since $\langle s_n \rangle$ is monotonically increasing sequence. Therefore

$$s_n \ge s_m > l - \varepsilon, \ \forall\, n \ge m. \qquad \ldots(1)$$

$$\text{Sup. } S = l \Rightarrow s_n < l < l + \varepsilon, \ \forall\, n. \qquad \ldots(2)$$

From (1) and (2), we have

$$l - \varepsilon < s_n < l + \varepsilon, \ \forall\, n \ge m$$

$$\Rightarrow \quad |s_n - l| < \varepsilon, \ \forall\, n \ge m$$

$\Rightarrow$ $\langle s_n \rangle$ converges to l.

Theorem 2. *Every bounded monotonically decreasing sequence converges.*

Proof. Let $\langle s_n \rangle$ be a bounded monotonically decreasing sequence. Consider a sequence $\langle t_n \rangle$ such that

$$t_n = -s_n, \ \forall\, n \in \mathbf{N}.$$

Then, $\langle t_n \rangle$ is bounded monotonically increasing sequence and therefore it converges [By Theorem 1]

If $\lim t_n = l$, then $\lim s_n = \lim(-t_n) = -l$.

Theorem 3. *A non-decreasing sequence (increasing), which is not bounded above diverges to* ∞.

Proof. Let $\langle s_n \rangle$ be a monotonic non-decreasing sequence, which is not bounded above. Let c be any positive number. Since, the sequence $\langle s_n \rangle$ is unbounded and monotonically increasing, therefore, there must exists a positive integer m such that

$$s_n \geq s_m > c, \ \forall\, n > m$$

$$\Rightarrow \quad s_n > c, \ \forall\, n > m.$$

Hence, the sequence $\langle s_n \rangle$ diverges to ∞.

Theorem 4. *A non-increasing sequence (decreasing), which is not bounded below diverges to* $-\infty$.

Proof. Proof is exactly on same lines and left as an exercise for the students.

SOLVED EXAMPLES

Example 1. *Show that the sequence* $\langle s_n \rangle$ *defined by*

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}$$

converges.

Solution. Since, the sequence $\langle s_n \rangle$ is defined by

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}$$

$$\Rightarrow \quad s_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n+2}.$$

Now
$$s_{n+1} - s_n = \left(\frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$$
$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$
$$= \frac{1}{2n+1} - \frac{1}{2n+2}$$
$$> 0, \ \forall\, n.$$

Hence, the sequence $\langle s_n \rangle$ is monotonically increasing.

Now
$$|s_n| = \left|\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right|$$
$$< \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} \qquad \text{(upto } n \text{ terms)}$$
$$= n \cdot \frac{1}{n} = 1$$

i.e., $\quad |s_n| < 1, \ \forall\, n.$

$\Rightarrow$ sequence $\langle s_n \rangle$ is bounded.

Then, by monotonic convergence criterion, the sequence $\langle s_n \rangle$ converges.

Example 2. *Show that* $\lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$ *exists and lies between 2 and 3.*

Solution. Let $s_n = \left(1 + \dfrac{1}{n}\right)^n$

$\therefore \quad s_1 = 2$

$$s_n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \ldots + \frac{n(n-1)\ldots 1}{n!} \cdot \frac{1}{n^n}$$

[By binomial theorem for positive integral index]

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right). \qquad \ldots(1)$$

Similarly

$$s_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \ldots + \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\ldots\left(1 - \frac{n}{n+1}\right).$$

Comparing (1) and (2), we see that $s_{n+1} \geq s_n, \ \forall\, n$.

$\Rightarrow$ The sequence $\langle s_n \rangle$ is monotonically increasing.

Now from (1), we have

$$2 < s_n < 1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \ldots + \frac{1}{n\,!}$$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}, \text{ which is a G.P.}$$

$$= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$$

$$= 3 - \frac{1}{2^{n-1}} < 3, \ \forall\, n.$$

$\Rightarrow$ The sequence $\langle s_n \rangle$ is bounded.

Thus, the sequence $\langle s_n \rangle$, being a monotonically increasing sequence bounded above by 3, is convergent.

Since $\quad 2 < s_n < 3, \ \forall\, n$

$\Rightarrow \quad 2 \leq \lim\limits_{n \to \infty} s_n \leq 3, \ \forall\, n.$

$\Rightarrow$ limit of the sequence $\langle s_n \rangle$ lies between 2 and 3.

Example 3. *Show that the sequence* $\langle s_n \rangle$ *defined by*

$$s_1 = \sqrt{2},\ s_{n+1} = \sqrt{(2s_n)}$$

converges to 2.

Solution. We have $s_{n+1} = \sqrt{(2s_n)}$

For $n = 1 \quad s_2 = \sqrt{(2s_1)}$

$s_2 = \sqrt{(2\sqrt{2})}.$

Since $\quad 1 < \sqrt{2} \Rightarrow 2 < 2\sqrt{2} \Rightarrow \sqrt{2} < \sqrt{(2\sqrt{2})}$

$\Rightarrow s_1 < s_2.$

Now, let us suppose that $s_m < s_{m+1}$

then $\quad \sqrt{(2s_m)} < \sqrt{(2s_{m+1})}$

$\Rightarrow \quad s_{m+1} < s_{m+2}.$

How, by the method of Mathematical induction, we have

$$s_n < s_{n+1},\ \forall\, n \in \mathbf{N}$$

i.e., $\langle s_n \rangle$ is monotonically increasing sequence.

Now, we shall show that $\langle s_n \rangle$ is bounded.

Since $\quad s_1 = \sqrt{2} < 2.$

Let us suppose that $s_m < 2$. Then $\sqrt{(2s_m)} < \sqrt{(2\,.\,2)} = 2$

$\Rightarrow \quad s_{m+1} < 2$

By the method of mathematical induction, we have

$$s_n < 2,\ \forall\, n \in \mathbf{N}$$

$\Rightarrow \ \langle s_n \rangle$ is bounded above by 2.

$\Rightarrow \ \langle s_n \rangle$ is monotonically increasing sequence which is bounded above.

Then, by monotone convergent criterion $\langle s_n \rangle$ is convergent.

Now, let $\quad \lim\limits_{n \to \infty} s_n = l \Rightarrow \lim\limits_{n \to \infty} s_{n+1} = l$

given that $\quad s_{n+1} = \sqrt{(2s_n)}$

$\Rightarrow \quad \lim s_{n+1} = \lim \sqrt{2s_n}$

$\Rightarrow \quad l = \sqrt{2l} \Rightarrow l\,(l - 2) = 0$

which gives $\quad l = 2,\ l = 0.$

But, since $\langle s_n \rangle$ is positive terms sequence with first term $= \sqrt{2}$. Hence l can not be equal to 0

$\Rightarrow \quad l = 2.$

Example 4. Prove that the sequence $\langle a_n \rangle$ is convergent where :

$$a_n = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \frac{1}{3\,!} + \ldots + \frac{1}{n\,!} + \ldots.$$

Solution. Since

$$a_n = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \frac{1}{3\,!} + \ldots + \frac{1}{n\,!} + \ldots$$

and $$a_{n+1} = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \frac{1}{3\,!} + \ldots + \frac{1}{n\,!} + \frac{1}{(n+1)\,!}$$

then $$a_{n+1} - a_n = \frac{1}{(n+1)\,!} > 0, \quad \forall\ n \in \mathbf{N}.$$

Thus $\langle a_n \rangle$ is monotonically increasing.

Further,

$$a_n = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \frac{1}{3\,!} + \ldots + \frac{1}{n\,!} + \ldots$$

$$\Rightarrow \quad 2 < a_n < 1 + 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{2^{n-1}}$$

$$\Rightarrow \quad 2 < a_n \le 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 3 - \frac{1}{2^{n-1}} < 3, \ \forall\, n$$

$\Rightarrow$ $\langle a_n \rangle$ is bounded.

Hence, $\langle a_n \rangle$ is convergent.

• SUMMARY

- A function $f : N \to S$ is known as a sequence.
- A sequence $\langle s_n \rangle$ is bounded iff $|s_n| < k\ \ \forall n$.
- Every bounded sequence has at least one limit point.
- A sequence $\langle s_n \rangle$ converges to l if for given $\varepsilon > 0$ $m \in N$ such that $|s_n - l| < \varepsilon\ \forall\ n \ge m$.
- A sequence $\langle s_n \rangle$ is a Cauchy sequence $\exists$ for given $\varepsilon > 0 \,\exists\, m, n$ in N such that $|s_n - s_m| < \in\ \forall\ n \ge m$.
- **Cauchy's first theorem on limit :** If $\lim\limits_{n \to \infty} s_n = l$, then $\lim\limits_{n \to \infty} \dfrac{s_1 + s_2 + \ldots\ldots s_n}{n} = l$.
- **Cauchy's second theorem on limit :** If $\lim\limits_{n \to \infty} s_n = l$, then $\lim\limits_{n \to \infty} (s_1\, s_2 \ldots\ldots s_n)^{1/n} = l$.
- A sequence $\langle s_n \rangle$ is monotonic if either $s_n \ge s_m$ or $s_n \le s_m\ \forall\ n > m$.

• STUDENT ACTIVITY

1. Prove that every convergent sequence is bounded.

2. Prove that every Cauchy sequence is convergent.

• TEST YOURSELF

1. Discuss the boundedness of the following sequence $\langle s_n \rangle$ where $\langle s_n \rangle$ is given by

   (i) $s_n = 6$ (ii) $s_n = (-1)^n . 4$

   (iii) $s_n = \dfrac{2n+3}{3n+4}$ (iv) $s_n = \left(1 + \dfrac{1}{n}\right)^n$.

   (v) $s_n = \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} + \ldots + \dfrac{1}{(2n)^2}$

   (vi) $s_n = n^3$ (vii) $s_n = 1 + (-1)^n$.
2. Discuss the convergence and divergence of sequences in Ques. 1.
3. Give examples of sequence $\langle s_n \rangle$ for which

$$\lim_{n \to \infty} \frac{s_{n+1}}{s_n} = 1$$

   and (i) $s_n \to \infty$ (ii) $s_n \to 2$ (iii) $s_n \to 0$.
4. Verify the following :

   (i) $\lim\limits_{n \to \infty} \dfrac{3n-5}{4-2n} = -\dfrac{3}{2}$ (ii) $\lim\limits_{n \to \infty} [(n^2+1)^{1/8} - (n+1)^{1/4}] = 0$

   (iii) $\lim\limits_{n \to \infty} \left[\dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} + \ldots + \dfrac{1}{(2n)^2}\right] = 0$

   (iv) $\lim\limits_{n \to \infty} \left(1 - \dfrac{1}{n}\right)^{-n} = e$ (v) $\lim\limits_{n \to \infty} \dfrac{n}{[n\,!]^{1/n}} = e$

   (vi) $\lim\limits_{x \to \infty} \dfrac{e^{1/x}}{e^{1/x}+1} = 1$.
5. Show that the sequences $\langle s_n \rangle$ defined by $s_1 = \dfrac{1}{2}$, $s_{n+1} = \dfrac{2s_n + 1}{3}$ $\forall\, n \in \mathbf{N}$ is convergent. Also find its limit.

ANSWERS

1. (i), (ii), (iii), (iv), (v), (vii) bounded (vi) unbounded.
2. (i), (iii), (iv), (v) converges (ii), (vii) oscillate (vi) diverges to ∞
3. (i) $s_n = n$ (ii) $s_n = \dfrac{2n+1}{n}$ (iii) $s_n = \dfrac{1}{n}$
5. $l = 1$

Fill in the Blanks :

1. Every convergent sequence is ....
2. Every bounded sequence is .... convergent.
3. The limit of a positive term sequence is always ....
4. Limit of the sequence is ....
5. A sequence is Cauchy if and only if it is ....
6. Every Cauchy sequence is ....

True or False :

Write T for true and F for false statement :

1. Every convergent sequence is bounded. *(T/F)*
2. Every bounded monotonically increasing sequence is convergent. *(T/F)*
3. If $\langle s_{n+1} - s_n \rangle$ oscillate finitely, then $\langle s_n \rangle$ oscillate. *(T/F)*
4. If given k (however large) we can find m for which $a_m > k$ then $s_n \to \infty$. *(T/F)*
5. If $\langle s_{n+1} - s_n \rangle$ oscillate infinitely, then $\langle s_n \rangle$ oscillate. *(T/F)*

Multiple Choice Questions :

Choose the most appropriate one :

1. An oscillatory sequence is :

   (a) always bounded (b) may or may not be bounded

   (c) never bounded (d) none of these.
2. Formula for s_n, for the given sequences $1, -1, 1, -1, \ldots$ is :

(a) $s_n = (-1)^n \; \forall\, n \in N$ (b) $s_n = (-1)^{n+1} \; \forall\, n \in N$

(c) $s_n = 1$ if n is even (d) none of these.

3. If the sequence $\langle s_n \rangle$ converges to l then the sequence $\langle | s_n | \rangle$ converges to :

(a) l (b) $|l|$ (c) $-l$ (d) none of these.

4. A sequence of $\langle s_n \rangle$ of real numbers such that $\langle | s_n | \rangle$ converges but $\langle s_n \rangle$ does not, is given by :

(a) $\langle (-1)^n \rangle$ (b) $\langle \frac{1}{n} \rangle$ (c) $\langle \frac{n}{n+1} \rangle$ (d) none of these.

ANSWERS

Fill in the Blanks :

1. Bounded **2.** not necessarily **3.** non-negative

4. unique **5.** convergent **6.** convergent

True and False :

1. T **2.** T **3.** F **4.** F **5.** F

Multiple Choice Questions :

1. (b) **2.** (a) **3.** (b) **4.** (a)

UNIT 2

INFINITE SERIES

STRUCTURE

- Definitions
- Sequence of Partial Sums
- Convergence, divergence or oscillation of a series
- Comparison tests
- Cauchy's Root test
- D'Alembert Ratio Test
- Raabe's Test
- Logarithmic test
- Cauchy's integral test
- Leibnitz Test
  - ❑ Summary
  - ❑ Student Activity
  - ❑ Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is an infinite series ?
- How to distinguishe the sequence and series.
- How to check whether a given series is convergent or divergent using the said tests.

• 2.1. DEFINITIONS

Let $\langle u_n \rangle$ be a sequence of real numbers, then an expression of the form

$$u_1 + u_2 + \ldots + u_n + \ldots \qquad \ldots(1)$$

is called an infinite series. In symbols it is generally written as

$$\sum_{n=1}^{\infty} u_n \text{ or } \Sigma u_n.$$

If all the terms of $\langle u_n \rangle$ after a certain number are zero then the expression $u_1 + u_2 + \ldots + u_m$, written as $\sum_{n=1}^{m} u_n$ is called a finite series.

The term u_n is called the n^{th} term or general term of the series (1). The sum of first n terms of the series is denoted by s_n. Thus,

$$s_n = u_1 + u_2 + \ldots + u_n.$$

• 2.2. SEQUENCE OF PARTIAL SUM OF AN INFINITE SERIES

An expression of the form $u_1 + u_2 + \ldots + u_n + \ldots$ which involves addition of infinitely many terms has in itself no meaning. In order to give a meaning to the value of such an infinite sum, we form a sequence of partial sums. It is the limit of such a sequence which gives meaning to the infinite series.

Let us associate to the infinite series $u_1 + u_2 + \ldots + u_n + \ldots$, a sequence $\langle s_n \rangle$ defined by

$$S_n = u_1 + u_2 + \ldots + u_n.$$

Then the sequence $\langle s_n \rangle$ is called the **sequence of partial sums** of the given series

$$u_1 + u_2 + \ldots + u_n + \ldots .$$

• 2.3. CONVERGENCE, DIVERGENCE OR OSCILLATION OF A SERIES

An infinite series $\sum_{n=1}^{\infty} u_n$ is said to be :

(i) Convergent if the sequence $\langle s_n \rangle$ of its partial sums converges to a real number S and in that case S is called the sum of the series $\sum_{n=1}^{\infty} u_n$ and we write $\sum_{n=1}^{m} u_n = S$. In this case, we also say that the series is convergent to S.

(ii) Converges absolutely, if $\sum_{n=1}^{\infty} |u_n|$ converges.

(iii) Converges conditionally, if $\sum_{n=1}^{\infty} u_n$ converges but $\sum_{n=1}^{\infty} |u_n|$ does not converge.

(iv) Diverges to ∞ (or $-\infty$) if the sequence $\langle s_n \rangle$ diverges to ∞ (or $-\infty$) and in that case $\sum_{n=1}^{\infty} u_n = \infty \left(\text{or } \sum_{n=1}^{\infty} u_n = -\infty \right)$.

(v) Oscillate finitely, if the sequence $\langle s_n \rangle$ oscillate finitely.

(vi) Oscillate infinitely, if the sequence $\langle s_n \rangle$ oscillate infinitely.

(vii) Oscillatory if S_n, the sum of its first n terms, neither tends to a definite finite limit nor to $+\infty$ or $-\infty$ as $n \to \infty$.

Examples :

(1) The series $1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots + \left(\frac{2}{3}\right)^{n-1} + \ldots$ is convergent.

(2) The series $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ is convergent.

(3) The series $1 + 2 + 3 + \ldots + n + \ldots$ is divergent.

(4) The series $3 - 3 + 3 - 3 + \ldots$ is oscillatory.

SOME IMPORTANT THEOREM

Theorem 1. (Necessary condition for convergence). *For a series $\sum u_n$ to be convergent, it is necessary that*

$$\lim u_n = 0.$$

Or

For every convergent series $\sum u_n$, we must have $\lim u_n = 0$.

Proof. Let us suppose, the series $\sum u_n$ be convergent. Let S_n denote the sum of n terms of the series $\sum u_n$.

$$\left.\begin{aligned} \Rightarrow \quad S_n &= u_1 + u_2 + \ldots + u_n \\ \Rightarrow \quad S_{n-1} &= u_1 + u_2 + \ldots + u_{n-1} \end{aligned}\right] \Rightarrow u_n = S_n - S_{n-1} \qquad \ldots(1)$$

The series $\sum u_n$ is convergent, therefore S_n and S_{n-1} both will tend to the same finite limit, say l as $n \to \infty$.

Now, from (1)

$$\lim u_n = \lim S_n - \lim S_{n-1} = l - l = 0.$$

Hence, for a convergent series, it is necessary that $\lim u_n = 0$.

Theorem 2. (Cauchy's General principle of convergence for series). *A necessary and sufficient condition for a series $\sum u_n$ to be convergent is that to each $\varepsilon > 0$, there exists a positive integer m such that*

$$|u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \varepsilon, \textit{ whenever } n \geq m \textit{ and } p \geq 1.$$

Proof. Let $\langle s_n \rangle$ be the sequence of partial sums of the series $\sum u_n$. The series $\sum u_n$ will converge if and only if the sequence $\langle s_n \rangle$ of its partial sums converges. But by Cauchy's general principle of convergence for sequences, we know that a necessary and sufficient condition for the convergence of $\langle s_n \rangle$ is that for each $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$|s_n - s_m| < \varepsilon, \ \forall\, n > m$$

$\Rightarrow$ $|u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \varepsilon, \ \forall\, n > m$ and $p \geq 1$.

Theorem 3. *A series of positive terms is convergent if S_n, the sum of n terms, is less than a fixed number for all values of n.*

Proof. Let $u_1 + u_2 + \ldots + u_n + \ldots$ be the series of positive terms.

Then $\qquad S_n = u_1 + u_2 + \ldots + u_n$.

Obviously if n increases, then S_n increases and may tend to a finite limit or to $+\infty$. The series can not oscillate.

If S_n remains less than a fixed number for all values of n it can not tend to infinity and so it must tend to a finite limit. Hence the series is convergent.

Theorem 4. *A series of positive term Σu_n is convergent if and only if the sequence $\langle s_n \rangle$ (where $s_n = u_1 + u_2 + \ldots + u_n$) of its partial sum is bounded above.*

Proof. Since, $u_n > 0$, $\forall\, n$, the sequence $\langle s_n \rangle$ of partial sums of the series is monotonically increasing.

Now the series Σu_n is convergent iff the sequence $\langle s_n \rangle$ is convergent.

i.e., iff the sequence $\langle s_n \rangle$ is bounded above.

($\because$ a monotonically increasing sequence is convergent iff it is bounded above)

Theorem 5. (Convergence of geometric series). *The geometric series*

$$1 + r + r^2 + \ldots + r^{n-1} + \ldots \text{ is}$$

(*i*) *Converges to* $\dfrac{1}{1-r}$ *if* $|r| < 1$.

(*ii*) *Diverges to* $+\infty$ *if* $r \geq 1$.

(*iii*) *Oscillate finitely if* $r = -1$.

and (*iv*) *Oscillate infinitely if* $r < -1$.

Proof. Here $\qquad S_n = 1 + r + r^2 + \ldots + r^{n-1}$

$$= \begin{cases} \dfrac{1-r^n}{1-r} & \text{if } r \neq 1 \\ n & \text{if } r = 1. \end{cases}$$

Now, there are following cases :

Case (i). If $|r| < 1$.

Then $$\lim_{n \to \infty} r^n = 0$$

so that $$\lim_{n \to \infty} S_n = \frac{1}{1-r}$$

which gives, the series is convergent to $\dfrac{1}{1-r}$.

Case (ii). If $r > 1$.

Then $$\lim_{n \to \infty} r^n = \infty$$

so that $$S_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} + \frac{r^n}{r-1} \to \infty \text{ as } n \to \infty.$$

Hence, the series is divergent to ∞.

If $r = 1$, then $S_n = 1 + 1 + \ldots + 1 + \ldots$ to n times $= n$

Thus, the sequence $\langle s_n \rangle$ diverges and hence the series diverges.

Case (iii). If $r = -1$.

Then, $$S_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

therefore the sequence $\langle s_n \rangle$ oscillate between 0 and 1.

$\Rightarrow$ The series oscillate finitely between 0 and 1.

Case (iv). If $r < -1$.

Let $r = -a$ where $a > 1$.

Then $$S_n = \frac{1}{1+a} - \frac{(-1)^n \cdot a^n}{1+a}$$

so that $S_{2n} \to -\infty$ and $S_{2n+1} \to \infty$.

Therefore, the sequence $\langle s_n \rangle$ oscillate infinitely between $-\infty$ and $+\infty$.

Hence, the series oscillate infinitely.

Theorem 6. *A positive terms series Σu_n either converges to a finite limit or diverges to ∞.*

Proof. Let

$$S_n = u_1 + u_2 + \ldots + u_n$$

$$\Rightarrow \quad S_{n+1} = u_1 + u_2 + \ldots + u_{n+1}$$

$$\Rightarrow \quad S_{n+1} - S_n = u_{n+1} > 0$$

$$\Rightarrow \quad S_{n+1} > S_n, \ \forall n$$

$\Rightarrow$ $\langle S_n \rangle$ is monotonically increasing sequence.

Since, a monotonically increasing sequence is either convergent to a finite limit or divergent to ∞, the sequence $\langle S_n \rangle$ of partial sums of the series Σu_n is either convergent to a finite limit or divergent to ∞.

Hence, the series Σu_n is either converges or diverges to ∞.

Theorem 7. $\left(\textit{The Auxiliary series } \Sigma \frac{1}{n^p}\right)$ *The infinite series*

$$\Sigma\left(\frac{1}{n^p}\right) = \frac{1}{1^p} + \frac{1}{2^p} + \ldots + \frac{1}{n^p} + \ldots$$

is convergent if $p > 1$ and divergent if $p \le 1$.

Proof. Case (i). $p > 1$.

We have $$\frac{1}{1^p} = 1.$$

Also, $$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = 2^{1-p}$$

and $$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p} = 4^{1-p}$$
$$= (2^{1-p})^2$$

... ...

... ...

$$\frac{1}{(2^n)^p} + \frac{1}{(2^n+1)^p} + \ldots + \frac{1}{(2^{n+1}-1)^p} < (2^n)^{1-p} = (2^{1-p})^n.$$

Adding, all the above inequalities, we have

$$S_{2^{n+1}-1} = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots + \left(\frac{1}{(2^n)^p} + \frac{1}{(2^n+1)^p} + \ldots + \frac{1}{(2^{n+1}-1)^p}\right)$$
$$< 1 + 2^{1-p} + 2^{(1-p)^2} + \ldots + 2^{(1-p)^n}.$$

This is a geometric series of n terms with common ratio

$$2^{1-p} = \frac{1}{2^{p-1}} \ (< 1 \text{ as } p > 1)$$

$$= \frac{1-(2^{1-p})^{n+1}}{1-2^{1-p}} = \frac{1}{1-2^{1-p}} - \frac{(2^{1-p})^{n+1}}{1-2^{1-p}}$$

$$< \frac{1}{1-2^{1-p}} = C \text{ (say)}.$$

Now, since the series is of positive terms and

$$2^{n+1} - 1 > 2^n > n, \ \forall n.$$

We have

$$S_n < S_{2^{n+1}-1} < C, \ \forall n.$$

$\Rightarrow$ the sequence $\langle S_n \rangle$ of partial sums of the series $\Sigma \frac{1}{n^p}$ is bounded above.

Hence, the given series is convergent.

Case (ii). When $p = 1$. Then the given series becomes

$$\Sigma \frac{1}{n^p} = 1 + \frac{1}{2} + \frac{1}{3} + \dots .$$

Now, this series may be written as follows

$$\Sigma \frac{1}{n^p} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \dots$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \dots$$

$$= 1 + \frac{1}{2} + \frac{2}{4} + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} \dots$$

Now since $\lim u_n = \frac{1}{2} \neq 0$, the series is divergent.

Case (iii). When $p < 1$. Then

$$2^p < 2,\ 3^p < 3,\ 4^p < 4 \text{ and so on.}$$

Hence, the given series reduces to

$$\Sigma \frac{1}{n^p} > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots .$$

Clearly, the series on the right hand side is divergent. [By case (ii)]

Hence, the given series is divergent when $p < 1$.

• 2.4. COMPARISON TESTS

The most important technique for deciding whether a series is convergent or not is to compare it with another suitable chosen series which is already known to be convergent or divergent.

First form. *Let Σu_n and Σv_n be two series of positive terms such that*

$$u_n < kv_n, \ \forall\, n$$

Then,

(i) Σv_n converges $\Rightarrow \Sigma u_n$ converges

(ii) Σu_n diverges $\Rightarrow \Sigma v_n$ diverges.

Proof. Firstly we shall prove **(i)** Σv_n convergent $\Rightarrow u_n$ is convergent

Now, $\quad u_n < kv_n, \ \forall\, n \in \mathbf{N}$

$$\Rightarrow \quad (u_1 + u_2 + \dots + u_n) < k\,(v_1 + v_2 + \dots + v_n) . \qquad \dots(1)$$

But the series Σv_n is given to be convergent.

$\Rightarrow$ By the fundamental result for positive term series, $\exists$ a positive number M such that

$$v_1 + v_2 + \dots + v_n < M, \ \forall\, n \in \mathbf{N}. \qquad \dots(2)$$

From (1) and (2), we have

$$u_1 + u_2 + \dots + u_n < k \,.\, M = k_1 \text{ (say)}, \ \forall\, n \in \mathbf{N}$$

$\Rightarrow \quad u_1 + u_2 + \dots + u_n < k_1 \ \forall\, n \in \mathbf{N}$, where $k_1 = mk > 0$

$\Rightarrow \quad \exists$ a positive number k such that

$$u_1 + u_2 + \dots + u_n < k_1, \ \forall\, n \in \mathbf{N}$$

$\Rightarrow$ by the fundamental result for the positive terms series, Σu_n is also convergent.

We shall now prove that if Σu_n is divergent, then Σv_n is also divergent.

Since, we are given Σu_n to be divergent.

$\Rightarrow$ The sequence $\langle s_n \rangle$ of its partial sums is also divergent.

$\Rightarrow$ $\exists$ a positive number k_2 (however large) and positive integer $m \in \mathbf{N}$ such that

$$s_n > k_2, \ \forall\, n > m$$

i.e., $$u_1 + u_2 + \dots + u_n > k_2, \ \forall\, n > m. \qquad \dots(3)$$

From (1) and (3), we have

$$k_2 < u_1 + u_2 + \dots + u_n < k\,(v_1 + v_2 + \dots + v_n), \ \forall\, n > m$$

$$\Rightarrow \quad v_1 + v_2 + \dots + v_n > \frac{k_2}{k} (= k_3), \ \forall\, n > m$$

$$\Rightarrow \quad T_n > k_3, \ \forall\, n > m$$

where $\qquad k_3 = \dfrac{k_2}{k}$ and $T_n = v_1 + v_2 + \ldots + v_n$

$\Rightarrow$ $\exists$ a positive number k_3 (however large) and a positive integer m such that $T_n > k_3$, $\forall\, n > m$ and thus T_n is divergent and cosequently Σv_n is divergent.

Second form. *Let Σu_n and Σv_n be two series of positive terms and let k_1 and k_2 be positive real number such that*

$$\boldsymbol{k_1 v_n \le u_n \le k_2 v_n, \quad \forall\, n}$$

Then, the series Σu_n and Σv_n converge or diverge together.

Proof. We have

$$k_1 v_n \le u_n \le k_2 v_n, \;\; \forall\, n. \qquad ...(1)$$

(i) If the series Σv_n is convergent, then $\Sigma k_2 v_n$ is convergent and hence, from second part of the (i) the series Σu_n is convergent.

(ii) If the series Σu_n is convergent, then from first part of the inequality (1), $\Sigma k_1 v_n$ is convergent and hence $\Sigma v_n \left(= \dfrac{1}{k_1} \Sigma k_1 v_n \right)$ is convergent.

(iii) If the series Σu_n is divergent, then from second part of inequality (1) $\Sigma k_2 v_n$ is divergent and hence Σv_n is divergent.

(iv) If the series Σv_n is divergent, then $\Sigma k_1 v_n$ is divergent and hence from first part of the inequality (1), Σu_n is divergent.

Third form. *If Σu_n and Σv_n be two given positive term series such that*

$$\boldsymbol{u_n \le k v_n, \quad \forall\, n > m, k > 0 \text{ and } m \in \mathbf{N}}$$

Then,

(i) Σv_n *is convergent* $\Rightarrow \Sigma u_n$ *is convergent*

(ii) Σu_n *is divergent* $\Rightarrow \Sigma v_n$ *is also divergent.*

Proof. (i) Let us suppose $\langle s_n \rangle$ and $\langle t_n \rangle$ be two sequences of partial sums of the two given positive terms series Σu_n and Σv_n respectively.

Therefore, $\quad s_n = u_1 + u_2 + \ldots + u_n, \;\; \forall\, n \in \mathbf{N}$

and $\quad t_n = v_1 + v_2 + \ldots + v_n, \;\; \forall\, n \in \mathbf{N}$

since $\quad u_n \le k v_n \;\; \forall\;\; n \ge m \Rightarrow s_n \le k t_n, \;\; \forall\, n \ge m$

$\Rightarrow \quad s_n - s_m \le k\,(t_n - t_m) = k t_n - k t_m$

$\Rightarrow \quad s_n \le k t_n + (s_m - k t_m) = k t_n + M$

where $\quad M = s_m - k t_m$, a fixed quantity. $\qquad ...(1)$

Now, if Σv_n is convergent $\Rightarrow \langle t_n \rangle$ is convergent and thus it is bounded above

$\Rightarrow \quad \exists$ a number A such that $t_n \le A, \;\; \forall\, n \in \mathbf{N}$. $\qquad ...(2)$

Now from (1) and (2), we have

$$s_n < k\,.\,A + M = k_1, \; \forall\, n \in \mathbf{N}$$

and therefore $\langle s_n \rangle$ is bounded above.

Moreover, $\langle s_n \rangle$ is a monotonically increasing sequence, therefore, $\langle s_n \rangle$ is monotonically increasing sequence which is bounded above and thus, it is convergent and hence Σv_n is convergent.

(ii) Now if Σu_n is divergent $\Rightarrow \langle s_n \rangle$ is divergent and therefore $\exists$ a positive number $\varepsilon > 0$ and $m' \in s_n$

$$s_n > B, \;\; \forall\, n \ge m'.$$

Let $m^* = \max\,\{m, m'\}$ so that $s_n > B, \;\; \forall\, n \ge m^*$.

Now from (i)

$$t_n > \frac{1}{k}\,(s_n - M) > \frac{1}{k}\,(B - M) = C, \;\; \forall\, n \ge m^*, \, C \ne 0.$$

$\Rightarrow$ $\langle t_n \rangle$ is divergent and hence Σv_n is divergent.

Fourth form. *Let Σu_n and Σv_n be two series of positive terms and let k_1 and k_2 be positive real numbers such that $k_1 v_n < u_n < k_2 v_n$, $\forall\, n > m$, m being a fixed positive integer. Then the series Σu_n and Σv_n converge or diverge together.*

Proof. Proof immediately follows from the second form of comparison test.

Fifth form. *Let $\sum u_n$ and $\sum v_n$ be two series of positive terms such that*

$$\lim_{n \to \infty} \frac{u_n}{v_n} = l, \text{ (finite and non-zero)}$$

then both the series converge or diverge together.

Proof. Since $\frac{u_n}{v_n} > 0, \ \forall\, n$

$$\therefore \quad \lim_{n \to \infty} \frac{u_n}{v_n} \geq 0 \text{ i.e., } l \geq 0.$$

But $l \neq 0$ (by assumption) : therefore $l > 0$.

Now, let $\varepsilon > 0$ be choosen in such a way that $l - \varepsilon > 0$.

Since $\lim_{n \to \infty} \frac{u_n}{v_n} = l$, therefore $\exists$ a positive integer m such that

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon, \ \forall\ n > m. \quad \text{...(1)}$$

Since, $v_n > 0 \ \forall\, n$, therefore, multiplying (1) by v_n, we obtain

$$(l - \varepsilon)\, v_n < u_n < (l + \varepsilon)\, v_n, \ \forall\ n > m.$$

Since $l - \varepsilon$ and $l + \varepsilon$ are both positive, therefore applying the fourth form of comparison test, we find that the series $\sum u_n$ and $\sum v_n$ converge or diverge together.

Sixth form. *Let $\sum u_n$ and $\sum v_n$ are two series of positive terms and $\exists$ a positive integer m such that*

$$\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}, \ \forall\, n \geq m$$

then $\sum u_n$ and $\sum v_n$ both converge or diverge together.

Proof. Let us suppose $\langle s_n \rangle$ and $\langle t_n \rangle$ are two sequences of partial sum of the series $\sum u_n$ and $\sum v_n$ respectively, such that

$$s_n = u_1 + u_2 + \ldots + u_n$$
$$t_n = v_1 + v_2 + \ldots + v_n \ \ \forall\, n.$$

Now for $n \geq m$, we have

$$\frac{u_m}{u_n} = \frac{u_m}{u_{m+1}} \cdot \frac{u_{m+1}}{u_{m+2}} \cdots \frac{u_{n-1}}{u_n}$$
$$\geq \frac{v_m}{v_{m+1}} \cdot \frac{v_{m+1}}{v_{m+2}} \cdots \frac{v_{n-1}}{v_n}$$
$$= \frac{v_m}{v_n}$$

$$\Rightarrow \quad u_n \leq \frac{u_m}{v_m} \cdot v_n.$$

Since, m is fixed positive integer, $\frac{u_m}{v_n}$ is a fixed number say k. Thus for $n \geq m$, we have

$$u_n \leq k v_n$$

$\Rightarrow$ $\sum u_n$ and $\sum v_n$ both converge or diverge together.

SOLVED EXAMPLES

Example 1. *Test the convergence of the series*

$$\frac{2}{1} + \frac{3}{4} + \frac{4}{9} + \ldots + \frac{n+1}{n^2} + \ldots.$$

Solution. Here $u_n = \frac{n+1}{n^2}$. Take $v_n = \frac{n}{n^2} = \frac{1}{n}$

Then $\frac{u_n}{v_n} = \frac{n+1}{n^2} \Big/ \frac{1}{n} = \frac{n+1}{n^2} \cdot \frac{n}{1} = \frac{n+1}{n}$

Therefore $\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n+1}{n} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)$

$= 1$, which is finite and non-zero.

Thus, by the comparison test the two series are either both convergent or both divergent. But, the auxiliary series $\Sigma v_n = \frac{1}{n}$ is divergent. Hence, the given Σu_n is also divergent.

Example 2. *Test the convergence of the series*

$$\frac{1}{1 . 2} + \frac{1}{2 . 3} + \ldots + \frac{1}{n(n+1)} + \ldots .$$

Solution. Here $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

If s_n is the partial sum of n terms of the series Σu_n, then

$$\begin{aligned} s_n &= u_1 + u_2 + \ldots + \ldots + u_n \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Now, $$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left[1 - \frac{1}{n+1}\right]$$

$= 1$, which is finite and non-zero.

Hence, the given series is convergent.

Example 3. *Show that the series*

$$1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \ldots$$

is convergent.

Solution. Since, we have $\frac{1}{2\,!} = \frac{1}{2}$

$$\frac{1}{3\,!} < \frac{1}{2^2}$$

$$\ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots$$

$$\frac{1}{n\,!} < \frac{1}{2^{n-1}}.$$

Therefore, $$1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \ldots + \frac{1}{n\,!} + \ldots < 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots .$$

The series on the right hand side is a geometric series with common ratio $\frac{1}{2}$ and hence convergent. So the series on the left hand side will also be convergent.

Example 4. *Test for convergence the series whose general term is*

$$[(n^3 + 1)^{1/3} - n].$$

Solution. Here, we have

$$\begin{aligned} u_n &= (n^3 + 1)^{1/3} - n \\ &= n\left[\left(1 + \frac{1}{n^3}\right)^{1/3} - 1\right] \\ &= n\left[\left(1 + \frac{1}{3n^3} + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2\,!} \cdot \frac{1}{n^6} + \ldots\right) - 1\right] \\ &= \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^3} + \ldots\right]. \end{aligned}$$

Let $v_n = \frac{1}{n^2}$, then the auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^2}$.

Now $\lim_{n \to \infty} \frac{u_n}{v_n} = \frac{1}{3} - \frac{1}{9n^3} + \ldots = \frac{1}{3}$, which is finite and non-zero.

Since the series $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent $(p = 2 > 1)$, therefore, the given series is also convegent.

Example 5. *Test for convergence the series whose n^{th} term is*

$$[\sqrt{(n^4+1)} - \sqrt{(n^4-1)}].$$

Solution. Here, we have

$$u_n = \sqrt{(n^4+1)} - \sqrt{(n^4-1)}$$

$$= n^2\left[\left(1+\frac{1}{n^4}\right)^{1/2} - \left(1-\frac{1}{n^4}\right)^{1/2}\right]$$

$$= n^2\left[\left(1+\frac{1}{2n^4}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2\,!}\cdot\frac{1}{n^8}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-1\right)}{3\,!}\cdot\frac{1}{n^{12}}+\ldots\right)\right.$$

$$\left.-\left(1-\frac{1}{2n^4}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2\,!}\cdot\frac{1}{n^8}-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-1\right)}{3\,!}\cdot\frac{1}{n^{12}}+\ldots\right)\right]$$

$$= n^2\left[\frac{1}{n^4}+\frac{1}{8n^{12}}+\ldots\right]$$

$$= \frac{1}{n^2}+\frac{1}{8n^{10}}+\ldots$$

Let $v_n = \frac{1}{n^2}$, then the auxiliary series is $\Sigma v_n = \Sigma \frac{1}{n^2}$, which is convergent.

Now
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{8n^{10}}+\ldots\right]\Big/\frac{1}{n^2}$$

$$= \lim_{n\to\infty}\left[1+\frac{1}{8n^8}+\ldots\right]$$

$= 1$, which is finite and non-zero.

Therefore, by comparison test, the given series is also convergent.

Example 6. *Test for convergence the series whose n^{th} term is*

$$\sqrt{n^3+1} - \sqrt{n^3}.$$

Solution. Here, we have

$$u_n = \sqrt{n^3+1} - \sqrt{n^3} = n^{3/2}\left[1+\frac{1}{n^3}\right]^{1/2} - n^{3/2}$$

$$= n^{3/2}\left[1+\frac{1}{2n^3}-\frac{1}{8n^6}+\ldots\right] - n^{3/2}$$

$$= \frac{1}{2n^{3/2}} - \frac{1}{8n^{9/2}} + \ldots .$$

Let us take $v_n = \frac{1}{n^{3/2}}$ (since, we know that, when u_n is in the form of series in powers of $1/n$, v_n is taken as the term of lowest power of $1/n$, by ignoring the numerical factor).

Then, we have

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{1}{2n^{3/2}}-\frac{1}{8n^{9/2}}+\ldots\right]\times\frac{n^{3/2}}{1}$$

$$= \lim_{n\to\infty}\left[\frac{1}{2}-\frac{1}{8n^3}+\ldots\right]$$

$= \frac{1}{2}$, which is finite and non-zero.

But the auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$ is convergent $(p = 3/2 > 1)$. Hence, the given series is also convergent.

EXERCISE 1

Test the convergence or divergence of the following series :

1. $\sum u_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$.

2. $\sum u_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$.

3. $\sum u_n = 1 + \frac{4}{5} + \frac{6}{10} + \frac{8}{17} + \dots + \frac{2n}{n^2+1} \dots$.

4. $\sum u_n = \sqrt{\frac{1}{2^3}} + \sqrt{\frac{2}{3^3}} + \sqrt{\frac{3}{4^3}} + \dots$.

5. $\sum u_n = \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{10} + \dots + \frac{\sqrt{n}}{n^2+1} + \dots$.

6. $\sum u_n = \frac{\sqrt{1}}{1+\sqrt{1}} + \frac{\sqrt{2}}{2+\sqrt{2}} + \frac{\sqrt{3}}{3+\sqrt{3}} + \dots$.

7. $u_n = \frac{n}{n^2+\sqrt{n}}$.

8. $u_n = \frac{n}{(a+nb)^2}$.

9. $u_n = \frac{\sqrt{n+1}+\sqrt{n-1}}{n}$.

ANSWERS

1. Divergent **2.** Divergent **3.** Divergent **4.** Divergent
5. Convergent **6.** Divergent **7.** Divergent **8.** Divergent
9. Convergent

• 2.5. CAUCHY'S ROOT TEST

Let $\sum u_n$ be a series of positive terms and let

$$\lim_{n \to \infty} u_n^{1/n} = l.$$

Then **if,**

(i) $l < 1$, $\sum u_n$ converges;

(ii) $l > 1$, $\sum u_n$ diverges;

(iii) $l = 1$, the test fails and the series may either converge or diverge.

Proof. Case (i) Let $u_n^{1/n} = l < 1$.

Since $l < 1$, we can choose an $\varepsilon > 0$ such that

$$l + \varepsilon < 1.$$

Let $l + \varepsilon = r$ then $0 < r < 1$.

Since $\lim_{n \to \infty} u_n^{1/n} = l$, therefore, there exists a positive integer m_1 such that

$$|u_n^{1/n} - l| < \varepsilon; \quad \forall\ n > m_1$$

$$\Rightarrow \quad l - \varepsilon < u_n^{1/n} < l + \varepsilon, \quad \forall\ n > m_1$$

$$\Rightarrow \quad (l - \varepsilon)^n < u_n < (l + \varepsilon)^n, \quad \forall\ n > m_1.$$

Since $u_n < r^n$, $\forall\ n > m_1$ and since $\sum r^n$ converges (being a geometric series with common ratio less than one). Then by comparison test, $\sum u_n$ converges.

Case (ii) Let $u_n^{1/n} = l > 1$

Since $l > 1$, we can choose $\varepsilon > 0$ such that

$$l - \varepsilon > 1$$

Let $\quad l - \varepsilon = R$ then $R > 1$.

Since $R^n < u_n$, $\forall\ n > m_2$, and since $\sum R^n$ diverges (being a G.P. with common ratio greater than one). Then, by comparison test, $\sum u_n$ diverges.

Case (iii) Let $u_n = \frac{1}{n}$.

Then $$u_n^{1/n} = \left(\frac{1}{n}\right)^{1/n}$$

Then $$\lim_{n \to \infty} u_n^{1/n} = 1.$$

Since $\Sigma\left(\frac{1}{n}\right)$ diverges, therefore we find that if

$$\lim_{n \to \infty} u_n^{1/n} = 1, \text{ then the series } \Sigma u_n \text{ may diverge.}$$

Again, let $u_n = \frac{1}{n^2}$. In this case also

$$\lim_{n \to \infty} u_n^{1/n} = 1$$

but the series Σu_n converges. Thus we find that if $\lim_{n \to \infty} u_n^{1/n} = 1$, then the series Σu_n may converge. The above two examples show that if

$$\lim_{n \to \infty} (u_n)^{1/n} = 1.$$

Then the test fails.

2.6. D'ALEMBERT RATIO TEST

If Σu_n be a series of positive terms such that

(a) $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = l$.

Then, if

(i) $l > 1$, the series converges;
(ii) $l < 1$, the series diverges;
(iii) $l = 1$, the series may converge or diverge and therefore the test fails.

(b) $\frac{u_n}{u_{n+1}} \to +\infty$ as $n \to \infty$. Then Σu_n converges.

Proof. (a) Case (i) When $l > 1$, Let $\varepsilon > 0$ be a positive number such that $l - \varepsilon > 1$.

Now, since $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = l$, therefore, $\exists$ a positive integer m such that

$$l - \varepsilon < \frac{u_n}{u_{n+1}} < l + \varepsilon, \text{ whenever } n > m.$$

Now, putting $n = m + 1, m + 2, \ldots, p - 1$, in succession in the above inequality, we get

$$l - \varepsilon < \frac{u_{m+1}}{u_{m+2}} < l + \varepsilon,$$

$$l - \varepsilon < \frac{u_{m+2}}{u_{m+3}} < l + \varepsilon,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$l - \varepsilon < \frac{u_{p-1}}{u_p} < l + \varepsilon.$$

Multiplying the corresponding sides of the first part of the above inequalities, we get

$$(l - \varepsilon)^{p-1-m} < \frac{u_{m+1}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+3}} \cdots \frac{u_{p-1}}{u_p}$$

$$\Rightarrow \quad (l - \varepsilon)^{p-1-m} < \frac{u_{m+1}}{u_p}$$

$$\Rightarrow \quad u_p < u_{m+1} (l - \varepsilon)^{m+1} \cdot (l - \varepsilon)^{-p}$$

$$\Rightarrow \quad u_p < k (l - \varepsilon)^{-p}, \ \forall \ p \geq m + 2 \text{ and } k = u_{m+1} (l - \varepsilon)^{m+1}.$$

Since, the series $\Sigma (l - \varepsilon)^{-p}$ converges (being a geometric series with common ratio $(l - \varepsilon)^{-1}$, which is certainly less than unity), then by comparison test it follows that Σu_n converges.

Case (ii) When $l < 1$, let $\varepsilon > 0$ be a positive number such that $l + \varepsilon < 1$.

Now since $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = l$, therefore, $\exists$ a positive integer m such that

$$l - \varepsilon < \frac{u_n}{u_{n+1}} < l + \varepsilon, \quad \forall\, n > m.$$

Putting $n = m + 1, m + 2, \ldots, p - 1$ in succession in the second part of the above inequality, we get

$$\frac{u_{m+1}}{u_{m+2}} < l + \varepsilon,$$

$$\frac{u_{m+2}}{u_{m+3}} < l + \varepsilon,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{u_{p-1}}{u_p} < l + \varepsilon.$$

Multiplying the corresponding sides of the above inequalities, we have

$$\frac{u_{m+1}}{u_p} < (l + \varepsilon)^{p-1-m}$$

$$\Rightarrow \quad u_p > u_{m+1}\,(l + \varepsilon)^{m+1}\,(l + \varepsilon)^{-p}$$

$$\Rightarrow \quad u_p > A(l + \varepsilon)^{-p}, \quad \forall\, p \geq m + 2 \text{ and } A = u_{m+1}\,(l + \varepsilon)^{m+1}.$$

Since, $\Sigma\,(l + \varepsilon)^{-p}$ is a divergent series (being a geometric series with common ratio $(l + \varepsilon)^{-1}$, which is certainly greater than unity), then by comparison test, it follows that Σu_n diverges.

Case (iii) Let $l = 1$.

Now, first consider the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

Then $$\frac{u_n}{u_{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n} \Rightarrow \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = 1.$$

Since, the harmonic series is divergent, we find that if $l = 1$, a series may diverge.

Now, consider the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} + \ldots.$$

Then $$\frac{u_n}{u_{n+1}} = \frac{(n+1)^2}{n^2} = \left(1 + \frac{1}{n}\right)^2 \Rightarrow \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = 1.$$

Since, the series $\Sigma \frac{1}{n^2}$ converges, we find that if $l = 1$, a series may converge.

(b) Let us suppose $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = +\infty$ then there exist positive integers m and p such that

$$\frac{u_n}{u_{n+1}} > p \; \forall\, n \geq m, \; p > 1.$$

Replacing n by $m, m + 1, m + 2, \ldots, n - 1$, we have

$$\frac{u_m}{u_{m+1}} > p$$

$$\frac{u_{m+1}}{u_{m+2}} > p$$

$$\ldots \quad \ldots \quad \ldots$$

$$\frac{u_{n-1}}{u_n} > p.$$

Multiplying the corresponding sides of the above inequalities, we have

$$\frac{u_m}{u_n} > p^{n-m}$$

$$\Rightarrow \quad u_n < p^{m-n}\,.\,u_m,$$

$$\Rightarrow \quad u_n < A\,.\,p^{-n} \quad \forall\, n > m \text{ and } A = p^m u_m.$$

Since Σp^{-n} is convergent, then by comparison test the series Σu_n is convergent.

• 2.7. RAABE'S TEST

If Σu_n be a series of positive terms is such that

$$\lim_{n \to \infty} \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} = l.$$

Then, if

(i) $l > 1$, the series converges,

(ii) $l < 1$, the series diverges,

(iii) $l = 1$, the series may either converge or diverge and therefore the test fails.

Proof. Case (i) When $l > 1$. We can write $l = 1 + r$, where $r > 0$ choosing $\varepsilon = r/2$, we can find a positive integer m such that

$$l - \varepsilon < n \left(\frac{u_n}{u_{n+1}} - 1 \right) < l + \varepsilon, \ \forall \ n \geq m.$$

Now, from the first part of the above inequality, we have

$$(1 + r) - \frac{1}{2} r < n \left(\frac{u_n}{u_{n+1}} - 1 \right), \ \forall \ n \geq m$$

$$\Rightarrow \qquad \frac{1}{2} r u_{n+1} < n u_n - (n+1) u_{n+1}, \ \forall \ n \geq m. \qquad ...(1)$$

Putting $n = m + 1, m + 2, \ldots, p - 1$ in succession in (1), we have

$$\frac{1}{2} r u_{m+2} < (m+1) u_{m+1} - (m+2) u_{m+2}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{1}{2} r u_p < (p-1) u_{p-1} - p u_p.$$

Now, adding the corresponding sides of the above inequalities, we have

$$\frac{1}{2} r [u_{m+2} + u_{m+3} + \ldots + u_p] < (m+1) u_{m+1} - p u_p,$$

$$\Rightarrow \qquad \frac{1}{2} r (u_{m+2} + \ldots + u_p] < (m+1) u_{m+1},$$

or $$u_1 + u_2 + \ldots + u_p < \frac{2(m+1)}{r} u_{m+1} + u_1 + u_2 + \ldots + u_{m+1}, \ \forall \ p \geq m + 2.$$

The above inequality shows that the sequence $\langle s_n \rangle$ of the partial sums of the series Σu_n is bounded and therefore Σu_n converges.

Case (ii) When $l < 1$. Let us choose $\varepsilon = 1 - l$, then we can find a positive integer m such that

$$l - \varepsilon < n \left(\frac{u_n}{u_{n+1}} - 1 \right) < 1 \ (= l + \varepsilon), \ \forall \ n \geq m$$

or $$n u_n < (n+1) u_{n+1}, \ \forall \ n \geq m.$$

Putting $n = m + 1, m + 2, \ldots, p - 1 \ (p \geq m + 2)$, in succession, we get

$$(m+1) u_{m+1} < (m+2) u_{m+2},$$

$$(m+2) u_{m+2} < (m+3) u_{m+3},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$(p-1) u_{p-1} < p u_p.$$

From the above inequality, we have by transitivity

$$(m+1) u_{m+1} < p u_p, \ \forall \ p \geq m + 2$$

or $$u_p > k (1/p), \ \forall \ p \geq m + 2 \text{ and } k = (m+1) u_{m+1}.$$

Now, since the series $\Sigma \left(\frac{1}{p} \right)$ diverges, then by comparison test the given series diverges.

Case (iii) When $l = 1$. In this case the test fails to give any definite information.

For example, consider the series $\Sigma \frac{1}{n}$ and $\Sigma \frac{1}{n (\log n)^2}$ then, we have

$$\lim_{n \to \infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = 1.$$

But the former series is divergent, while the latter is convegent.

• 2.8. LOGARITHMIC TEST

If Σu_n be a series of positive terms such that

$$\lim_{n \to \infty} \left(n \log \frac{u_n}{u_{n+1}}\right) = l.$$

then Σu_n converges if $l > 1$ and diverges when $l < 1$.

Proof. Case (i) When $l > 1$. In this case, we can choose $\varepsilon > 0$ such that $l - \varepsilon > 1$. Let $l - \varepsilon = p$ (say).

Since $$\lim_{n \to \infty} \left(n \log \frac{u_n}{u_{n+1}}\right) = l.$$

Therefore, we can find a positive integer m such that

$$l - \varepsilon < n \log \frac{u_n}{u_{n+1}} < l + \varepsilon, \ \forall\, n \geq m.$$

Consider the first part of the above inequality, we have

$$n \log \frac{u_n}{u_{n+1}} > p, \ \forall\, n \geq m$$

$$\frac{u_n}{u_{n+1}} > e^{p/n}, \ \forall\, n \geq m. \qquad \ldots(1)$$

Since, $a_n = \left(1 + \frac{1}{n}\right)^n$ defines a monotonically increasing sequence converging to e, therefore,

$$e \geq \left(1 + \frac{1}{n}\right)^n, \ \forall\, n. \qquad \ldots(2)$$

From (1) and (2), we have

$$\frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)^p, \ \forall\, n \geq m$$

$$\Rightarrow \qquad \frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}, \ \forall\, n \geq m, \qquad \ldots(3)$$

where $$v_n = \frac{1}{n^p}.$$

Now since $p > 1$, therefore Σv_n converges and from (3) it then follows by comparison test that Σu_n converges.

Case (ii) When $l < 1$. Let the comparison series $\Sigma v_n = \Sigma \frac{1}{n^p}$ be divergent, *i.e.*, $p < 1$.

$\therefore \quad \Sigma u_n$ will be divergent if $\frac{v_n}{v_{n+1}} > \frac{u_n}{u_{n+1}}$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} < \left(1 + \frac{1}{n}\right)^p \Rightarrow \log\left(\frac{u_n}{u_{n+1}}\right) < p \log\left(1 + \frac{1}{n}\right) = p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\right]$$

$$\therefore \quad n \log\left(\frac{u_n}{u_{n+1}}\right) = p\left[1 - \frac{1}{2n} + \frac{1}{3n^3} \ldots\right]$$

$$\therefore \quad \lim_{n \to \infty} \left[n \log \frac{u_n}{u_{n+1}}\right] = p < 1$$

$\therefore \quad \Sigma u_n$ will be divergent if $l < 1$.

Some Important Limits :

(i) $\lim\limits_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$	(ii) $\lim\limits_{n \to \infty} n^{1/n} = 1$
(iii) $\lim\limits_{n \to \infty} \frac{\log n}{n} = 0$	(iv) $\lim\limits_{n \to \infty} \left(1 + \frac{x}{n}\right)^p = 1$ if p is finite
(v) $\lim\limits_{n \to \infty} \left(1 + \frac{x}{n}\right)^{n+p} = e^x$, if p is finite.	

Some Other Important Test :

(1) De Morgan's and Bertrand's test :

The series Σu_n of positive terms is convergent or divergent according as

$$\lim \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} \log n\right] > 1 \text{ or } < 1.$$

(2) Alternative to Bertrand's test :

The series Σu_n of positive terms is convergent or divergent according as

$$\lim \left[\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n\right] > 1 \text{ or } < 1.$$

SOLVED EXAMPLE

(i) Based on D'Alembert's Ratio Test.

Example 1. *Test for convergence the series*

$$1 + \frac{2^p}{2\,!} + \frac{3^p}{3\,!} + \frac{4^p}{4\,!} + \dots .$$

Solution. Here, we have

$$u_n = \frac{n^p}{n\,!} \Rightarrow u_{n+1} = \frac{(n+1)^p}{(n+1)\,!}.$$

Now $$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{(n+1)^p}{(n+1)\,!} \cdot \frac{n\,!}{n^p} = \lim_{n \to \infty} \left[1 + \frac{1}{n}\right]^p \cdot \frac{1}{(n+1)}.$$

Hence, by Ratio test the series Σu_n is convergent.

Example 2. *Test for convergence the series*

$$\frac{2\,!}{3} + \frac{3\,!}{3^2} + \frac{4\,!}{3^3} + \dots + \frac{(n+1)\,!}{3^n} + \dots .$$

Solution. Here, we have

$$u_n = \frac{(n+1)\,!}{3^n} \Rightarrow u_{n+1} = \frac{(n+2)\,!}{3^{n+1}}.$$

Now $$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{3}{n+2} = 0.$$

Hence, by ratio test, the given series is divergent.

Example 3. *Test the series*

$$x + \frac{x^3}{3\,!} + \frac{x^5}{5\,!} + \frac{x^7}{7\,!} + \dots$$

for convergence, for all positive value of x.

Solution. Since x is positive. Hence the given series is of positive term series

Here $$u_n = \frac{x^{2n-1}}{(2n-1)\,!},\ u_{n+1} = \frac{x^{2n+1}}{(2n+1)\,!}$$

$$\Rightarrow \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{x^{2n-1}}{(2n-1)\,!} \cdot \frac{(2n+1)\,!}{x^{2n+1}} = \lim_{n \to \infty} \frac{2n\,(2n+1)}{x^2}.$$

$$= +\infty,\ \forall \text{ positive values of } x.$$

Then, by ratio test the given series converges for all positive values of x.

Example 4. *Test for convergence the series*

$$1+\frac{x}{2^2}+\frac{x^2}{3^2}+\frac{x^3}{4^2}+\dots.$$

Solution. Here we have

$$u_n=\frac{x^{n-1}}{n^2}$$

$$\Rightarrow \qquad u_{n+1}=\frac{x^n}{(n+1)^2}$$

$$\Rightarrow \qquad \frac{u_n}{u_{n+1}}=\frac{x^{n-1}(n+1)^2}{n^2 . x^n}=\frac{1}{x}\cdot\left(1+\frac{1}{n}\right)^2$$

$$\Rightarrow \qquad \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{1}{x}\left(1+\frac{1}{n}\right)^2=\frac{1}{x}.$$

Hence, by ratio test the series converges if $\frac{1}{x}>1$ *i.e.*, $x<1$ diverges if $x>1$ and the test fails if $x=1$.

For $\boldsymbol{x=1}$, $u_n=\frac{1}{n^2}$. Therefore in the case the series $\Sigma u_n=\Sigma\frac{1}{n^2}$ is convergent.

(ii) Based on Cauchy's Root Test :

Example 5. *Test the convergence of the series* $x+2x^2+3x^3+4x^4+\dots.$

Solution. Here, we have

$$u_n=nx^n$$

$$\Rightarrow \qquad (u_n)^{1/n}=n^{1/n}.x$$

$$\Rightarrow \qquad \lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}(x.n^{1?n})=x.1=x$$

$$=x.1=x \qquad [\because \log n^{1/n}=1]$$

Then, by Cauchy's root test, Σu_n is convergent if $x<1$ and is divergent if $x>1$.

For $\boldsymbol{x=1}$, the Cauchy's root test fails.

In this case, the given series becomes

$$1+2+3+\dots.$$

s_n = sum of n terms of the series $=\frac{1}{2}n(n+1)$.

Thus the given series is convergent if $x<1$ and is divergent if $x\geq 1$.

Example 6. *Test the convergence of the series*

$$\frac{1}{2}+\left(\frac{2}{3}\right)x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\dots\infty,\ x>0.$$

Solution. *Omitting the first term of the series (because it will not affect the convergence or divergence of the series)*, we have

$$u_n=\left(\frac{n+1}{n+2}\right)^n.x^n.$$

$$\text{Therefore } \lim_{n\to\infty}u_n^{1/n}=\lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n}\right)x}{1+\left(\frac{2}{n}\right)}\right]=x.$$

Therefore, by Cauchy's root test, the given series Σu_n converges if $x<1$, divergent if $x>1$.

For $\boldsymbol{x=1}$, test fails

$$\therefore \qquad \lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}=\frac{e}{e^2}=\frac{1}{e}>0.$$

$\therefore$ The series Σu_n diverges if $x=1$.

Hence, the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

Exmaple 7. *Test the series for convergence*

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ldots .$$

Solution. Here, we have

$$u_n = \frac{1}{n^n}$$

$$\Rightarrow \qquad \lim_{n \to \infty} (u_n)^{1/n} = \lim_{n \to \infty} \frac{1}{n} = 0 < 1.$$

Hence by Cauchy's root test the given series is convergent.

(iii) Based on Raabe's Test.

Example 8. *Test the convergence of the series*

$$1 + \frac{3}{7}x + \frac{3\,.\,6}{7\,.\,10}x^2 + \frac{3\,.\,6\,.\,9}{7\,.\,10\,.\,13}x^3 + \ldots .$$

Solution. After leaving the first term we have

$$u_n = \frac{3\,.\,6\,.\,9 \ldots 3n}{7\,.\,10\,.\,13 \ldots (3n+4)}x^n$$

$$\Rightarrow \qquad u_{n+1} = \frac{3\,.\,6\,.\,9 \ldots 3n\,(3n+3)}{7\,.\,10\,.\,13 \ldots (3n+4)\,(3n+7)}x^{n+1}.$$

Now, $$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \left(\frac{3n+3}{3n+7}\right)x$$

$$= \lim_{n \to \infty} \left(\frac{3 + 3/n}{3 + 7/n}\right)x$$

$$= x.$$

Then, by D'Alembert ratio test the series is convergent if $x < 1$, divergent if $x > 1$ and the test fails if $x = 1$.

For $x = 1$, we have

$$\frac{u_n}{u_{n+7}} = \frac{3n+7}{3n+3}$$

or $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{3n+7}{3n+3} - 1\right) = \frac{4}{3n+3}$$

$$\Rightarrow \qquad \lim_{n \to \infty} \left[\left(\frac{u_n}{u_{n+1}} - 1\right)\right] = \lim_{n \to \infty} \frac{4}{3n+3} = \lim_{n \to \infty} \frac{4}{3 + 3/n}$$

$$= \frac{4}{3} > 1.$$

Therefore, by Raabe's test the series is convergent when $x = 1$.

Hence, the given series is convergent when $x \leq 1$ and divergent when $x > 1$.

Example 9. *Test the convergence of the series*

$$\frac{a}{b} + \frac{(1+a)}{(1+b)} + \frac{(1+a)(2+a)}{(1+b)(2+b)} \ldots .$$

Solution. Here, we have

$$u_n = \frac{(1+a)(2+a) \ldots (n-1+a)}{(1+b)(2+b) \ldots (n-1+b)}$$

$$\Rightarrow \qquad u_{n+1} = \frac{(1+a)(2+a) \ldots (n+a)}{(1+b)(2+b) \ldots (n+b)}$$

$$\therefore \qquad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \left[\frac{n+b}{n+a}\right]$$

$$= \lim_{n \to \infty} \left[\frac{1 + \frac{b}{n}}{1 + \frac{a}{n}}\right]$$

$$= 1.$$

Hence, the D'Alembert ratio test fails.

Now, consider

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{n+b}{n+a} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{b-a}{n+b}\right]$$

$$= \lim_{n\to\infty}\left[\frac{b-a}{1+b/n}\right]$$

$$= (b-a).$$

Then by Raabe's test the given series is convergent if $b - a > 1$ *i.e.*, $b > a + 1$ and divergent if $b < a + 1$.

The test fails for $b = a + 1$.

Now for $b = a + 1$, the given series becomes

$$\frac{a}{a+1} + \frac{1+a}{2+a} + \ldots = \Sigma \frac{1+a}{n+a}.$$

Taking $v_n = \dfrac{1}{n}$, by comparison test, we can easily show that the series is divergent.

Hence, the given series is convergent if $b > a + 1$ and divergent if $b \le a + 1$.

(iv) Based on Logarithmic Test.

Example 10. *Test the convergence of the series*

$$1 + \frac{1}{2}x + \frac{2\,!}{3^2}x^2 + \frac{3\,!}{4^3}x^3 + \ldots.$$

Solution. Here, we have

$$u_n = \frac{(n-1)\,!}{n^{n-1}}x^{n-1}$$

$$\Rightarrow \qquad u_{n+1} = \frac{n\,!}{(n+1)^n}x^n$$

$$\therefore \qquad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{(n+1)^n\,(n-1)\,!\,x^{n-1}}{n\,!\,x^n\,.\,n^{n-1}}$$

$$= \lim_{n\to\infty}\left[1+\frac{1}{n}\right].\frac{1}{x}$$

$$= \frac{e}{x}.$$

Hence, the given series is convergent if $\dfrac{e}{x} > 1$ *i.e.*, if $x < e$, divergent if $x > e$ and the test fails if $x = e$. In this case

$$\lim_{n\to\infty}\left[n\log\frac{u_n}{u_{n+1}}\right] = \lim_{n\to\infty}\left[n\log\frac{\left(1+\frac{1}{n}\right)^n}{e}\right]$$

$$= \lim_{n\to\infty}\left[n^2\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\right) - n\right]$$

$$= \lim_{n\to\infty}\left[-\frac{1}{2} + \frac{1}{3n} - \ldots\right]$$

$$= -\frac{1}{2} < 1.$$

Hence, by log test the series Σu_n is diverget if $x = e$.

Thus the given series Σu_n is convergent if $x < e$ and divergent if $x \ge e$.

Example 11. *Test the convergence of the series*

$$x + \frac{2^2x^2}{2\,!} + \frac{3^3x^3}{3\,!} + \frac{4^4x^4}{4\,!} + \ldots.$$

Solution. Here, we have

$$u_n = \frac{n^n x^n}{n\,!}$$

$$\Rightarrow \qquad u_{n+1} = \frac{(n+1)^{n+1} \cdot x^{n+1}}{(n+1)\,!}.$$

Therefore, $$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{(n+1)\,!\, n^n x^n}{(n+1)^{n+1} x^{n+1} \cdot n\,!}$$

$$= \lim_{n \to \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n x} = \frac{1}{ex}.$$

Thus, by D'Alembert's ratio test the series is convergent if $ex < 1$ *i.e.*, $x < \frac{1}{e}$, divergent if $x > \frac{1}{e}$ and the test fails if $\frac{1}{ex} = 1$ *i.e.*, $x = \frac{1}{e}$.

In this case

$$\lim_{n \to \infty} n\left[\log \frac{u_n}{u_{n+1}}\right] = \lim_{n \to \infty} n \log\left[\frac{e}{\left(1 + \frac{1}{n}\right)^n}\right]$$

$$= \lim_{n \to \infty} n\left[\log e - n \log\left(1 + \frac{1}{n}\right)\right]$$

$$= \lim_{n \to \infty} n\left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^2} - \ldots\right)\right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{2} - \frac{1}{3n} + \ldots\right] = \frac{1}{2} < 1.$$

Hence, by Logarithmic test, the series is divergent if $x = \frac{1}{e}$. Thus the given series Σu_n is convergent if $x < \frac{1}{e}$ and divergent if $x \geq \frac{1}{e}$.

Example 12. *Test the convergence of the series*

$$\frac{(a+x)}{1\,!} + \frac{(a+2x)^2}{2\,!} + \frac{(a+3x)^3}{3\,!} + \ldots.$$

Solution. Here, we have

$$u_n = \frac{(a+nx)^n}{n\,!}$$

$$\Rightarrow \qquad u_{n+1} = \frac{[a + (n+1)\,x]^{n+1}}{(n+1)\,!}$$

$$\Rightarrow \qquad \frac{u_n}{u_{n+1}} = \frac{\left[1 + \frac{a/x}{n}\right]^n}{\left[1 + \frac{1}{n}\right]^n \left[1 + \frac{a/x}{n+1}\right]^{n+1}} \cdot \frac{1}{x}.$$

$$\Rightarrow \qquad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \left[\frac{\left[1 + \frac{a/x}{n}\right]^n}{\left[1 + \frac{1}{n}\right]^n \left[1 + \frac{a/x}{n+1}\right]^{n+1}} \cdot \frac{1}{x}\right]$$

$$= \frac{e^{a/x}}{x \cdot e \cdot e^{a/x}} = \frac{1}{ex}.$$

Hence, by D'Alembert's ratio test the given series is convergent if $\frac{1}{ex} > 1$ *i.e.*, $x < \frac{1}{e}$ and divergent if $x > \frac{1}{e}$ and the test fails if $x = \frac{1}{e}$.

In this case

$$\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty} n\log\left[\frac{\left[1+\frac{ae}{n}\right]^n e}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{ae}{n+1}\right)^{n+1}}\right]$$

$$= \lim_{n\to\infty} n\left[n\log\left(1+\frac{ae}{n}\right)+\log e - n\log\left(1+\frac{1}{n}\right)-(n+1)\log\left(1+\frac{ae}{n+1}\right)\right]$$

$$= \lim_{n\to\infty} n\left[n\left(\frac{ae}{n}-\frac{a^2e^2}{2n^2}+\frac{a^3e^3}{3n^3}\dots\right)+1-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\right)\right.$$

$$\left.-(n+1)\left(\frac{ae}{n+1}-\frac{a^2e^2}{2(n+1)^2}+\frac{a^3e^3}{3(n+1)^3}\right)\right]$$

$$= \lim_{n\to\infty}\left[-\frac{a^2e^2}{2}+\frac{1}{2}+\frac{a^2e^2}{2\left(1+\frac{1}{n}\right)} + \text{terms containing } n \text{ in the denominator}\right]$$

$$= -\frac{a^2e^2}{2}+\frac{1}{2}+\frac{a^2e^2}{2}$$

$$= \frac{1}{2} < 1.$$

Hence, by logarithmic test, the series is divergent.

Thus the given series is convergent if $x < \frac{1}{e}$ and divergent if $x \ge \frac{1}{e}$.

EXERCISE 2

Test the convergence of the following series :

1. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\dots$
2. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\dots+\frac{n^2}{n!}+\dots$
3. $1+\frac{2!}{2^2}+\frac{3!}{3^2}+\frac{4!}{4^2}+\dots+\frac{n!}{n^2}+\dots$
5. $\frac{1^2.2^2}{1!}+\frac{2^2.3^2}{2!}+\frac{3^2.4^2}{3!}+\dots$
6. $\Sigma\left(\frac{\sqrt{n}}{n^2+1}\right)$.
7. $1+\frac{2}{3}\left(\frac{1}{4}\right)+\frac{2.4}{3.5}\left(\frac{1}{6}\right)+\frac{2.4.6}{3.5.7}\left(\frac{1}{8}\right)+\dots$

ANSWERS

1. Convergent	2. Convergent	3. Convergent	5. Convergent
6. Convergent	7. Convergent		

• 2.10. CAUCHY'S INTEGRAL TEST

Let $f(x)$ is a non-negative monotonically decreasing integrable function on $[1, \infty[$ then the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f(x)\,dx$ converge or diverge together.

Proof. Let $f(x)$ is a monotonically decreasing on $[1, \infty[$.

Then we have

$$f(n) \ge f(x) \ge f(n+1), \text{ where } n \le x \le n+1.$$

Also, $f(x)$ is non-negative and integrable, we have

$$\int_n^{n+1} f(n)\, dx \geq \int_n^{n+1} f(x)\, dx \geq \int_n^{n+1} f(n+1)\, dx$$

or
$$f(n) \geq \int_n^{n+1} f(x)\, dx \geq f(n+1). \qquad ...(1)$$

Now, putting $n = 1, 2, \ldots (n-1)$ in (1) and adding all these, we get

$$f(1) + f(2) + \ldots + f(n-1) \geq \int_1^2 f(x)\, dx + \int_2^3 f(x)\, dx + \ldots$$

$$+ \int_{n-1}^{n} f(x)\, dx \geq f(2) + f(3) + \ldots + f(n). \qquad ...(2)$$

Let us suppose

$$S_n = f(1) + f(2) + \ldots + f(n)$$

and
$$I_n = \int_1^n f(x)\, dx.$$

Then (2) can be written as

$$S_n - f(x) \geq I_n \geq S_n - f(1)$$

or
$$f(n) \leq S_n - I_n \leq f(1). \qquad ...(3)$$

Let
$$u_n = S_n - I_n \quad \forall\, n \in N.$$

Then
$$\begin{aligned} u_{n+1} - u_n &= (S_{n+1} - I_{n+1}) - (S_n - I_n) \\ &= (S_{n+1} - S_n) - (I_{n+1} - I_n) \\ &= f(n+1) - \int_n^{n+1} f(x)\, dx \\ &\leq 0 \qquad \text{[By using (1)]} \end{aligned}$$

Hence, we have $\langle u_n \rangle$ is monotonically decreasing sequence.

Now, from (3) $u_n \geq f(n) \geq 0$, $\forall\, n \in \mathbf{N}$. Therefore sequence $\langle u_n \rangle$ is bounded below. Hence $\langle u_n \rangle$ is a convergent sequence and it has a finite limit.

Now, since $S_n = u_n + I_n$, the sequence $\langle S_n \rangle$ and $\langle I_n \rangle$ converge or diverge together. Hence, the series $\Sigma f(n)$ and the integral $\int_1^{\infty} f(x)\, dx$ converge or diverge together.

Alternating Series. *A series, whose terms are alternatively positive and negative is called an alternating series.*

Thus, a series of the form

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots$$

where $u_n > 0\ \forall\, n$, is an alternating series.

Absolute Convergence. *A series Σu_n is said to be absolutely convergent if the series $\Sigma \mid u_n \mid$ is convergent.*

Conditional Convergence. *A series Σu_n is said to be conditionally convergent if Σu_n is convergent but $\Sigma \mid u_n \mid$ is divergent.*

REMARK

- The conditional convergence of a series is also known as **semi-convergent or non-absolutely convergent.**

SOME IMPORTANT THEOREMS

Theorem 1. *An absolutely convergent series is convergent.*

Proof. Let us suppose, the series Σu_n is absolutely convergent. Then by definition $\Sigma \mid u_n \mid$ is convergent.

Now
$$u_n + \mid u_n \mid = \begin{cases} 2u_n, & \text{if } u_n \text{ is positive} \\ 0, & \text{if } u_n \text{ is negative.} \end{cases}$$

Therefore, every term of the series $\Sigma(u_n + |u_n|)$ is ≥ 0 and less than equal to the corresponding term of the convergent series $\Sigma 2|u_n|$.

Hence $\Sigma(u_n + |u_n|)$ is convergent. Hence Σu_n is convergent.

Theorem 2. *If the terms of a convergent series of positive terms are rearranged, the series remains convergent and its sum is unaltered.*

Proof. Let us suppose Σu_n be a convergent series, and let the terms be rearranged in any manner. Denote the new series by Σv_n, so that every u is a v and every v is *a* u.

Let $\quad s_n = u_1 + u_2 + \ldots + u_n$

and $\quad t_n = v_1 + v_2 + \ldots + v_n$.

Then, for any definite value of n, s_n contains n terms each of which occurs, sooner or later, in the v series and so we can find a corresponding m such that t_m contains all the terms of s_n (and possibly other not contained in s_n).

Now, since each term is positive,

$$s_n \leq t_m.$$

Also, suppose that the first m terms of Σv_n are among the first $(n + p)$ terms of Σu_n. Therefore,

$$s_n \leq t_m \leq s_{n+p}$$

and m tends to infinity with n.

Let Σu_n converges to s, so that

$$\lim s_n = \lim s_{n+p} = s$$

$$\therefore \quad \lim t_m = s.$$

Hence, Σv_n is convergent and has the same sum as Σu_n.

Theorem 3. *If the terms of an absolutely convergent series are rearragned, the series remains convergent and its sum is unaltered.*

Proof. Let Σu_n be an absolutely convergent series, and let its terms be rearranged in a different order. Let, the new series is denoted by Σv_n so that every v occurs somewhere in the u series and every u occurs somewhere in the v series.

Now, we have $u_n + |u_n| = 2u_n$ or 0 according as u_n is positive or negative. Now $\Sigma |u_n|$ is a convergent series of positive terms, so also is the series $\Sigma(u_n + |u_n|)$, because its terms are less than equal to be corresponding terms of the series $\Sigma 2|u_n|$.

Let $\quad \Sigma |u_n| = s \quad$ and $\quad \Sigma(u_n + |u_n|) = s'$

so that $\quad \Sigma u_n = s' - s$.

Also, since $\Sigma |u_n|$ and $\Sigma(u_n + |u_n|)$ are convergent series of positive terms, their sum remains unchanged by any rearragement of terms (By Theorem 2).

Accordingly,

$$\Sigma |v_n| = s$$

and $$\Sigma(v_n + |v_n|) = s'.$$

Hence $\Sigma v_n = s' - s = \Sigma u_n$, as asserted.

• 2.11. LEIBNITZ TEST

If the alternative series

$$u_1 - u_2 + u_3 - \ldots \ (u_n > 0, \forall\, n \in \mathbf{N})$$

is such that

(i) $\boldsymbol{u_{n+1} \leq u_n \quad \forall\, n \in \mathbf{N}}$

(ii) $\boldsymbol{\lim_{n \to \infty} u_n = 0.}$

Then the series converges.

Proof. Let $S_n = u_1 - u_2 + u_3 - \ldots + (-1)^n u_n$ so that $\langle S_n \rangle$ is a sequence of partial sums of the given series.

Now for all n

$$S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \geq 0 \qquad \text{[By (1)]}$$

which gives that $\langle S_{2n} \rangle$ is a monotonically increasing sequence.

Further, $\quad S_{2n} = u_1 - u_2 + u_3 - .. + u_{2n-1} - u_{2n}$

$$= u_1 - (u_2 + u_3) - (u_4 - u_5) - \ldots - u_{2n}$$

$$= u_1 - [(u_2 - u_3) + \ldots + u_{2n}]$$
$$= u_1 - \text{some positive number}$$
$$\leq u_1.$$

Therefore, the monotonically increasing sequence $\langle S_{2n} \rangle$ is bounded above and consequently it is convergent.

Let $\lim_{n \to \infty} S_{2n} = S.$

Now $S_{2n+1} = S_{2n} + u_{2n+1}$

$\therefore \lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} u_{2n+1}$

$$= S + 0 \qquad \left[\because \lim_{n \to \infty} u_n = 0\right]$$
$$= S.$$

Thus, the subsequences $\langle S_{2n} \rangle$ and $\langle S_{2n} + 1 \rangle$ both converge to the same limits. Now we shall show that the sequence $\langle S_n \rangle$ also converges to S.

Let $\varepsilon > 0$ be given. Since, the sequenes $\langle S_{2n} \rangle$ and $\langle S_{2n+1} \rangle$ both converge to S, there exist positive integers m_1, m_2 such that

$$| S_{2n} - S | < \varepsilon \quad \forall\, n \geq m_1,$$

and $$| S_{2n+1} - S | < \varepsilon \quad \forall\, n \geq m_2.$$

Let $$m = \max (m_1, m_2).$$

Then $$| S_n - S | < \varepsilon \quad \forall\, n \geq 2m$$

which gives that the sequence $\langle S_n \rangle$ converges to S.

Hence, the given series $\Sigma (-1)^{n-1} u_n$ converges.

SOLVED EXAMPLES

Example 1. *Show that*

$$\lim_{n \to \infty} \left[1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right] \textit{ exists.}$$

Solution. Let $f(x) = \frac{1}{x}, \quad x \in [1, \infty[.$

Then $f(x) > 0$ and monotonically decreasing on $[1, \infty[$.

Let $$S_n = f(1) + f(2) + \ldots + f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

and $$I_n = \int_1^n f(x)\, dx = \int_1^n \frac{1}{x}\, dx = [\log x]_1^n = \log n.$$

It can be easily shown that

$$f(n) \leq S_n - I_n \leq f(1) \quad \forall\, n \in \mathbf{N}$$

or $$0 < \frac{1}{n} \leq S_n - I_n \leq 1 \quad \forall\, n \in \mathbf{N}$$

which gives that the sequence $\langle u_n \rangle$, where $u_n = S_n - I_n$, is bounded below.

Now, it can also be shown easily that the sequence $\langle u_n \rangle$ is a monotonically decreasing sequence. Therefore it converges.

Hence, $\lim_{n \to \infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)$ exist.

Example 2. *Show by integral test that $\Sigma \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

Solution. Let $f(x) = \frac{1}{x^p}$, $p > 0$. Then $f(x)$ is positive valued and monotonically decreasing.

Therefore by Cauchy's integral test $\Sigma \frac{1}{n^p}$ and $\int_1^\infty f(x)\, dx$ converges and diverges together.

Let $$I_n = \int_1^n \frac{1}{x^p}\, dx = \int_1^n x^{-p}\, dx$$

$$= \begin{cases} \left(\dfrac{n^{1-p}}{1-p} - \dfrac{1}{1-p}\right), & \text{if } p \neq 1 \\ \log n, & \text{if } p = 1. \end{cases}$$

If $\quad n \to \infty,\ n^{1-p} = \dfrac{1}{n^{p-1}} \to 0$ as $p > 1$

and tends to ∞ if $p < 1$ and $\log n \to \infty$

$\therefore \qquad \lim\limits_{n \to \infty} I_n = -\dfrac{1}{1-p} = \dfrac{1}{p-1}$, if $p > 1$

and $\qquad \lim\limits_{n \to \infty} I_n = \infty$, if $p \leq 1$.

Hence, $\displaystyle\int_1^\infty f(x)\,dx$ converges if $p > 1$ and diverges if $p \leq 1$. Then by Cauchy's integral test the series $\Sigma \dfrac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example 3. *Show that Cauchy's integral test that the series* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}$ *converges if* $p > 1$ *and diverges if* $0 < p \leq 1$.

Solution. Let us suppose

$$f(x) = \frac{1}{x\,(\log x)^p}, \ p > 0$$

and $x \in [2, \infty[$; then obviously $f(x)$ is monotonically decreasing on $[2, \infty[$ and positive valued.

Let $\qquad I_n = \displaystyle\int_2^n \frac{dx}{x\,(\log x)^p}$

Then $\qquad I_n = \left[\dfrac{(\log x)^{1-p}}{1-p}\right]_2^n, \ p \neq 1$

$$= \frac{1}{(1-p)}\,[(\log n)^{1-p} - (\log 2)^{1-p}], \quad p \neq 1$$

and $\qquad I_n = [\log \log x]_2^n, \quad p = 1$

$$= [\log \log n - \log \log 2], \quad p = 1.$$

Therefore, we have

$$\lim_{n \to \infty} I_n = \lim_{n \to \infty} \int_2^n f(x)\,dx = \infty, \text{ if } p < 1$$

and $\qquad \lim\limits_{n \to \infty} I_n = -\dfrac{1}{(1-p)}\,(\log 2)^{1-p}$, if $p > 1$.

Thus the integral $\displaystyle\int_2^\infty f(x)\,dx$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Hecce, by Cauchy's integral test, the series

$$\sum_{n=2}^{\infty} f(x) = \sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}$$

converges if $p > 1$ and diverges if $0 < p \leq 1$.

Example 4. *Test the convergence of the series*

$$\frac{1}{x} - \frac{1}{x+a} + \frac{1}{x+2a} \ldots; \qquad x > 0, a > 0.$$

Solution. Since, the given series is an alternating series.

$\therefore$ the n^{th} term

$$t_n = (-1)^{n-1} \frac{1}{x+(n-1)\,a}\, u_n, \text{ where } u_n = \frac{1}{x+(n-1)\,a} > 0.$$

Now $\qquad u_{n+1} - u_n = \dfrac{1}{x+na} - \dfrac{1}{x+(n-1)\,a}$

$$= \frac{[x+(n-1)a]-[x+na]}{[x+na][x+(n-1)a]}$$

$$= \frac{-a}{[x+na][x+(n-1)a]} < 0$$

$\therefore \qquad u_{n+1} < u_n.$

Also, $\qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{x+(n-1)a} = 0.$

Hence, by Leibnitz test, the given series is convergent.

Example 5. *Test the convergence of the series*

$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \dots .$$

Solution. The given series is an alternating series.

Here, the n^{th} term

$$t_n = (-1)^n u_n, \text{ where } u_n = \frac{\log(n+1)}{(n+1)^2} > 0$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\log(n+1)}{(n+1)^2} = \lim_{n\to\infty} \frac{\log(n+1)}{(n+1)} \cdot \frac{1}{(n+1)} = 0.$$

Now, we shall show that

$$u_{n+1} \le u_n \quad \forall\, n.$$

Let $\qquad f(x) = \frac{\log x}{x^2}.$

Then $\qquad f'(x) = \frac{x^2 \cdot \frac{1}{x} - 2x\log x}{x^4} = \frac{1-2\log x}{x^3} < 0 \text{ when } x > e^{1/2}.$

Therefore, the function $f(x)$ is monotonically decreasing for all $x > e^{1/2}$. We know that

$$2 < e < 3 \Rightarrow 2^{1/2} < e^{1/2} < 3^{1/2}$$

$$\Rightarrow 1 < e^{1/2} < 2$$

so $f(n+2) \le f(n+1)$ for all n

i.e., $\qquad u_{n+1} \le u_n \quad \forall\, n.$

Hence, by Leibnitz test the given series is convergent.

Example 6. *Show that the series*

$$\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \dots$$

is conditionally convergent.

Solution. The given series is an alternating series.

$\therefore$ the n^{th} term

$$t_n = (-1)^{n-1} u_n \quad \text{where } u_n = \frac{1}{\sqrt{n}} > 0.$$

Now $\qquad u_{n+1} - u_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}$

$$= \frac{\sqrt{n}-\sqrt{n+1}}{\sqrt{n}\sqrt{n+1}} < 0.$$

$\therefore \qquad u_{n+1} < u_n.$

Also $\qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$

$\therefore$ by Leibnitz test the given series is convergent.

But the series $\Sigma \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \Sigma \frac{1}{\sqrt{n}}$ is divergent $\left(\because\ p = \frac{1}{2} < 1 \right)$.

Hence, the given series is conditionally convergent.

• SUMMARY

- $\sum_{r=1}^{n} u_r$ is known as the partial sum of the infinite series $\sum_{N=1}^{\infty} u_n$.
- If $\lim u_n = 0$, then Σu_n is need not be convergent.
- If $\lim u_n \neq 0$ then the Σu_n is divergent.
- **Cauchy's root test :** If $\lim_{n \to \infty} (u_n)^{1/n} = l$, then
  (i) Σu_n is convergent if $l < 1$
  (ii) Σu_n is divergent if $l > 1$
  (iii) If $l = 1$, then the test fails.
- **D'Alembert Ratio Test :** If $\lim \dfrac{u_n}{u_{n+1}} = l$, then
  (i) Σu_n converges if $l > 1$.
  (ii) Σu_n diverges if $l < 1$.
  (iii) If $l = 1$, then the test fails.
- **Raabe's Test :** If $\lim_{n \to \infty} n\left(\dfrac{u_n}{u_{n+1} - 1}\right) = l,\ u_n > 0$, then
  (i) Σu_n converges if $l > 1$.
  (ii) Σu_n diverges if $l < 1$.
  (iii) If $l = 1$, thn the test fails.
- **Logarithmic Test :** If $\lim_{n \to \infty}\left(n \log \dfrac{u_n}{u_{n+1}}\right) = l,\ u_n > 0$, then
  (i) Σu_n converges if $l > 1$
  (ii) Σu_n diverges if $l < 1$
- **De Morgan's and Bertrand's test :** If $\lim_{n \to \infty}\left[n\left(\dfrac{u_n}{u_{n+1}} - 1\right) - 1\right] \log n = l,\ u_n > 0$ then
  (i) Σu_n converges if $l > 1$
  (ii) Σu_n diverges if $l < 1$.

• STUDENT ACTIVITY

1. Prove that $\sum_{n=1}^{\infty} \dfrac{1}{n^b}$ is convergent if $p > 1$.

2. Test the convergence of the series $\dfrac{1}{x} - \dfrac{1}{x+a} + \dfrac{1}{x+2a}$ ...., $x > 0, a > 0$.

• TEST YOURSELF

1. Test the convergence of the following series.
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots .$$
2. Prove that the following series is absolutely convergent

$$\left(\frac{\sqrt{2}-1}{1}\right)-\left(\frac{\sqrt{3}-\sqrt{2}}{2}\right)+\left(\frac{\sqrt{4}-\sqrt{3}}{3}\right)-\dots.$$

3. Show that the series $\Sigma (-1)^{n-1} \sin \frac{1}{n}$ is conditionally convergent.

4. Test for convergence the series $\Sigma \left(\frac{n^{n-1} . x^{n-1}}{n!}\right)$.

5. Show that the series

$$\frac{2}{1^2}-\frac{3}{2^2}+\frac{4}{3^2}-\frac{5}{4^2}+\dots.$$

converge conditionally.

6. Show that the series $\Sigma (-1)^n [\sqrt{n^2+1}-n]$ is conditionally convergent.

7. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} . n}{n^2+1}$ is not absolutely convergent.

ANSWERS

1. Convergent 4. Convergent if $x \leq \frac{1}{e}$ and divergent if $x > \frac{1}{e}$

Fill in the Blanks :

1. Every absolutely convergent series is ....
2. The sum of an absolutely convergent series is .... of the order of terms.
3. A series whose terms are alternatively positive and negative is called an ....
4. If Σu_n is convergent, and $\Sigma | u_n |$ is divergent then series Σu_n is said to be ....

True or False :

Write T for true and F for false statement.

1. For every convergent series, it is necessary that $\lim u_n = 0$. (T/F)
2. The series $\Sigma \frac{1}{n}$ is convergent. (T/F)
3. If Σu_n is a series of positive terms then $u_n > 0, \ \forall\, n \in \mathbf{N}$. (T/F)
4. If $\lim u_n > 0$ then series is convergent. (T/F)
5. If $\lim u_n = 0$, then the series may or may not be convergent. (T/F)
6. If $\lim u_n = 0$, then the series is always convergent. (T/F)

Multiple Choice Questions :

Choose the most appropriate one.

1. If $\lim u_n = 0$ (u_n is the n^{th} term of the given series) then :
   (a) series is necessarily convergent (b) series is necessairly divergent
   (c) may or may not be convergent (d) none of these.
2. If Σu_n converges to l_1 and Σv_n converges to l_2, then $\Sigma (u_n + v_n)$ converges to :
   (a) l_1 (b) l_2 (c) $l_1 + l_2$ (d) $l_1 - l_2$.
3. If Σu_n and Σv_n are two divergent series having all positive terms, then $\Sigma (u_n + v_n)$ is :
   (a) convergent (b) divergent (c) oscillatory (d) none of these.
4. The nature of the given series will be change if :
   (a) the sign of all terms are changed
   (b) a finite no. of terms are added or omitted
   (c) each term of the series is multiplied or divided by a non-zero number
   (d) none of these.

ANSWERS

Fill in the Blanks :

1. Convergent 2. Independent
3. Alternating series 4. Conditionally or semi-convergent.

True or False :

1. T 2. F 3. T 4. F 5. T 6. F

Multiple Choice Questions

1. (c) 2. (c) 3. (b) 4. (d)

UNIT 3

UNIFORM CONVERGENCE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is pointwise convergence ?
- What is uniform convergence.
- How to determine that the given sequence or series of functions is uniformly convergent ?

3.1. POINTWISE CONVERGENCE

Let $\langle t_n \rangle$ be a sequence of real valued functions on a metric space (X, d). Let the function f_n be tends to a definite limit for all values of $x \in X$ as $n \to \infty$. Therefore, to each point $t \in X$, there corresponds a sequence of numbers $\langle f_n(t) \rangle$ with terms

$$f_1(t), f_2(t), f_3(t) \ldots$$

Let this sequence $\langle f_n(t) \rangle$ converges to $f(t)$. Then pointwise converges can be defined as follows:

Definition. Let (X, d) be a metric space and f be a function from X to R. Also, for each $n \in \mathbf{N}$ let $f_n : X \to \mathbf{R}$. Then, the sequence of functions $\langle f_n \rangle$ conveges pointwise to the function f, if for each $x \in X$, the sequence of real nubmers $\langle f_n(x) \rangle$ converges to the real number $f(x)$.

Therefore $\langle f_n \rangle$ converges pointwise to f if

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \forall \; x \in X.$$

For example :

(i) For each $n \in \mathbf{N}$. Let us define $f_n : \mathbf{R} \to \mathbf{R}$ by $f_n(x) = \frac{x}{n} \;\; \forall \, x \in \mathbf{R}$

Then $\langle f_n(x) \rangle$ converges to $f(x) = 0 \;\; \forall \; x \in \mathbf{R}$.

(ii) The sequence $\langle f_n(x) \rangle = \langle x^n \rangle$ converges pointwise to the function $f : [0, 1] \to \mathbf{R}$ defined by $f(x) = \begin{cases} 0 & \text{if } x \in \,]0, 1[\\ 1 & \text{if } x = 1. \end{cases}$

(iii) The sequence $\langle x(1-x)^n \rangle$ converges pointwise to the function f that vanish identically.

(iv) The sequence $\langle f_n = 1 + \frac{x}{1+nx} \rangle$ converges, pointwise to the function f defined by $f(x) = 1 \;\; \forall \; x \in \,]0, \infty[$.

(v) The geometric series $1 + x + x^2 + x^3 + \ldots$ converges to $(1-x)^{-1} \;\; \forall \; x \in \,]-1, 1[$.

Theorem wihout proof. Let (X, d) be a metic space and f be a function fro X to R and $f_n : X \to R \; \forall \; n \in N$. The sequence of function $\langle f_n \rangle$ converes pointwise to f if and only if for each $x \in X$ and for each positiv real number, ε, $\exists$ a positive integer m such that

$n \geq m \Rightarrow |f_n(x) - f(x)| < \varepsilon.$

SOLVED EXAMPLE

Example 1. *Let $\langle f_n \rangle$ be the sequence defined by $f_n : \mathbf{R} \to \mathbf{R}$ such that*

$$f_n(x) = \frac{x}{n} \quad \forall \; x \in \mathbf{R}, n \in \mathrm{N}.$$

Show that the sequence converges pointwise to the zero function.

Solution. Here, we want to show the given sequence converges pointwise to the zero function *i.e.*, $f(x) = 0$, $x \in \mathbf{R}$, then we must show that given $\varepsilon > 0$, we can find $m \in \mathbf{N}$ such that

$$\forall \; n \geq m \Rightarrow \left| \frac{x}{n} - 0 \right| = \frac{|x|}{n}. \qquad ...(1)$$

Let us choose $m > \frac{|x|}{\varepsilon}$.

Then (1) gives

$$\forall \; n \geq m \Rightarrow \left| \frac{x}{n} - 0 \right| = \frac{|x|}{n} < \varepsilon.$$

Here, the given sequence converges pointwise to the zero function.

• 3.2. UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS

Let us suppose the sequence $\langle f_n(x) \rangle$ converges for every point x in X. Therefore, f_n tends to a definite limit as $n \to \infty$ for every $x \in X$. The limit is also a function of x.

Then by definition of limit, we must have that for every $\varepsilon > 0 \; \exists$ a positive integer m such that

$$n \geq m \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Here, it must be noted that the integer m depends upon x as well as ε.

Definition. *The sequence $\langle f_n(x) \rangle$ of functions is said to converge, uniformly on X to a function f, if for every $\varepsilon > 0$, we can find a positive integer m such that*

$$n \geq m \Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall \; x \in X.$$

Some Examples :

(1) The sequence of function $\langle f_n \rangle$ defined on $\mathbf{R}$ such that $f_n(x) = \frac{x}{n} \; \forall \; n \in \mathbf{N}$ converges pointwise to the zero function (*i.e.*, $f(x) = 0$) while, this sequence does not converges uniformly to this function.

We will prove that convergence is not uniform.

Let us suppose the sequence $\langle \frac{x}{n} \rangle$ converges uniformly to the zero function on $\mathbf{R}$, then there is some $m \in \mathbf{N}$ (m depending only on $\varepsilon = 1$) such that

$$n \geq m \Rightarrow |f_n(x) - f(x)| = \frac{|x|}{n} < 1 \quad \forall \; x \in \mathbf{R}$$

which is not true for all $x \in \mathbf{R}$ for if $n = m$ and $x = m$, then $\frac{|x|}{m} = 1$.

(2) Let $\langle f(x) \rangle = \langle x^n \rangle$ be the given sequence of function defined on $[0, 1]$. Then we can easily verify that the given sequence $\langle f_n(x) \rangle$ converges pointwise to the limit function f, defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

for every $x \in [0, 1]$.

To check that the convergence is uniform, we consider the interval $[0, 1]$. Let $\varepsilon > 1$ be given. Then, we have

$$|f_n(x) - f(x)| < \varepsilon \Rightarrow |x^n - 0| < \varepsilon \Rightarrow x^n < \varepsilon$$

$$\Rightarrow \quad \frac{1}{x^n} > \frac{1}{\varepsilon} \Rightarrow n \log \frac{1}{x} > \log \frac{1}{\varepsilon}$$

i.e., $$n > \frac{\log(1/\varepsilon)}{\log(1/x)}. \qquad \ldots(1)$$

Therefore, when $x \neq 1$, $m \in N$, such that

$$m > \frac{\log(1/\varepsilon)}{\log(1/x)}.$$

In particular, when $x = 0$, $m = 1$.

Now as x increases from 0 to 1, it is clear from (1) that $n \to \infty$.

Therefore, it is not possible to find $m \in \mathbf{N}$ such that

$$n \geq m \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

for all $x \in [0, 1]$. Hence, the given sequence is not uniformly convergent on [0, 1].

Note. If we consider the interval $[0, k]$, where $0 < k < 1$, then the greatest value of $\log(1/\varepsilon)/\log(1/x)$ is $\log(1/\varepsilon)/\log(1/k)$ so that if we take $m > (\log 1/\varepsilon)/\log(1/k) \in \mathbf{N}$, we have

$$n \geq m \Rightarrow |f_n(x) - f(x)| < \varepsilon \;\; \forall \; x \in [0, k].$$

Therefore, $\langle f_n(x) \rangle$ converges uniformly on $[0, k]$.

(3) The sequence of function $\langle 1/(1 + nx^2) \rangle$ does not converges uniformly on $\mathbf{R}$ to the function f defined by

$$f(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0. \end{cases}$$

(4) Let a be any positive real number and for each $n \in \mathbf{N}$.

Define $$f_n(x) = \frac{1}{1 + nx^2} \quad \forall \; x \in [a, \infty[.$$

The sequence $\langle f_n(x) \rangle$ converges uniformly to the zero function *i.e.*, $f(x) = 0$ on $[a, b[$, because of $m \in \mathbf{N}$, $m > (1 - \varepsilon)/a^2$, then

$$\begin{aligned} n \geq m \Rightarrow |f_n(x) - 0| &= \frac{1}{1 + nx^2} \\ &\leq \frac{1}{1 + mx^2} \\ &\leq \frac{1}{1 + ma^2} \\ &< \varepsilon \;\; \forall \; x \in [a, \infty[. \end{aligned}$$

Point of Non-uniform convergence. A point such that the sequence does not converge uniformly in any neighbourhood of it, however small, is said to be a point of non-uniform converges of the sequence.

Sum function of a series. Consider the series

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots, x \in X$$

of real valued function defined on a metric space (X, d). This series gives rise to a sequence of function $\langle f_n(x) \rangle$ where

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x).$$

The series $\sum u_n(x)$ is said to be convergent on X if the corresponding sequence $\langle f_n(x) \rangle$ is convergent on X and the limit function $s(x)$ of the sequence is said to the sum function or the sum of the series.

Uniform Convergence of a Series of Functions :

Definition. *The series* $\sum_{n=1}^{\infty} u_n(x)$ *is said to converge uniformly on X if the sequence* $\langle f_n(x) \rangle$, *where* $f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$, *converges uniformly on X.*

• 3.3. CAUCHY'S GENERAL PRINCIPLE OF UNIFORM CONVERGENCE

Theorem 1. *Let* $\langle f_n \rangle$ *be a sequence of real valued function defined on X. Then* $\langle f_n \rangle$ *converges uniformly on X if and only if for every* $\varepsilon > 0$, *there exists a positive integer m such that*

$$n \geq m, p \geq m, x \in X \Rightarrow |f_n(x) - f_p(x)| < \varepsilon. \qquad \ldots(1)$$

Proof. The only if part. Let us first suppose, the sequence $\langle f_n \rangle$ converges uniformly to the function f on X. Then, by definition.

For given $\varepsilon > 0$, $\exists$ a positive integer m such that

$$|f_n(x) - f(x)| < \varepsilon/2 \quad \forall \ n \geq m, \forall \ x \in X.$$

Therefore, if $p, n \geq m$, we have for any $x \in X$

$$\begin{aligned} |f_n(x) - f_p(x) &= |f_n(x) - f(x) + f(x) - f_p(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_p(x)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence (1) holds for this m.

The if part. Let $\langle f_n \rangle$ be a sequence of function from X to B such that for given $\varepsilon > 0 \ \exists$ a positive integer m such that (1) holds.

To show $\exists$ a function f on X such that the sequence $\langle f_n \rangle$ converges uniformly to f on X.

Now, for each fixed $x \in X$, (1) gives that the sequence of real numbers $\langle f_n(x) \rangle$ is a Cauchy sequence and therefore $\lim_{n \to \infty} f_n(x)$ exists for every $x \in X$

($\because$ Every Cauchy sequence of real numbers is convergent)

Define $f : X \to \mathbf{R}$ by $f(x) = \lim_{n \to \infty} f_n(x) \ \forall \ x \in X$.

We want to show that the sequence $\langle f_n \rangle$ converges uniformly to f.

If $x \in \mathbf{R}$, $\varepsilon > 0$, then there is some $m \in \mathbf{N}$ such that

$$n, p \geq m \Rightarrow |f_n(x) - f_p(x)| < \varepsilon/2 \text{ for all } x \in X.$$

For any fixed p, $p \geq m$ and fixed $x \in X$, consider the sequence $\langle |f_n(x) - f_p(x)| : n \in \mathbf{N} \rangle$. Since

$$\lim_{n \to \infty} f_n(x) = f(x) \text{ and } |f_n(x) - f_p(x)| < \varepsilon/2$$

for $n \geq m$, we have

$$\lim_{n \to \infty} |f_n(x) - f_p(x)| = |f(x) - f_p(x)| < \varepsilon/2.$$

Therefore, if $p \geq m$, the

$$|f(x) - f_p(x)| < \varepsilon \ \forall \ x \in X.$$

Hence, the sequence $\langle f_n(x) \rangle$ converges uniformly to f on X.

Theorem 2. *The series $\sum u_n(x)$ converges uniformly on X if and only if for every $\varepsilon > 0 \ \exists$ a positive integer m such that*

$$n \geq m \Rightarrow |u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+p}(x)| < \varepsilon, \ p = 1, 2, \ldots$$

for all $x \in X$.

Proof. Let $s_n(x)$ denotes the sequence of partial sum of the given series such that

$$s_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x), \ x \in X.$$

Then, $s_{n+p}(x) - s_n(x) = u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+p}(x)$.

The series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $f(x)$ on X if and only if $\langle f_n \rangle$ converges uniformly on X.

But $\langle s_n(x) \rangle$ converges to $s(x)$ on X if and only if for given $\varepsilon > 0 \ \exists$ a positive integer m such that

$$n \geq m \Rightarrow |s_{n+p}(x) - s_n(x)| < \varepsilon, \ p = 1, 2, \ldots$$

for all $x \in X$. Hence,

$$n \geq m \Rightarrow |u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+p}(x)| < \varepsilon \ \ p = 1, 2, \ldots$$

for all $x \in X$.

• 3.4. UNIFORM CONVERGENCE OF A SEQUENCE OF CONTINUOUS FUNCTION

Theorem 3. *Let $\langle f_n \rangle$ be a sequence of continuous real valued function defined on the compact metric space (X, d) such that*

$$f_1(x) \geq f_2(x) \geq \ldots \geq f_n(x) \geq \ldots \qquad \ldots(1)$$

for every $x \in X$. If $\langle f_n \rangle$ pointwise converges on X to the continuous function f on X, then $\langle f_n \rangle$ converges uniformly to f on X.

Proof. Let $\quad g_n = f_n - f$ for each $n \in \mathbf{N}$,

Then, from (1), we get

$$f_1(x) \geq g_2(x) \geq \dots g_n(x) \geq \dots \geq 0. \qquad \dots(2)$$

Also, since $\langle f_n \rangle$ converges to f on X, we have

$$\lim_{n \to \infty} g_n(x) = 0 \quad \forall \ x \in X. \qquad \dots(3)$$

To show, $\langle g_n \rangle$ converges uniformly to 0 on X. Let $\varepsilon > 0$ be given.

If $x \in X$, then from (3) $\exists$ a positive integer $m(x)$ such that

$$0 \leq g_m(x) \leq \varepsilon/2.$$

Since $g_m(x)$ is continuous at x, therefore, $\exists$ an open sphere $S(x, r)$ such that $y \in s(x) \Rightarrow g_{m(x)}(y) < \varepsilon$. Therefore, the collection

$$C = \{S(x, r) : x \in X, r > 0\}$$

forms an open cover of X. Since X is compat, therefore, by definition $\exists$ a finite subcover of C *i.e.*, $\exists$ a finite number of open spheres $S(x, r)$ say $S(x_1, r_1), S(x_2, r_2) \dots S(x_k, r_k)$, which also cover X.

Now, let

$$m = \max \{m(x_1), m(x_2), \dots, m(x_k)\}.$$

If y is any point of X, then $y \in S(x_i, r)$ for some $i = 1, 2, \dots, k$.

Therefore, $\quad g_{m(x_i)}(y) < \varepsilon.$

But since $m(x_i) \leq m$, therefore from (2), we have

$$g_m(y) = g_{m(x_i)}(y)$$

$$\Rightarrow \quad 0 \leq g_m(y) < \varepsilon \quad \forall \ y \in X.$$

Thus, from (2), we have

$$0 \leq g_n(y) < \varepsilon \quad \forall \ n \geq m, y \in X.$$

Hence, $\langle g_n \rangle$ converges uniformly to 0 on X. This implies that $\langle f_n \rangle$ converges uniformly on Y to the function f.

• 3.5. TESTS FOR UNIFORM CONVERGENCE

Theorem 1. (M_n-test). *Let $\langle f_n \rangle$ be a sequence of function defined on a metric space X. Let $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in X$ and let*

$$M_n = \sup \{|f_n(x) - f(x)| : x \in X\}.$$

Then $\langle f_n \rangle$ converges uniformly to f if and only if $M_n \to 0$ as $n \to \infty$.

Proof. Necessary condition. Let us suppose, the sequence $\langle f_n \rangle$ of functions converges uniformly to f on X. Then by definition, for a given $\varepsilon > 0$ $\exists$ a positive integer m (independent of x) such that

$$n \geq m \Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall \ x \in X.$$

Also, M_n is the supremum of $|f_n(x) - f(x)|$.

Therefore

$$|f_n(x) - f(x)| < \varepsilon \quad \forall \ n \geq m \quad \forall \ x \in X$$

$$\Rightarrow \quad M_n = \sup_{x \in X} |f_n(x) - f(x)| < \varepsilon \quad \forall \ n \geq m$$

$$\Rightarrow \quad M_n \to 0, \text{ as } n \to \infty.$$

Sufficient condition. Let us assume that $M_n \to 0$ as $n \to \infty$. Then for a given $\varepsilon > 0$ $\exists$ a positive integer m such that

$$|M_n - 0| < \varepsilon \quad \forall \ n \geq m \quad \forall \ x \in X$$

$$\Rightarrow \quad M_n < \varepsilon \quad \forall \ n \geq m, \quad \forall \ x \in X$$

$$\Rightarrow \quad \sup_{x \in X} |f_n(x) - f(x)| = M_n < \varepsilon, \ \forall \ n \geq m$$

$$\Rightarrow \quad |f_n(x) - f(x)| \leq M_n < \varepsilon \quad \forall \ n \geq m, \quad \forall \ x \in X$$

$$\Rightarrow \quad \langle f_n \rangle \text{ converges uniformly to } f \text{ on } X.$$

SOLVED EXAMPLES

Example 1. *Show that the sequence $\langle f_n \rangle$ where*

$$f_n(x) = nx(1-x)^n$$

does not converge uniformly on [0, 1].

Solution. Here, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \qquad \left| \text{Form } \frac{\infty}{\infty} \right.$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)} \qquad \text{[Using L-Hospital rule]}$$

$$= \lim_{n\to\infty} -\frac{x(1-x)^n}{\log(1-x)}$$

$$= 0 \qquad [\because\ (1-x)^n \to 0\ \forall\ x \in [0, 1]\ ]$$

$\Rightarrow$ $\quad f(x) = 0\ \forall\ x \in [0, 1]$.

Now $\quad M_n = \sup\{|f_n(x) - f(x)|\} : x \in [0, 1]\}$

$$= \sup\{nx(1-x)^n : x \in [0, 1]\}$$

$$= \sup (1-x)^n\ \forall\ x \in x[0, 1].$$

Therefore, $$M_n \geq n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right)^n \qquad \left(\text{Taking } x = \frac{1}{n} \in [0, 1]\right)$$

$$= \left(1 - \frac{1}{n}\right)^n \to \frac{1}{e} \text{ as } n \to \infty.$$

Hence, by M_n-test $\langle f_n \rangle$ does not converge uniformly on [0, 1]. Therefore, 0 is a point of non-uniform convergence, since $x = \frac{1}{n} \to 0$ as $n \to \infty$.

Theorem 2. (Weirstrass's M-test). *A series $\sum_{n=1}^{\infty} u_n(x)$ of functions will converge uniformly on X if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constants such that*

$$|u_n(x)| \leq M_n \ \forall\ n \text{ and } \forall\ x \in X.$$

Proof. Since $\sum M_n$ is convergent, therefore, by definition for a given $\varepsilon > 0$ we can find a positive integer m such that

$$n \geq m \Rightarrow M_{n+1} + M_{n+2} + \ldots + M_{n+p} < \varepsilon \qquad \ldots(1)$$

(for $p = 1, 2, 3, \ldots$)

Since $\quad |u_n(x)| \leq M_n\ \forall\ n$ and $\forall\ x \in X$. $\qquad \ldots(2)$

From (1) and (2), we conclude that

$$|u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+p}(x)| \leq |u_{n+1}(x)| + |u_{n+2}(x)| + \ldots + |u_{n+p}(x)|$$

$$\leq M_{n+1} + M_{n+2} + \ldots + M_{n+p}$$

$$< \varepsilon, \text{ for every } n \geq m \text{ and } \forall\ x \in X.$$

Hence, $\sum u_n(x)$ converges uniformly on X.

SOLVED EXAMPLES

Example 1. *Show that the series*

$$\frac{\cos x}{1^p} + \frac{\cos 2x}{2^p} + \frac{\cos 3x}{3^p} + \ldots + \frac{\cos nx}{n^p} + \ldots$$

converges uniformly on **R** *if $p > 1$. Also give the interval of convergence.*

Solution. Here, we have

$$\left|\frac{\cos nx}{n^p}\right| \leq \frac{1}{n^p}\ \forall\ x \in \mathbf{R}.$$

Also, the series $\sum \frac{1}{n^p}$ is known to be convergent for $p > 1$.

Hence, by Weirstrass's M-test the given series converges uniformly on **R** for $p > 1$.
Above is true for all $x \in R$.

So the interval of uniform convergence is $[a, b]$ where a, b are any finite distinct real numbers.

Theorem 3. (Abel's Test). *The series $\sum u_n(x)\, v_n(x)$ will converges uniformly in $[a, b]$ if*

(*i*) $\sum u_n(x)$ *is uniformly convergent in* $[a, b]$

(*ii*) *the sequence* $\langle v_n(x) \rangle$ *is monotonic for every* $x \in [a, b]$

(*iii*) *the sequence* $\langle v_n(x) \rangle$ *is uniformly bounded in* $[a, b]$ *by k i.e.,*

$$|v_n(x)| < k \quad \forall \ x \in [a, b] \ \text{ and } \ \forall\, n \in \mathbf{N}.$$

Proof. Let $R_{n,p}(x)$ be denote the partial remainder of the series $\sum u_n(x)\, v_n(x)$ and $r_{n,p}(x)$ that of the series $\sum u_n(x)$. Then

$$\begin{aligned} R_{n,p}(x) &= u_{n+1}(x)\, v_{n+1}(x) + u_{n+2}(x)\, v_{n+2}(x) + \dots u_{n+p}(x)\, v_{n+p}(x) \\ &= r_{n,1}(x)\, v_{n+1}(x) + \{r_{n,2}(x) - r_{n,1}(x)\}\, v_{n+2}(x) + \{r_{n,3}(x) - r_{n,2}(x)\}\, v_{n+3}(x) + \dots \\ &\qquad s+ \{r_{n,p}(x) - r_{n,p-1}(x)\}\, v_{n+p}(x) \\ &= r_{n,1}(x)\, \{v_{n+1}(x) - v_{n+2}(x)\} + r_{n,2}(x)\, \{v_{n+2}(x) - v_{n+3}(x)\} + \dots \\ &\qquad + r_{n,p-1}(x)\, \{v_{n+p-1}(x) - v_{n+p}(x)\} + r_{n,p}(x)\, v_{n+p}(x) \quad \text{....(1)} \end{aligned}$$

Given that $\langle v_n(x) \rangle$ is monotonic, therefore,

$$\{v_{n+1}(x) - v_{n+2}(x)\}, \{v_{n+2}(x) - v_{n+3}(x)\}, \dots, \{v_{n+p-1}(x) - v_{n+p}(x)\}$$

all have the same sign for fixed value of x in $[a, b]$. ...(A)

Also, given that $\langle v_n(x)\}$ is uniformly bounded by k, therefore

$$|v_n(x)| < k \ \text{ for all } x \in [a, b] \text{ and } \forall\, n \in \mathbf{N}. \qquad \text{...(2)}$$

Also, since the given series $\sum u_n(x)$ is uniformly convergent in $[a, b]$ for a given $\varepsilon > 0$, $\exists$ a positive integer m, independent of x such that for $n \geq m$

$$|r_{n,p}(x)| = |u_{n+1}(x) + u_{n+2}(x) + \dots + u_{n+p}(x)| < \frac{\varepsilon}{3k}. \qquad \text{...(3)}$$

From (1) and (3), we have

$$\begin{aligned} |R_{n,p}(x)| &< \frac{\varepsilon}{3k} |\ v_{n+1}(x) - v_{n+2}(x)| + \frac{\varepsilon}{3k} |v_{n+2}(x) - v_{n+3}(x)| \dots \\ &\qquad + \frac{\varepsilon}{3k} |v_{n+1}(x) - v_{n+p}(x)| + \frac{\varepsilon}{3k} |v_{n+p}(x)| \\ &= \frac{\varepsilon}{3k} |v_{n+1}(x) - v_{n+p}(x)| + \frac{\varepsilon}{3k} |v_{n+p}(x)|. \qquad \text{...(4)} \end{aligned}$$

Using (A), we have

$$\begin{aligned} &|v_{n+1}(x) - v_{n+2}(x)| + |v_{n+2}(x) - v_{n+3}(x)| + \dots + |v_{n+p-1}(x) - v_{n+p}(x)| \\ &\quad = |v_{n+1}(x) - v_{n+2}(x) + v_{n+2}(x) - v_{n+3}(x) + \dots + v_{n+p-1}(x) - v_{n+p}(x)| \\ &\quad = |v_{n+1}(x) - v_{n+p}(x)|. \end{aligned}$$

Now
$$\begin{aligned} |v_{n+1}(x) - v_{n+p}(x)| &\leq |v_{n+1}(x)| + |-v_{n+p}(x)| \\ &\leq k + k \\ &< 2k. \qquad \text{...(5)} \end{aligned}$$

Then (4) can be written as

$$|R_{n,p}(x)| < \frac{\varepsilon}{3k} \cdot 2k + \frac{\varepsilon}{3k} \cdot k = \varepsilon \qquad \text{...(6)}$$

i.e., $\quad |u_{n+1}(x) \cdot v_{n+1}(x)| + \dots + u_{n+p}(x)\, v_{n+p}(x)| < \varepsilon \quad \forall\, n \geq m \quad \forall \ x \in [a, b].$

Hence, from (6), the given series $\sum u_n(x)\, v_n(x)$ converges uniformly on $[a, b]$.

SOLVED EXAMPLE

Example 1. *Test the series* $\sum \frac{(-1)^{n-1}}{n} \cdot x^n$ *for uniform convergence in* $[0, 1]$.

Solution. Let us suppose $v_n(x) = x^n$ and $u_n(x) = \frac{(-1)^{n-1}}{n}$.

Clearly, the sequence $\langle v_n(x) \rangle$ is uniformly bounded and monotonically increasing on $[0, 1]$.

Also, the series $\sum u_n(x) = \frac{\sum (-1)^{n-1}}{n}$ is convergent. Hence, by Abel's test the series

$$\sum u_n(x)\, v_n(x) = \frac{\sum (-1)^{n-1}}{n} \cdot x^n$$

is uniformly convergent on [0, 1].

Theorem 4. (Dirichlet's Test). *The series $\sum u_n(x)\, v_n(x)$ will be uniformly convergent on $[a, b]$ if*

(i) The sequence $\langle v_n(x) \rangle$ is a positive monotonic decreasing sequence converging uniformly to zero for all $x \in [a, b]$.

(ii) $f_n(x) = \sum_{r=1}^{n} u_r(x)$ is uniformly bounded in $[a, b]$ i.e.,

$$|f_n(x)| = \left| \sum_{r=1}^{n} u_r(x) \right| < k$$

for every value of x in $[a, b]$ and for all positive integral values of n, where k is a fixed number, independent of x.

Proof. Proceed as in previous theorem, we have

$$\begin{aligned} R_{n,p}(x) &= u_{n+1}(x)\, v_{n+1}(x) + u_{n+2}(x)\, v_{n+2}(x) + \ldots + u_{n+p}(x)\, v_{n+p}(x) \\ &= [s_{n+1}(x) - s_n(x)]\, v_{n+1}(x) + [s_{n+2}(x) - s_{n+1}(x)]\, v_{n+2}(x) + \ldots \\ &\qquad + [s_{n+p}(x) - s_{n+p-1}(x)]\, v_{n+p}(x) \\ &= s_{n+1}(x)\,[v_{n+1}(x) - v_{n+2}(x)] + s_{n+2}(x)\,[v_{n+2}(x) - v_{n+3}(x)] + \ldots \\ &\quad + s_{n+p-1}(x)\,[v_{n+p-1}(x) - v_{n+p}(x)] + s_{n+p}(x)\, v_{n+p}(x) - s_n(x)\, v_{n+1}(x) \quad \ldots(1) \end{aligned}$$

Now, since $\langle v_n(x) \rangle$ is a positive monotonic decreasing sequence, therefore, $v_1(x), v_2(x), v_3(x) \ldots$ are all positive and

$$v_1(x) > v_2(x) > v_3(x) > \ldots > v_n(x) > \ldots .$$

Also $|f_n(x)| < k$ for all x in $[a, b]$ and for all $n \in \mathbf{N}$.

$\therefore$ From (1), we have

$$\begin{aligned} |R_{n,p}(x)| &\le |f_{n+1}(x)\,[v_{n+1}(x) - v_{n+2}(x)] + \ldots + |f_{n+p-1}(x)|\,[v_{n+p-1}(x) - v_{n+p}(x)] \\ &\qquad + |f_{n+p}(x)|\, v_{n+p}(x)| + s_n(x)\,|\, v_{n+1}(x)| \\ &< k\,[v_{n+1}(x) - v_{n+p}(x) + v_{n+p}(x) + v_{n+1}(x)] \\ &= 2k\, v_{n+1}(x). \end{aligned} \quad \ldots(2)$$

Also, since $\langle v_{n+1}(x) \rangle$ converges to zero, we have

$$|v_n(x)| < \frac{\varepsilon}{2k} \quad \forall \; n \ge m$$

i.e.,
$$v_n(x) < \frac{\varepsilon}{2k} \quad \forall \; n \ge m. \quad \ldots(3)$$

From (2) and (3), we conclude that

$$|R_{n,p}(x)| < 2k \cdot \frac{\varepsilon}{2k} \quad \text{for} \quad n \ge m$$

$$\Rightarrow \quad |R_{n,p}(x)| < \varepsilon \quad \text{for} \quad n \ge m \quad \forall \; x \in [a, b].$$

Hence, the series $\sum u_n(x)\, v_n(x)$ is uniformly convergent in $[a, b]$.

SOLVED EXAMPLE

Example 1. *Show that the series* $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot x^n$ *converges uniformly in* $0 \le x \le k < 1$.

Solution. Let $u_n = (-1)^{n-1}$, $v_n(x) = x^n$.

Since
$$s_n(x) = \sum_{r=1}^{n} u_r = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

$\Rightarrow$ $s_n(x)$ is bounded for all $n \in \mathbf{N}$.

Also $\langle v_n(x) \rangle$ is positive monotonic decreasing sequence, converging to a zero for all values of x in $0 \le x \le k < 1$.

Hence, by Dirichlet's test, the given series is uniformly convergent in $0 \le x \le k < 1$.

Example 2. *Show that the sequence* $\langle f_n \rangle$, *where* $f_n(x) = \dfrac{nx}{1+n^2x^2}$ *does not converge uniformly on* **R**.

Solution. Here, we have

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{1+n^2x^2} = 0 \quad \forall \; x \in \mathbf{R}.$$

Let if possible, the sequence converges uniformly on **R**, then for a given $\varepsilon > 0$, $\exists$ a positive integer m such that

$$n \ge m, x \in \mathbf{R} \Rightarrow |f_n(x) - f(x)| = \frac{n|x|}{1+n^2x^2} < \varepsilon. \qquad \text{...(1)}$$

If we take $\varepsilon = \dfrac{1}{3}$ and $x = \dfrac{1}{n}$ $(n = 1, 2, 3, \ldots)$, then

$$|f_n(x) - f(x)| = \frac{n\frac{1}{n}}{1+n^2\frac{1}{n^2}} = \frac{1}{2} < \frac{1}{3} = \varepsilon.$$

Thus, there is no single m such that (1) holds simultaneously for all $x \in \mathbf{R}$.

For if, such an m existed, we would have (on taking $n = m$)

$$|f_m(x) - f(x)| < \frac{1}{3} \quad \forall \; x \in \mathbf{R}$$

but if we take $x = \dfrac{1}{m}$, we get a contradiction $\left(\because \text{ in this case } \dfrac{1}{2} < \dfrac{1}{3}\right)$ and therefore, the sequence is not uniformly convergent on **R**. Also since $\dfrac{1}{m} \to 0$, therefore, 0 is a point of non-uniform convergence.

Example 3. *Discuss the series*

$$\sum_{n=1}^{\infty} \left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$$

for uniform convergence.

Solution. Here, we have

$$u_1(x) = \frac{x}{1+x^2} - 0$$

$$u_2(x) = \frac{2x}{1+2^2x^2} - \frac{x}{1+x^2}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$u_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{(1+(n-1)^2x^2)}.$$

On adding, we get

$$f_n(x) = \frac{nx}{1+n^2x}.$$

Now do same as example (1).

Example 4. *Show that the sequence* $\langle f_n \rangle$ *where*

$$f_n(x) = \frac{x}{1+nx^2}$$

converges uniformly on **R**.

Solution. Here, we have

$$f(x) = \lim_{n \to \infty} \frac{x}{1+nx^2} = 0 \quad \forall \; x \in \mathbf{R}.$$

Let $\qquad y = f_n(x) - f(x) = \dfrac{x}{1+nx^2}.$

For maxima and minima of y, we must have

$$\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{(1+nx^2)-2nx^2}{(1+nx^2)^2}=0$$

$$\Rightarrow \quad \frac{1-nx^2}{(1+nx^2)^2}=0$$

$$\Rightarrow \quad x=\pm\frac{1}{\sqrt{n}}.$$

Clearly, $\frac{d^2y}{dx^2}$ is negative when $x=\frac{1}{\sqrt{n}}$.

$\therefore$ Maximum value of $y=\dfrac{1/\sqrt{n}}{1+n\left(\frac{1}{n}\right)}=\dfrac{1}{2\sqrt{n}}$.

Also,
$$\frac{1}{2\sqrt{n}}-|y|=\frac{1}{2\sqrt{n}}-\frac{|x|}{1+nx^2}=\frac{1+nx^2-2\sqrt{n}\,|x|}{2\sqrt{n}\,(1+nx^2)}$$
$$=\frac{(1-|x|\sqrt{n})^2}{2\sqrt{n}\,(1+nx^2)}\geq 0.$$

Now,
$$M_n=\sup_{x\in\mathbf{R}}|f_n(x)-f(x)|$$
$$=\sup_{x\in\mathbf{R}}\left|\frac{x}{1+nx^2}\right|=\sup_{x\in\mathbf{R}}|y|$$
$$=\max.\ y=\frac{1}{2\sqrt{n}}\to 0 \text{ as } n\to\infty.$$

Hence, by M_n-test the sequence is uniformly convergent on **R**.

Example 5. *Show that* 0 *is a point of non-uniformly convergence of the sequence* $\langle f_n(x)\rangle$, *where* $f_n(x)=nx\,e^{-nx^2}$, $x\in\mathbf{R}$.

Solution. Here, we have

$$f(x)=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}nx\,e^{-nx^2}$$
$$=\lim_{n\to\infty}\frac{nx}{e^{nx^2}} \qquad \left|\text{Form }\frac{\infty}{\infty}\right.$$
$$=\lim_{n\to\infty}\frac{x}{x^2e^{nx^2}} \qquad \text{(By L-Hospital rule)}$$
$$=0.$$

Let if possible, the sequence be uniformly convergent in a neighbourhood $]0,k[$ of 0, where $k\in\mathbf{N}$.

Then, for a given $\varepsilon>0$, $\exists$ a positive integer m such that

$$n\geq m,\ x\in\,]0,k[\ \Rightarrow\ |f_n(r)-f(x)|=nx\,e^{-nx^2}<\varepsilon. \qquad ...(1)$$

In particular, the inequality (1) must be true for $x=\frac{1}{\sqrt{n}}$, where n is a positive integer greater than m such that

$$0<\frac{1}{\sqrt{n}}<k.$$

Then (1) gives

$$\frac{\sqrt{n}}{e}<\varepsilon.$$

Now, since $x\to 0$, when $n\to\infty$, we see that on taking x sufficiently near 0, we can take n so large that $\frac{\sqrt{n}}{e}>\varepsilon$, which is a contradiction.

Hence, 0 is a point of non-uniform convergence of the sequence.

Aliter. Let $y = f_n(x) - f(x) = nx\, e^{-nx^2}$.

For maxima and minima of y, we must have

$$\frac{dy}{dx} = 0 \Rightarrow ne^{-nx^2} - 2n^2x^2\, e^{-nx^2} = 0$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{2n}}.$$

Also, $$\frac{d^2y}{dx^2} = -\text{ ve, when } x = \frac{1}{\sqrt{2n}}.$$

Therefore, maximum $$y = n \cdot \frac{1}{\sqrt{2n}} e^{-n \cdot \frac{1}{2n}} = \sqrt{\frac{n}{2e}}$$

$$\Rightarrow \quad M_n = \sup_{x \in \mathbf{R}} |f_n(x) - f(x)|$$

$$= \sup_{x \in \mathbf{R}} n\,|x|\, e^{-nx^2}$$

$$= \sup |y|$$

$$= \text{Max. } y$$

$$= \sqrt{\frac{n}{2e}} \to \infty \text{ as } n \to \infty$$

$\Rightarrow$ M_n does not tends to zero as $n \to \infty$.

Hence by M_n-test, the given sequence is not uniformly convergent.

Also $x \to 0$ as $n \to \infty$, therefore, 0 is a pair of non-uniformly convergence.

Example 6. *Show that the sequence* $\langle f_n \rangle$, *where* $f_n = x^{n-1}(1-x)$. *Converges uniformly in the interval* [0, 1].

Solution. Here, we have

$$f(x) = \lim_{n \to \infty} f_n(x)$$

$$= \lim_{n \to \infty} x^{n-1}(1-x) = 0 \quad \forall\; x \in [0, 1].$$

Let $$y = |f_n(x) - f(x)| = x^{n-1}(1-x).$$

For maxima or minima of y, we must have $\frac{dy}{dx} = 0$

$$\Rightarrow \quad (n-1)x^{n-2}(1-x) - x^{n-1} = 0$$

$$\Rightarrow \quad x^{n-2}[(n-1)(1-x) - x] = 0$$

$$\Rightarrow \quad x = 0, \frac{n-1}{n}.$$

Also, we can see that $\frac{d^2y}{dx^2}$ is negative, when $x = \frac{n-1}{n}$.

Now $$M_n = \sup_{x \in [0,1]} |f_n(x) - f(x)|$$

$$= \sup_{x \in [0,1]} |x^{n-1}(1-x)|$$

$$= \sup_{x \in [0,1]} |y|$$

$$= \text{Max. } y$$

$$= \left(1 - \frac{1}{n}\right)^{n-1}\left(1 - \frac{n-1}{n}\right)$$

$$\to \frac{1}{e} \times 0 = 0 \text{ as } n \to \infty.$$

Hence, by M_n-test, the sequence is uniformly convergent on [0, 1].

Example 7. *Show that the sequence* $\langle f_n \rangle$, *where* $f_n(x) = \frac{n}{n+x}$, $x \geq 0$ *is uniformly convergent in any finite interval.*

Solution. Here, we have

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{n}{n+x} \qquad \left|\text{Form } \frac{\infty}{\infty}\right.$$

$$= \lim_{n \to \infty} \frac{1}{1 + x/n} = 1 \ \forall \ x \geq 0.$$

For an arbitrary choosen positive number ε, we have

$$|f_n(x) - f(x)| < \varepsilon$$

if $$\left| \frac{n}{n+x} - 1 \right| < \varepsilon$$

i.e., if $$\left| \frac{-x}{n+x} \right| < \varepsilon$$

i.e., if $$\frac{x}{n+x} < \varepsilon$$

i.e., if $$n > x\left(\frac{1}{\varepsilon} - 1\right).$$

Obviously, n increase with x and tends to ∞ as $x \to \infty$.

Therefore, converges is not uniform in $[0, \infty[$.

But if $]0, k[$ is any finite interval, where $k > 0$, however large then m is any positive integer $\geq k\left(\frac{1}{\varepsilon} - 1\right)$ such that

$$n \geq m, x \in [0, k] \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Hence, the sequence is uniformly convergent on $[0, k]$.

Example 8. *Show that the series* $\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots$ *converges uniformly on* **R**. *Give the interval of uniform convergence.*

Solution. Let $$\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

Then, we have

$$|u_n(x)| = \left| \frac{\cos nx}{n^2} \right| \leq \frac{1}{n^2} \ \forall \ x \in \mathbf{R}.$$

Taking $M_n = \frac{1}{n^2}$, the series $\sum M_n = \sum \frac{1}{n^2}$ is convergent.

Hence, by weirstrass's M-test, the given series converges uniformly on **R**.

Also, the interval of uniform convergence is $a \leq x \leq b$, where a and b are any finite unequal real numbers.

Example 9. *The sum to n terms of a series is* $f_n(x) = \frac{n^2 x}{1 + n^4 x^2}$.

Show that it converges non-uniformly in the interval [0, 1].

Solution. Here, we have

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{n^2 x}{1 + n^4 x^2} = 0 \ \forall \ x \in [0, 1],$$

Let if possible, the sequence $\langle f_n(x) \rangle$ converges uniformly on [0, 1]. Then, by definition for a given $\varepsilon > 0$, $\exists \, m \in \mathbf{N}$ such that

$$n \geq m, x \in [0, 1] \Rightarrow |f_n(x) - f(x)| = \frac{n^2 |x|}{1 + n^4 x^2} < \varepsilon. \quad \ldots(1)$$

If $x = \frac{1}{n^2}$ $(n \in \mathbf{N})$, then

$$|f_n(x) - f(x)| = \frac{n^2 \cdot \frac{1}{n^2}}{1 + n^4 \cdot \frac{1}{n^4}} = \frac{1}{2}.$$

If we take $\varepsilon = \frac{1}{2}$, there is no single m such that (1) holds simultaneously for all $x \in [0, 1]$. For if such m exists, we would have

$$|f_m(x) - f(x)| < \frac{1}{2} \;\; \forall \;\; x \in [0, 1].$$

In particular, when $x = \frac{1}{m^2}$, we get a contradiction

$$\left(\because \text{ in this case we would have } \frac{1}{2} < \frac{1}{2}\right)$$

Hence, convergence is non-uniform on [0, 1].

• SUMMARY

- A sequence $\langle f_n(x) \rangle$ is said to be **pointwise convergent** to $f(x)$ if for given $\varepsilon > 0 \; \exists$ a positive integer m depending on x such that $|f_n(x) - f(x)| < \varepsilon \; \forall \; n \geq m$.
- A sequence $\langle f_n(x) \rangle$ is said to be **uniformly convergent** to $f(x), f(x) = \lim f_n(x)$ if for given $\varepsilon > 0 \; \exists$ a positive integer m not depending on x such that $|f_n(x) - f(x)| < \varepsilon \; \forall \; n \geq m \;\; \forall \; x$.
- **Cauchy's general principle of uniform convergence :** A sequence $\langle f_n(x) \rangle$ converges uniformly on X if f for every $\varepsilon > 0 \; \exists$ a positive integer m such that
$$|f_n(x) - f_p(x)| < \varepsilon \;\; \forall \; n \geq m, p \geq m.$$
- **M_n-test :** Let $f(x) = \lim f_n(x)$ and $M_n = \sup \{|f_n(x) - f(x)| : x \in X\}$. Then $\langle f_n(x) \rangle$ converges uniformly to $f(x)$ iff $M_n \to 0$ as $n \to \infty$.
- **Weirstrass's M-Test :** A series $\sum_{n=1}^{\infty} u_n(x)$ of functions, will converge uniformly on X if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constants such that $|u_n(x)| < M_n \; \forall \; n$ and $\forall \; x \in X$.

• STUDENT ACTIVITY

1. Show that the sequence $\langle f_n(x) \rangle$, where $f_n(x) = nx(1-x)^n$, is not uniformly convergent on [0, 1].

2. Show that the sequence $\langle f_n(x) \rangle$, where $f_n(x) = \frac{x}{1 + nx^2}$ converges uniformly on **R**.

• TEST YOURSELF

1. Test the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+x^2}$ for uniform convergence for all values of x.
2. Show that 0 is the point of non-uniform convergence of the sequence $\langle f_n(x) \rangle$ when

$$f_n(x) = e^{-nx},\ x \geq 0.$$

3. Show that 0 is a point of non-uniform convergence of the sequence $\langle f_n \rangle$ where

$$f_n(x) = 1 - (1 - x^2)^n.$$

4. Show that the sequence $\langle f_n(x) \rangle$ on $X = [0, 1]$ is convergent on every point of the metric space convergent on every point of metric space X but is not uniformly convergent on X, when $f_n(x) = x^n$ and

$$\lim_{n\to\infty} x^n = 0, \text{ when } 0 < x \leq 1$$

$$\lim_{n\to\infty} x^n = 1, \text{ when } x = 1.$$

5. Show that the sequence $\langle f_n \rangle$ where

$$f_n(x) = x^n(1 - x)$$

converges uniformly in $[0, 1]$.

ANSWERS

1. Uniformly convergent for all x.

UNIT 4

RIEMANN INTEGRAL

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is Riemann integral ?
- How to check whether the given function is Riemann integrable ona given interval.

4.1. SOME DEFINITIONS

Partition of a Closed Interval :

Let $I = [a, b]$ be a closed and bounded interval. Then, a finite set of points $P = \{x_0, x_1, x_2, ..., x_n\}$ such that

$$a = x_0 < x_1 < x_2 .... < x_{n-1} < x_n = b$$

is called a partition or division of the interval $I = [a, b]$.

Segments of Partitions :

The closed sub-intervals $I_1 = [a = x_0, x_1]$, $I_2 = [x_1, x_2]$ ..., $I_n = [x_{n-1}, x_n = b]$ are called the segments of the partition.

Length of the Subinterval :

The length of the subinterval I_r is denoted by Δx_r or δ_r defined by

$$\delta_r = \Delta x_r = x_r - x_{r-1}.$$

Norm of the Partition :

The norm of a partition P is the maximum of the lengths of the segments of a partition P, denoted by $\|P\|$, defined by

$$\|P\| = \max\{\Delta x_r, r = 1, 2, ..., n\}.$$

Refinement of Partition :

If a partition P^* is a refinement of a closed and bounded interval $[a, b]$ then

$$P^* = P_1 \cup P_2$$

is called the common refinement of P_1 and P_2.

Family of Partitions :

The family of all partitions of the closed interval $[a, b]$ is denoted by $\mathbf{P}(a, b)$.

Lower Riemann Sum, Upper Riemann Sum and Oscillatory Sum :

Let f be a bounded real valued function defined on a bounded and closed interval $[a, b]$ and $P = \{a = x_0, x_1, ..., x_n = b\}$ be any partition of $[a, b]$. Also, let m_r and M_r denotes the infimum and supermum of the function f on the subinterval $[x_{r-1}, x_r]$ respectively, then *the two sums*

$$L(P,f) = \sum_{r=1}^{n} m_r \, \delta x_r \quad \text{and} \quad U(P,f) = \sum_{r=1}^{n} M_r \, \delta x_r$$

are respectively called the ***lower Riemann sum and upper Riemann sum*** *of f on* $[a, b]$ *with respect to partition P.*

Also,
$$U(P,f) - L(P,f) = \sum_{r=1}^{n} [M_r - m_r] \, \delta x_r$$
$$= \sum_{r=1}^{n} \omega_r \, \delta x_r \quad \text{where } \omega_r = (M_r - m_r)$$

Then sum $\sum_{r=1}^{n} \omega_r \, \delta x_r$ *is called the* ***oscillatory sum*** *for the function f with respect to partition P of* $[a, b]$.

Upper and Lower Integrals :

The infimum of the set of the upper sums is called the upper integral of f over $[a, b]$ *and is denoted by* $U = \overline{\int_a^b} f(x)\, dx$.

Also, *the supremum of the set of the lower sums is called the lower integral of f over* $[a, b]$ *and is denoted by* $L = \underline{\int_a^b} f(x)\, dx$.

• 4.2. RIEMANN INTEGRAL

From the above discussion, it is clear that the supremum of the set of upper sums is $M(b - a)$ and the infimum of the set of the lower sums is $m\,(b - a)$, where M and m be the bounds of f on $[a, b]$ such that for every value of r

$$m \le m_r \le M_r \le M.$$

Definition. *A bounded function f is said to be Riemann integrable, or simply integrable over* $[a, b]$, *if its upper and lower integrals are equal; and their common value being called Riemann integral or simply the integral denoted by*

$$\int_a^b f(x)\, dx.$$

• 4.3. SOME THEOREMS

Theorem 1. *Let f be a bounded function defined on* $[a, b]$ *and let m and M be the infimum and supremum of* $f(x)$ *in* $[a, b]$, *then for every partition P of* $[a, b]$, *we have*

$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a).$$

Proof. Let $P = \{a = x_0, x_1, ..., x_{n-1}, x_n = b\}$ be any partition of $[a, b]$. Also, let $I_r = [x_{r-1}, x_r]$, $r = 1, 2, ..., n$ be the subintervals of $[a, b]$.

If M and m be the least upper bound and greatest lower bound of f on $[a, b]$, then we have

$$m \le f(x) \le M \quad \forall\, x \in [a, b] \qquad \text{(By definition of supremum and infimum)}$$

Now let M_r and m_r by the supremum and infimum of f in I_r.

Then $\quad m \le m_r \le M_r \le M$ for $r \in \mathbf{N}$

$$\Rightarrow \quad m\,\delta_r \le m_r\,\delta_r \le M_r\,\delta_r \le M\,\delta_r \qquad \text{(Multiplying by } \delta_r\text{)}$$

$$\Rightarrow \quad \sum_{r=1}^{n} m\,\delta_r \le \sum_{r=1}^{n} m_r\,\delta_r \le \sum_{r=1}^{n} M_r\,\delta_r \le \sum_{r=1}^{n} M\,\delta_r \qquad ...(1)$$

(By summing the above result)

Now $\sum_{r=1}^{n} m\,\delta_r = m \sum_{r=1}^{n} \delta_r = m \sum_{r=1}^{n} (x_r - x_{r-1})$ $\quad (\because\ \delta_r = x_r - x_{r-1})$

$$= m\,[(x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + \ldots + (x_n - x_{n-1})]$$

$$= m(x_n - x_0) = m(b - a) \qquad (\because\ x_0 = a, x_n = b)$$

Similarly, we may find that

$$\sum_{r=1}^{n} M\delta_r = M(b - a).$$

Also, by definition of lower sum and upper sums, we get

$$\sum_{r=1}^{n} m_r\,\delta_r = L(P, f), \quad \sum_{r=1}^{n} M_r\,\delta_r = U(P, f).$$

Using all these values in (1), we get

$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a) \quad \forall\ P \in \mathbf{P}\,[a, b].$$

Theorem 2. *If f_1 and f_2 are two real valued bounded functions defined on $[a, b]$, then*

(*i*) $L(P, f_1 + f_2) \ge L(P, f_1) + L(P, f_2)$

and (*ii*) $U(P, f_1 + f_2) \le U(P, f_1) + U(P, f_2) \quad \forall\ P \in \mathbf{P}\,[a, b]$

Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_{n-1}, x_n = b\}$ be any partition of $[a, b]$. Also, let $I_r = [x_{r-1}, x_r]$, $r = 1, 2, ..., n$ be the subintervals of $[a, b]$.

Since, f_1, f_2 both are bounded.

$\Rightarrow \quad f_1 + f_2$ is bounded. $\quad$ ($\because$ Sum of two bounded functions is also bounded)

Let M_r, m_r, M_{1r}, m_{1r} and M_{2r}, m_{2r} be the least upper bounds and greatest lower bounds of the functions $f_1 + f_2, f_1$ and f_2 in I_r for $r = 1, 2, ..., n$ respectively.

(i) By definition of infimum, we have

$$f_1(x) \ge m_{1r}$$

and $\quad f_2(x) \ge m_{2r} \quad \forall\ x \in I_r$

Therefore, $\quad f_1(x) + f_2(x) \ge m_{1r} + m_{2r}$

$\Rightarrow \quad (f_1 + f_2)\,(x) \ge m_{1r} + m_{2r}$

$\Rightarrow$ $(m_{1r} + m_{2r})$ is a lower bound of $(f_1 + f_2)\,(x)$ on I_r. But, since m_r to be the greatest lower bound of $(f_1 + f_2)$ on I_r, therefore,

$$m_r \ge m_{1r} + m_{2r}$$

$\Rightarrow \quad m_r\,\delta_r \ge m_{1r}\,\delta_r + m_{2r}\,\delta_r$ $\quad$ (Multiplying by δ_r)

$\Rightarrow \quad \sum_{r=1}^{n} m_r\delta_r \ge \sum_{r=1}^{n} m_{1r}\,\delta_r + \sum_{r=1}^{n} m_{2r}\,\delta_r$ $\quad$ (By summing the above result)

$\Rightarrow \quad L\,[P, f_1 + f_2] \ge L(P, f_1) + L(P, f_2).$

(ii) By definition of supremum, we have

$$f_1(x) \le M_{1r}$$

and $\quad f_2(x) \le M_{2r} \quad \forall\ x \in I_r$

$\Rightarrow \quad f_1(x) + f_2(x) \le M_{1r} + M_{2r}$

$\Rightarrow \quad (f_1 + f_2)\,(x) \le M_{1r} + M_{2r}$

$\Rightarrow \quad M_{1r} + M_{2r}$ is an upper bound of $(f_1 + f_2)\,(x)$ on I_r

$\Rightarrow \quad M_r \le M_{1r} + M_{2r}$

$\Rightarrow \quad M_r\,\delta_r \le M_{1r}\,\delta_r + M_{2r}\,\delta_r$ $\quad$ (Multiplying by δ_r)

$\Rightarrow \quad \sum_{r=1}^{n} M_r\,\delta_r \le \sum_{r=1}^{n} M_{1r}\,\delta_r + \sum_{r=1}^{n} M_{2r}\,\delta_r$ $\quad$ (By summing the above result)

$\Rightarrow \quad U(P, f_1 + f_2) \le U(P, f_1) + U(P, f_2).$

Theorem 3. *If f be a real valued bounded function defined on $[a, b]$ and $P_1, P_2 \in \mathbf{P}(a, b)$, such that P_2 is the refinement of P_1, then*

$$L(P_1, f) \le L(P_2, f),\ U(P_2, f) \le U(P_1, f).$$

Proof. Let $P_1 = \{a = x_0 < x_1 < x_2 \ldots < x_{r-1}, x_r ..., x_n = b\}$ be any partition of $[a, b]$ and let P_2 be any other partition of $[a, b]$ such that

$$P_2 = [a = x_0, x_1, x_2, ..., x_{r-1}, \alpha, x_r, ..., x_n = b]$$

contains just one point α more than $P_1 (x_{r-1} < \alpha < x_r)$.

Now let, the least upper bounds of f in the subinterval $[x_{r-1}, x_r]$, $[x_{r-1}, \alpha]$ and $[\alpha, x_r]$ be m_r, m_{1r} and m_{2r} respectively. Then, by the definition of least upper bound, it is clear that

$$M_r < M_{1r} \text{ and } M_r < M_{2r}. \qquad ...(1)$$

From the definition of lower Darboux sum, we find that $M_r(x_r - x_{r-1})$ is the contribution of the closed interval $[x_{r-1}, x_r]$ to $L(P_1, f)$ and $M_{1r}(\alpha - x_{r-1}) + M_{2r}(x_r - \alpha)$ that of the closed interval $[x_{r-1}, x_r]$ to $L[P_2, f]$.

Since α is the only extra point in P_2, which is not in P_1 and $x_{r-1} < \alpha < x_r$ therefore, the contribution of each subinterval except $I_r = [x_{r-1}, x_r]$ to $L(P_1, f)$ and $L(P_2, f)$ is the same. Thus,

$$L(P_2, f) \geq L(P_1, f)$$

$$\Rightarrow \qquad L(P_1, f) \leq L(P_2, f). \qquad ...(2)$$

In a similar manner taking the greatest lower bounds of f in the subintervals $[x_{r-1}, x_r]$, $[x_{r-1}, \alpha]$ and $[\alpha, x_r]$ as m_r, m_{1r} and m_{2r} respectively, we may prove that

$$U(P_2, f) \leq U(P_1, f). \qquad ...(3)$$

Also, we know that

$$L(P_2, f) \leq U(P_2, f). \qquad ...(4)$$

From (2), (3) and (4), we conclude that

$$L(P_1, f) \leq L(P_2, f) \leq U(P_2, f) \leq U(P_1, f).$$

Theorem 4. *Let f be a real valued function, defined on $[a, b]$ and $P_1, P_2 \in \mathbf{P}[a, b]$, then*

(*i*) $L(P_1, f) \leq U(P_2, f)$

(*ii*) $L(P_2, f) \leq U(P_1, f)$.

Proof. Let P_1 and P_2 be two partitions of the interval $[a, b]$. Then, it is clear that $P_1 \cup P_2$ is the common refinement of P_1 and P_2.

Also $\qquad P_1 \subseteq P_1 \cup P_2$ and $P_2 \subseteq P_1 \cup P_2$

Then, from above theorem, we have

$$L(P_1, f) \leq L(P_1 \cup P_2, f) \qquad ...(1)$$

and

$$U(P_1, f) \geq U(P_1 \cup P_2, f). \qquad ...(2)$$

Using, theorem (3), equation (1) and (2) gives

$$L(P_1, f) \leq L(P_1 \cup P_2, f) \leq U(P_1 \cup P_2, f) \leq U(P_2, f). \qquad ...(3)$$

Similarly, we may prove that

$$L(P_2, f) \leq L(P_1 \cup P_2, f) \leq U(P_1, f). \qquad ...(4)$$

From (3) and (4), we conclude that

$$L(P_1, f) \leq U(P_2, f) \text{ and } L(P_2, f) \leq U(P_1, f).$$

• 4.4. LOWER AND UPPER RIEMANN INTEGRALS

If f is bounded on the interval $[a, b]$, then for every $P \in \mathbf{P}(a, b)$, $U(P, f)$ and $L(P, f)$ exist and are bounded. Then the lower Riemann integral is defined as

$$\underline{\int_a^b} f = \sup_{\mathbf{P}} L(P, f)$$

and the upper Riemann integral is defined as

$$\overline{\int_a^b} f = \inf_{\mathbf{P}} U(P, f).$$

Riemann Integrable Function :

Definition I. *A real valued function $f(x)$ is said to be Riemann integrable on $[a, b]$ if and only if their lower and upper Riemann integrals are equal.*

i.e., iff $\qquad \underline{\int_a^b} f = \overline{\int_a^b} f.$

The common value of these integrals is known as the **Riemann integral** of f on $[a, b]$ and is denoted by $\int_a^b f(x)\,dx$

i.e.,
$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Definition II. *A function f is said to be Riemann integrable over $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a positive number δ and a number I such that for every partition*
$$P = [a = x_0, x_1, x_2, ..., x_n = b]$$
with $\| P \| < \delta$ and for every $t_r \in [x_{r-1}, x_r]$
$$\left| \sum_{r=1}^{n} f(t_r)\,(x_r - x_{r-1}) - I \right| < \varepsilon.$$

Here I is said to be the integral of f over $[a, b]$ and the class of all bounded functions f which are Riemann integrable on $[a, b]$ is denoted by $\mathbf{R}\,[a, b]$.

Theorem 1. (Darboux Theorem). *Assume that f is a bounded function defined on $[a, b]$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that*
$$U(P, f) < \overline{\int_a^b} f + \varepsilon \quad \text{and} \quad L(P, f) > \underline{\int_a^b} f - \varepsilon$$
for every partition P with $\| P \| \le \delta$.

Proof. Given that, f is bounded on $[a, b]$, then by definition of boundedness there exist $K > 0$ such that
$$|f(x)| \le K \quad \forall\, x \in [a, b].$$

Also, since $\inf U(P, f)$ is defined as $\overline{\int_a^b} f$

$\therefore$ for every $\varepsilon > 0$ there exists a partition $P_1 = [a = x_0, x_1, x_2, ..., x_n = b]$ such that
$$U(P_1, f) < \overline{\int_a^b} f + \varepsilon/2. \qquad ...(1)$$

If $x_0 = a$ and $x_n = b$, then the partition P has $(n-1)$ points. Let $\delta_1 > 0$ be any number such that
$$2k\,(n-1)\,\delta_1 = \varepsilon/2. \qquad ...(2)$$

Now, let P be any partition with $\| P \| < \delta_1$

Also, let $P_2 = P \cup P_1$, then clearly P_2 is a refinement of P and P_1 then P_2 has atmost $(n-1)$ more points than P. Therefore,
$$U(P, f) - 2K(n-1)\,\delta_1 \le U(P_2, f)$$
$$\le U(P_1, f) < \overline{\int_a^b} f + \varepsilon/2 \qquad \text{[using (1)]}$$
$$\Rightarrow \quad U(P, f) < \overline{\int_a^b} f + \varepsilon/2 + \varepsilon/2 \qquad \text{[using (2)]}$$
$$= \overline{\int_a^b} f + \varepsilon \text{ for all partition } P \text{ with } \| P \| < \delta_1$$

Similarly, we may easily shown that there exists a positive number δ_2 such that
$$L(P, f) > \underline{\int_a^b} f - \varepsilon \text{ for all partition } P \text{ with } \| P \| < \delta_2.$$

Define $\delta = \min\{\delta_1, \delta_2\}$

Then for all partition P of $[a, b]$ with $\|P\| < \delta$, we have:

$$U(P,f) < \overline{\int_a^b} f + \varepsilon \quad \text{and} \quad L(P,f) > \underline{\int_a^b} f - \varepsilon.$$

Theorem 2. (Necessary and Sufficient Condition for Integrability).

A necessary and sufficient condition for **R**-*integrability of a bounded function* $f: [a, b] \to \mathbf{R}$ *over* $[a, b]$ *is that for every* $\varepsilon > 0$, *there exists a partition* P *of* $[a, b]$ *such that*

$$0 \le U(P,f) - L(P,f) < \varepsilon \quad \forall \; \|P\| < \delta.$$

Proof. (i) Necessary Condition. Let us first suppose f be Riemann integrable on $[a, b]$. Therefore

$$\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f. \qquad ...(1)$$

Let $\varepsilon > 0$ be given, then by Darboux theorem, there exists $\delta > 0$ such that for every partition P with $\|P\| < \delta$

$$U(P,f) < \overline{\int_a^b} f + \varepsilon/2 \qquad ...(2)$$

and

$$L(P,f) > \underline{\int_a^b} f - \varepsilon/2, \qquad ...(3)$$

Adding inequalities (2) and (3), we get

$$U(P,f) + \underline{\int_a^b} f - \varepsilon/2 < L(P,f) + \overline{\int_a^b} f + \varepsilon/2$$

which gives $\quad U(P,f) - L(P,f) < \varepsilon \quad$ [using (1)]

which is the required necessary condition.

(ii) Sufficient Condition. For every $\varepsilon > 0$ and for a partition **P** of $[a, b]$ with $\|P\| \le \delta$, we have

$$U(P,f) - L(P,f) < \varepsilon.$$

By definition of upper and lower integrals, we have

$$L(P,f) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(P,f)$$

$$\Rightarrow \quad \overline{\int_a^b} f - \underline{\int_a^b} f \le U(P,f) - L(P,f) < \varepsilon$$

$$\Rightarrow \quad \overline{\int_a^b} f - \underline{\int_a^b} f \le 0 \qquad [\because \ \varepsilon \text{ is arbitrary}] \ ...(4)$$

Also, we know that lower Riemann integral can never exceed the upper Riemann integral, therefore

$$\underline{\int_a^b} f - \overline{\int_a^b} f \le 0. \qquad ...(5)$$

From (4) and (5), we conclude that

$$\underline{\int_a^b} f - \overline{\int_a^b} f = 0$$

$$\Rightarrow \qquad \underline{\int_a^b} f = \overline{\int_a^b} f.$$

Hence, the function f is Riemann integrable over $[a, b]$.

Theorem 3. *Let f be a bounded function defined on interval $[a, b]$ and P is a partition of $[a, b]$ then*

$$\lim_{\|P\| \to 0} L(P,f) = \underline{\int_a^b} f \quad \text{and} \quad \lim_{\|P\| \to 0} U(P,f) = \overline{\int_a^b} f.$$

Proof. Since given that f is a bounded function defined on interval $[a, b]$ and P is a partition of $[a, b]$ and $\underline{\int_a^b} f$ is the supremum of $L(P,f)$ for all partitions P

$$\Rightarrow \qquad L(P,f) \le \underline{\int_a^b} f \qquad \text{...(1)}$$

and $\overline{\int_a^b} f$ is the infimum of $U(P,f)$ for all partitions P

$$\Rightarrow \qquad U(P,f) = \overline{\int_a^b} f. \qquad \text{...(2)}$$

Now by Darboux theorem we know that for all $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$U(P,f) < \overline{\int_a^b} f + \varepsilon \qquad \text{...(3)}$$

$$L(P,f) > \underline{\int_a^b} f - \varepsilon \qquad \forall \text{ partition } P \text{ with } \|P\| \le \delta \quad \text{...(4)}$$

From equation (1) and (4), we have

$$\underline{\int_a^b} f - \varepsilon < L(P,f) \le \int_a^b f$$

$$\Rightarrow \qquad \underline{\int_a^b} f - \varepsilon < L(P,f) \le \int_a^b f < \underline{\int_a^b} f + \varepsilon$$

$$\Rightarrow \qquad \underline{\int_a^b} f - \varepsilon < L(P,f) < \underline{\int_a^b} f + \varepsilon$$

$$\Rightarrow \qquad \lim_{\|P\| \to 0} L(P,f) = \underline{\int_a^b} f.$$

Similarly from equation (2) and (3), we have

$$\overline{\int_a^b} f - \varepsilon < U(P, f) < \overline{\int_a^b} f + \varepsilon$$

$$\Rightarrow \qquad \lim_{\|P\| \to 0} U(P, f) = \overline{\int_a^b} f.$$

Theorem 4. *If $f : [a, b] \to R$ is bounded function then*

$$U(P, -f) = -L(P, f) \text{ and } L(P, f) = -U(P, f).$$

Proof. Consider a partition $P = \{x_0, x_1, \ldots, x_n\}$ in interval $[a, b]$, where $a = x_0$ and $x_n = b$.

Let M_r and m_r be the supremum and infimum of f in I_r.

Since f is bounded on $[a, b]$ thus $-f$ is also bounded on interval $[a, b]$ and $-m_r$ and $-M_r$ will be supremum and infimum of $-f$ in I_r.

Now $\quad L(P, -f) = \sum_{r=1}^{n} (-M_r)\, \delta x_r \qquad$ (Lower Riemann sum)

$$= -\sum_{r=1}^{n} M_r\, \delta x_r$$

$$= -U(P, f) \qquad \left\{ \because \sum_{r=1}^{n} M_r\, \delta x_r = U(P, f) \text{ the upper Riemann sum of } f \right\}$$

Similarly. $\quad U(P, f) = \sum_{r=1}^{n} (-m_r)\, \delta x_r$

$$= -\sum_{r=1}^{n} m_r\, \delta x_r \qquad \text{(Upper Riemann sum)}$$

$$= -L(P, f) \qquad \left\{ \because L \text{ is the lower Riemann sum of } f \text{ in } [a, b] \text{ such that } L(P, f) = \sum_{r=1}^{n} m_r\, \delta x_r \right\}$$

SOLVED EXAMPLES

Example 1. *Find $L(P, f)$ and $U(P, f)$ if $f(x) = x$ for $x \in [0, 3]$ and let $P = [0, 1, 2, 3]$ be the partition of* $[0, 3]$.

Solution. Let partition P divided the interval $[0, 3]$ into the subinterval $I_1 = [0, 1]$, $I_2 = [1, 2]$ and $I_3 = [2, 3]$.

The length of these intervals are given by

$$\delta_1 = 1 - 0 = 1$$
$$\delta_2 = 2 - 1 = 1$$
$$\delta_3 = 3 - 2 = 1.$$

Let M_r and m_r be respectively the l.u.b. and g.l.b. of the function f in $[x_{r-1}, x_r]$, then we get

$$M_1 = 1, m_1 = 0, M_2 = 2, m_2 = 1, M_3 = 3 \text{ and } m_3 = 2$$

Therefore, $\quad U(P, f) = \sum_{r=1}^{3} M_r\, \delta_r = M_1 \delta_1 + M_2 \delta_2 + M_3 \delta_3$

$$= 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 = 1 + 2 + 3 = 6$$

and $\quad L(P, f) = \sum_{r=1}^{3} m_r\, \delta_r = m_1 \delta_1 + m_2 \delta_2 + m_3 \delta_3$

$$= 0 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 = 0 + 1 + 2 = 3.$$

Example 2. *Let $f(x) = x$, $0 \le x \le 1$ and let $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ be a partition of $[0, 1]$, find $U(P, f)$ and $L(P, f)$.*

Solution. Let the partition P divides the interval $[0, 1]$ into the subintervals

$$I_1 = \left[0, \frac{1}{4}\right],\ I_2 = \left[\frac{1}{4}, \frac{1}{2}\right],\ I_3 = \left[\frac{1}{2}, \frac{3}{4}\right],\ I_4 = \left[\frac{3}{4}, 1\right]$$

Clearly, the length of each subinterval is $\frac{1}{4}$.

Now, let M_r and m_r respectively be the l.u.b. and g.l.b. of the function f in $[x_{r-1}, x_r]$, then, we get

$$M_1 = \frac{1}{4},\ M_2 = \frac{1}{2},\ M_3 = \frac{3}{4},\ M_4 = 1$$

and
$$m_1 = 0, m_2 = \frac{1}{4},\ m_3 = \frac{1}{2},\ m_4 = \frac{3}{4}.$$

Therefore,
$$U[P,f] = \sum_{r=1}^{4} M_r \delta_r = M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + M_4\delta_4$$
$$= \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + 1\cdot\frac{1}{4}$$
$$= \frac{1}{16} + \frac{1}{8} + \frac{3}{16} + \frac{1}{4} = \frac{5}{8}$$

and
$$L[P,f] = \sum_{r=1}^{4} m_r \delta_r = m_1\delta_1 + m_2\delta_2 + m_3\delta_3 + m_4\delta_4$$
$$= 0\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4}$$
$$= 0 + \frac{1}{16} + \frac{1}{8} + \frac{3}{16} = \frac{3}{8}.$$

Example 3. *Let* $f(x) = x$ *on* $[0, 1]$.

Find $\underline{\int_0^1} x\,dx$ *and* $\overline{\int_0^1} x\,dx$, *by partitioning* $[0, 1]$ *into* n *equal parts. Also, show that* $f \in \mathbf{R}[0, 1]$.

Solution. Let the partition P divides the interval $[0, 1]$ into n subintervals such that

$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r-1}{n}, \frac{r}{n}, \ldots, \frac{n}{n} = 1\right\}.$$

Clearly, here we have

$$m_r = \frac{r-1}{n},\ M_r = \frac{r}{n} \text{ and } \quad \delta_r = \frac{1}{n} \text{ for } r = 1, 2, \ldots, n.$$

Now, by definition, we have

$$L[P,f] = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} \frac{r-1}{n}\cdot\frac{1}{n} = \frac{1}{n^2}\sum_{r=1}^{n}(r-1)$$
$$= \frac{1}{n^2}[1 + 2 + 3 + \ldots + (n-1)] = \frac{(n-1)\,.\,n}{2n^2} = \frac{n-1}{2n}$$

and
$$U[P,f] = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} \frac{r}{n}\cdot\frac{1}{n}$$
$$= \frac{1}{n^2}\sum_{r=1}^{n} r = \frac{1}{n^2}[1 + 2 + 3 + \ldots + n]$$
$$= \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}.$$

Therefore,
$$\underline{\int_0^1} x\,dx = \lim_{\|P\|\to 0} L(P,f) = \lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}$$

and
$$\overline{\int_0^1} x\,dx = \lim_{\|P\|\to 0} U(P,f) = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$$

From above, it is clear that

$$\underline{\int}_0^1 x\,dx = \overline{\int}_0^1 x\,dx = \frac{1}{2}$$

Hence, $$\int_0^1 x\,dx = \frac{1}{2}.$$

Example 4. *Let* $f(x) = x^2$ *on* $[0, a]$, $a > 0$, *show that* $f \in \mathbf{R}\,[0, a]$. *Also, find* $\int_0^a f$.

Solution. Let $P = \left[\frac{ra}{n} : r = 0, 1, ..., n\right]$ be any partition of $[0, a]$. Then, clearly, we have

$$m_r = \frac{(r-1)^2 a^2}{n^2} \quad \text{and} \quad M_r = \frac{r^2 a^2}{n^2}$$

Also, $$\delta_r = \frac{a}{n}$$

Now, $$L\,[P, f] = \sum_{r=1}^{n} m_r\,\delta_r$$

$$= \sum_{r=1}^{n} \frac{(r-1)^2 a^2}{n^2}\,\frac{a}{n} = \frac{a^3}{n^3} \sum_{r=1}^{n} (r-1)^2$$

$$= \frac{a^3}{n^3}\left[\frac{n\,(n-1)\,(2n-1)}{6}\right] = \frac{a^3}{6}\left[\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

and $$U\,(P, f) = \sum_{r=1}^{n} M_r\,\delta_r = \sum_{r=1}^{n} \frac{r^2 a^2}{n^2}\,\frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} r^2 = \frac{a^3}{n^3}\,\frac{n\,(n+1)\,(2n+1)}{6}$$

$$= \frac{a^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

Hence, $$\underline{\int}_0^a f = \lim_{\|P\| \to 0} L\,(P, f)$$

$$= \lim_{n \to \infty} \frac{a^3}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) = \frac{a^3}{3}.$$

and $$\overline{\int}_0^a f = \lim_{\|P\| \to 0} U\,(P, f)$$

$$= \lim_{n \to \infty} \frac{a^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) = \frac{a^3}{3}$$

Therefore, $$\underline{\int}_0^a f = \overline{\int}_0^a f$$

which implies $f \in \mathbf{R}\,[0, a]$ and $\int_0^a f = \frac{a^3}{3}$.

• TEST YOURSELF

1. Show that if f is defined on $[a, b]$ by $f(x) = c \quad \forall\, x \in [a, b]$, where $c \in \mathbf{R}$ then $f \in \mathbf{R}\,[a, b]$ and $\int_a^b c = c\,(b - a)$.

2. Show that if f is defined on $[0, a]$, $a > 0$ by $f(x) = x^3$, then $f \in \mathbf{R}\,[0, a]$ and $\int_0^a f = \frac{a^4}{4}$.

3. Let f be the function defined on $[0, 1]$ by $f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$

Show that $f \notin \mathbf{R}\,[0, 1]$.

• 4.5. INTEGRABILITY OF CONTINUOUS AND MONOTONE FUNCTIONS

Theorem 1. *Every continuous function is R-integrable.*

Proof. Let f be a continuous function on $[a, b]$, then clearly f is bounded.

[$\because$ Every continuous function is bounded]

Also, f is uniformly continuous on $[a, b]$ [being the continuous function in a closed interval].

Let $\varepsilon > 0$ be given. Then there exists a partition

$$P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$$

of $[a, b]$ such that the oscillation $(M_r - m_r)$ of the partition f in the sub interval (x_{r-1}, x_r) is less than $\frac{\varepsilon}{b-a}$ for $r = 1, 2, \ldots, n$. Now, consider

$$U(P,f) - L(P,f) = \sum_{r=1}^{n} M_r (x_r - x_{r-1}) - \sum_{r=1}^{n} m_r (x_r - x_{r-1})$$

$$= \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$$

$$< \sum_{r=1}^{n} \frac{\varepsilon}{b-a} (x_r - x_{r-1}) \qquad \left(\because\ M_r - m_r = \frac{\varepsilon}{b-a}\right)$$

$$\Rightarrow\ U(P,f) - L(P,f) < \frac{\varepsilon}{b-a} \sum_{r=1}^{n} (x_r - x_{r-1})$$

$$\Rightarrow\ U(P,f) - L(P,f) < \frac{\varepsilon}{b-a} [(x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})]$$

$$\Rightarrow\ U(P,f) - L(P,f) < \frac{\varepsilon}{b-a} (x_n - x_0) = \frac{\varepsilon}{b-a} (b - a) \qquad [\because\ x_n = b \text{ and } x_n = a]$$

$$\Rightarrow\ U(P,f) - L(P,f) < \varepsilon.$$

Hence, the continuous function f is **R**-integrable.

Theorem 2. *Every monotonic function f is* **R**-*integrable.*

Proof. Let f be the monotonically increasing function on $[a, b]$

i.e., $\qquad f(a) \le f(x) \le f(b) \ \ \forall\, x \in [a, b].$

Now, for a given positive number ε there exist a partition

$$P = [a = x_0, x_1, \ldots, x_n = b] \text{ of } [a, b]$$

such that the length of each subinterval is less than $\frac{\varepsilon}{[f(b) - f(a) + 1]}$

i.e., $$(x_r - x_{r-1}) < \frac{\varepsilon}{[f(b) - f(a) + 1]} \quad \text{for } r = 1, 2, \ldots, n. \qquad \ldots(1)$$

Now, since the function f is monotonically increasing on $[a, b]$ then it is bounded and monotonically incresing on each subinterval $[x_{r-1}, x_r]$.

Let M_r and m_r be the bounds of f on the subinterval $[x_{r-1}, x_r]$ then,

$$M_r = f(x_r) \text{ and } m_r = f(x_{r-1}). \qquad \ldots(2)$$

For the partition P, consider

$$U(P,f) - L(P,f) = \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$$

$$< \frac{\varepsilon}{[f(b) - f(a) + 1]} \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] \qquad [\text{using (1) and (2)}]$$

$$\Rightarrow\ U(P,f) - L(P,f) < \frac{\varepsilon}{[f(b) - f(a) + 1]} [f(x_n) - f(x_0)]$$

$$\Rightarrow\ U(P,f) - L(P,f) < \frac{\varepsilon}{[f(b) - f(a) + 1]} [f(b) - f(a)] \qquad [\because\ x_0 = a, x_n = b]$$

$$\Rightarrow\ U(P,f) - L(P,f) < \varepsilon.$$

Therefore, the function f is Riemann integrable on $[a, b]$. Similarly, we may prove that the function f is **R**-integrable on $[a, b]$ if f is monotonically decreasing function.

Hence, every monotonic function f is **R**-integrable.

Theorem 3. *A bounded function f is* **R**-*integrable in $[a, b]$ if the set of its points of discontinuity is finite.*

Proof. Given that f is discontinuous on $[a, b]$, let $[x_1, x_2, \ldots, x_k]$ be a finite set of points of discontinuity. Also, suppose that M and m be the supremum and infinium of $f(x)$ respectively on $[a, b]$. Let $\varepsilon > 0$ be an arbitrary positive number.

Now, let the above points of discontinuity of the function f be enclosed in k non-overlapping intervals $[x_1', x_1'']$, $[x_2', x_2'']$ ..., $[x_k', x_k'']$ such that the sum of the lengths of these subinterval be less than

$$\frac{\varepsilon}{2\,(M-m)} \quad (\text{with } M - m \neq 0).$$

Since, as in each of these intervals the oscillations of the function f is less then equal to $(M - m)$, therefore, their total contribution to these oscillatory sum

$$\leq \frac{\varepsilon}{2\,(M-m)}\,(M-m) \quad i.e., \quad \leq \varepsilon/2.$$

Now, consider $(k + 1)$ subintervals $[a, x_1']$, $[x_1'', x_2']$, $[x_2'', x_3']$, ..., $[x_k'', b]$.

The function f is continuous in each of these subintervals. Now, each of the above $(k + 1)$ subintervals can be further subdivided so that contribution of each of them separately to the oscillatory sum of these $(k + 1)$ subintervals is less than $\dfrac{\varepsilon}{2\,(k+1)}$.

Therefore, there exists a partition of $[a, b]$ such that the oscillatory sum

$$< \varepsilon/2 + \frac{\varepsilon}{2\,(k+1)} \cdot (k+1)$$

i.e., $\qquad$ sum $< \varepsilon/2 + \varepsilon/2$

$\Rightarrow$ $\qquad$ sum $< \varepsilon$.

Hence, the function f is Riemann-integrable in $[a, b]$.

Theorem 4. *Let f be a bounded function on $[a, b]$ and let the set of its discontinuities have a finite number of limit points, then $f \in$* **R** *$[a, b]$.*

Proof. Let $\{x_1, x_2, ..., x_k\}$ be the finite set of limit points of the set of discontinuities of f on $[a, b]$ such that

$$x_1 < x_2 < \ldots < x_k.$$

Let $\varepsilon > 0$ be given. Now let the above points of discontinuity of the function f be enclosed in k non-overlapping intervals $[x_1', x_1'']$, $[x_2', x_2'']$ … $[x_k', x_k'']$

such that the sum of their length is $< \dfrac{\varepsilon}{2\,(M-m)}$

where M = supremum of f and m = infimum of f.

Now the partition P of $[a, b]$ is given by

$$P = [a, x_1', x_1'', x_2', x_2'' \ldots x_k', x_k'', b]$$

which has $(2k + 1)$ component intervals of two types.

(i) k intervals. $[x_i', x_i'']$, $i = 1, 2, \ldots k$ each of which contain a point x_i in its interior.

The total contribution to the oscillatory sum by these intervals is

$$= \sum_{i=1}^{k} (M_i - m_i)\,(b - a) \leq \sum_{i=1}^{k} (M - m)\,(b_i - a_i)$$

$$= (M - m) \sum_{i=1}^{k} (b_i - a_i)$$

$$< (M - m)\,\frac{\varepsilon}{2\,(M-m)} = \varepsilon/2.$$

(ii) $(k + 1)$ subintervals. $[a, x_1']$, $[x_1'', x_2']$, $[x_2'', x_3']$ … $[x_k'', b]$.

In the above subintervals, the function f has only a finite number of points of discontinuity. Hence, these exists a partition $P_r : r = 1, 2, \ldots, (k + 1)$ respectively of these subintervals such that the oscillatory sum is less than $\varepsilon/2\,(k + 1)$ for $r = 1, 2, \ldots, k + 1$.

Hence, the total contribution to the oscillatory sum by these subintervals is less than equal to

$\frac{\varepsilon}{2(k+1)}(k+1)$

i.e., $\leq \varepsilon/2.$

Therefore, for any partition P the total oscillatory sum

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, $f \in \mathbf{R}[a, b].$

• 4.6. ALGEBRA OF *R*-INTEGRABLE FUNCTIONS

Theorem 1. *If f is* **R**-*integrable on* $[a, b]$, *then* $|f|$ *is also* **R**-*integrable on* $[a, b]$.

Proof. Since the function f is **R**-integrable on $[a, b]$, therefore f is bounded on $[a, b]$

[$\because$ Every integrable function is bounded]

$\Rightarrow$ $|f(x)| \leq \lambda\ \forall\ x \in [a, b]$ for any positive number λ.

Also, since f is **R**-integrable on $[a, b]$, therefore there exists a partition P of $[a, b]$ such that for any positive number ε

$$U(P, f) - L(P, f) < \varepsilon. \qquad ...(1)$$

Let the upper and lower bounds of $|f|$ and f in $\delta_r = [x_{r-1}, x_r]$ be respectively given by M_r, m_r and M_r', m_r'.

Then for all y, z in $[x_{r-1}, x_r]$, we have

$$[|f(z)| - |f(y)|] \leq |f(z) - f(y)|$$

$$\Rightarrow \quad M_r - m_r \leq M_r' - m_r' \qquad \text{(By taking supremum)}$$

$$\Rightarrow \quad \sum_{r=1}^{n} (M_r - m_r)\,\delta_r \leq \sum_{r=1}^{n} (M_r' - m_r')\,\delta_r$$

$$\Rightarrow \quad \sum_{r=1}^{n} M_r\,\delta_r - \sum_{r=1}^{n} m_r\,\delta_r \leq \sum_{r=1}^{n} M_r'\,\delta_r - \sum_{r=1}^{n} m_r'\,\delta_r$$

$$\Rightarrow \quad \{U(P, |f|) - L(P, |f|)\} \leq U(P, f) - L(P, f)$$

$$\Rightarrow \quad U(P, |f|) - L(P, |f|) < \varepsilon \qquad \text{[using (1)]}$$

$\Rightarrow$ $|f|$ is **R**-integrable on (a, b).

Theorem 3. *If f_1 and f_2 are* **R**-*integrable functions on* $[a, b]$ *then* $f_1 \pm f_2$ *is also* **R**-*integrable on* $[a, b]$.

Proof. Let f_1, f_2 be two **R**-integrable functions on $[a, b]$.

Now f_1 is **R**-integrable on $[a, b]$

$\Rightarrow$ For given $\varepsilon > 0$ there exists a partition P_1 such that

$$U(P_1, f_1) - L(P_1, f_1) < \varepsilon/2. \qquad ...(1)$$

Also, f_2 is **R**-integrable

$\Rightarrow$for given $\varepsilon > 0$ there exists a partition P_2 such that

$$U(P_2, f_2) - L(P_2, f_2) < \varepsilon/2. \qquad ...(2)$$

Define the common refinement P of the partitions P_1 and P_2 such that

$$P = P_1 \cup P_2.$$

Clearly $P \in \mathbf{P}[a, b]$, where $\mathbf{P}[a, b]$ denotes the family of all partitions on $[a, b]$.

Consider

$$U(P, f_1 + f_2) - L(P, f_1 + f_2) \leq [\{U(P, f_1) - L(P, f_1)\} + \{U(P, f_2) - L(P, f_2)\}]$$

$$< \varepsilon/2 + \varepsilon/2 \qquad \text{[using (1) and (2)]}$$

$\Rightarrow$ $U(P, f_1 + f_2) - L(P, f_1 + f_2) < \varepsilon$

$\Rightarrow$ $f_1 + f_2$ is **R**-integrable on $[a, b]$.

Similarly we can show that $f_1 - f_2$ is **R**-integrable on $[a, b]$.

Theorem 3. *If f is* **R**-*integrable on* $[a, b]$, *then cf is also* **R**-*integrable on* $[a, b]$, *where* $c \in \mathbf{R}$.

Also $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Proof. Given that the function f is **R**-integrable on $[a, b]$ therefore, there exists a partition P on $[a, b]$, such that

$$U(P, f) - L(P, f) < \varepsilon. \quad ...(1)$$

Let $c \in \mathbf{R}$ be any constant, then we know that

$$(cf)(x) = cf(x).$$

Therefore, $U(P, cf) = cU(P, f)$ and $L(P, cf) = cL(P, f)$.

Now consider

$$U(P, cf) - L(P, cf) = c\,[U(P, f) - L(P, f)] < c\,\varepsilon$$

$$\Rightarrow \quad cf \in \mathbf{R}\,[a, b]$$

Also $$U(P, cf) < \int_a^b cf(x)\,dx + \varepsilon$$

and $$cU(P, f) < \int_a^b cf(x)\,dx + \varepsilon.$$

Now using (1), we get

$$cU(P, f) = U(P, cf) < \int_a^b cf(x)\,dx + \varepsilon$$

$$\Rightarrow \quad c\int_a^b f(x)\,dx \geq \int_a^b cf(x)\,dx. \quad ...(2)$$

Replacing f by $-f$ in (2), we get

$$c\int_a^b -f(x)\,dx \geq \int_a^b -cf(x)\,dx$$

$$\Rightarrow \quad c\int_a^b f(x)\,dx \leq \int_a^b cf(x)\,dx. \quad ...(3)$$

From (2) and (3), we conclude that

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Theorem 4. *If the function f is* **R**-*integrable and if M and m the supremum and infimum of f on $[a, b]$, then*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \text{ if } b \geq a$$

and $$m(b-a) \geq \int_b^a f(x)\,dx \geq M(b-a) \text{ if } b \leq a.$$

Proof. Let $\mathbf{P}[a, b]$ denotes the family of all partitions on $[a, b]$. If $b > a$, then for all $P \in \mathbf{P}[a, b]$, we have

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

$$\Rightarrow \quad m(b-a) \leq L(P, f) \leq M(b-a)$$

$$\Rightarrow \quad m(b-a) \leq \underline{\int_a^b} f(x)\,dx \leq M(b-a)$$

$$\Rightarrow \quad m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \qquad \left[\because \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \text{ for } f \in \mathbf{R}\,[a, b]\right]$$

If $b < a$, then in a similar way, we may get

$$m(a-b) \leq \int_b^a f(x)\,dx \leq M(a-b)$$

$$\Rightarrow \quad -m(b-a) \leq -\int_a^b f(x)\,dx \leq -M(b-a)$$

$$\Rightarrow \quad m(b-a) \geq \int_b^a f(x)\,dx \geq M(b-a).$$

Theorem 5. *If the function $f(x)$ is bounded and* **R**-*integrable over* $[a, b]$ *and* $f(x) \geq 0 \quad \forall\, x \in [a, b]$, *then* $\int_a^b f(x)\, dx \geq 0$.

Proof. Let M and m be the supremum and infimum of f on $[a, b]$. Then by above theorem if $b \geq a$, we have

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a). \qquad \text{...(1)}$$

Here, it is given that $f(x) \geq 0 \quad \forall\, x \; [a, b]$.

Therefore $\quad m \geq 0$.

Also $\quad b \geq a \Rightarrow b - a \geq 0$.

Hence, from (1), we conclude that $\int_a^b f(x)\, dx \geq 0$.

Theorem 6. (First Mean Value Theorem). *If the function f is* **R**-*integrable over* $[a, b]$ *and M, m be suprenum, infimum respectively of f on* $[a, b]$, *then there exists a number K,* $(m \leq K \leq M)$ *such that*

$$\int_a^b f(x)\, dx = k(b-a).$$

Also, if the function f is continuous on $[a, b]$, *then there exists $c \in [a, b]$ such that*

$$\int_a^b f(x)\, dx = (b-a)\, f(c).$$

Proof. We know that (From Theorem 8)

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a), \text{ if } b \geq a$$

and

$$m(b-a) \geq \int_b^a f(x)\, dx \geq M(b-a), \text{ if } b \leq a.$$

If $m \leq k \leq M$, then we conclude that

$$\int_a^b f(x)\, dx = k(b-a). \qquad \text{...(1)}$$

Also, if the function f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that $f(c) = k$, where $m \leq k \leq M$.

Hence, from (1), we conclude that

$$\int_a^b f(x)\, dx = (b-a)\, f(c).$$

Theorem 7. *If f and g are* **R**-*integrable over* $[a, b]$, *then fg is also integrable over* $[a, b]$.

Proof. Since f and g both are **R**-integrable over $[a, b]$, therefore f and g both are bounded on $[a, b]$

$\Rightarrow \;\exists\, M > 0$ such that $|f(x)| \leq M$ and $|g(x)| \leq M, \quad \forall\, x \in [a, b]$

Consider $\;|(fg)(x)| = |f(x) \cdot g(x)| \quad \forall\, x \in [a, b]$

$$\leq M^2 \quad \forall\, x \in [a, b]. \qquad \text{...(1)}$$

$\Rightarrow \; fg$ is bounded on $[a, b]$.

Now, let $\varepsilon > 0$ be given.

Since $f \in$ **R** (a, b) therefore, there exists a partition P_1 of $[a, b]$ such that

$$U(P_1, f) - L(P_1, f) < \varepsilon/2M. \qquad \text{...(2)}$$

Similarly $g \in$ **R** (a, b), therefore, there exsits a partition P_2 of $[a, b]$ such that

$$U(p_2, g) - L(P_2, g) < \varepsilon/2M. \qquad \text{...(3)}$$

Let $P = P_1 \cup P_2$ be a refinement of P_1 and P_2, then we have

$$\left.\begin{array}{r} U(P, f) - L(P, f) < \varepsilon/2M \\ \text{and} \quad U(P, g) - L[P, g] < \varepsilon/2M \end{array}\right]. \qquad \text{...(4)}$$

Let $m_r, M_r, m_r', M_r', m_r'', M_r''$ be the infimum and supremum of f, g and $f \cdot g$ respectively over the subinterval $I_r = [x_{r-1}, x_r]$. Then for all $x, y \in I_r$ we have

$$\begin{aligned}|(fg)(x) - (fg)(y)| &= |f(x) \cdot g(x) - f(y) \cdot g(y)| \\ &= |f(x) \cdot g(x) - f(y)\, g(x) + f(y)\, g(x) - f(y)\, g(y)| \\ &= |g(x)\, [f(x) - f(y)] + f(y)\, [g(x) - g(y)]| \\ &\le |g(x)|\, |f(x) - f(y)| + |f(y)|\, |g(x) - g(y)| \\ &\le M\, |f(x) - f(y)| + M\, |g(x) - g(y)|. \qquad \ldots(5)\end{aligned}$$

Now, $\qquad |f(x) - f(y)| \le M_r - m_r \qquad \ldots(6)$

and $\qquad |g(x) - g(y)| \le M_r' - m_r'. \qquad \ldots(7)$

$\therefore \qquad M_r'' - m_r'' < M(M_r - m_r) + M(M_r' - m_r'). \qquad \ldots(8)$

Multiplying both sides of (8) by δ_r and adding on respective sides, we get

$$\begin{aligned}U(P, fg) - L(P, fg) &\le M\, [U(P, f) - L(P, f)] + M\, [U(P, g) - L(P, g)] \\ &< M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M} = \varepsilon.\end{aligned}$$

Hence, fg is **R**-integrable.

Theorem 8. *If f and g are two **R**-integrable function on $[a, b]$ and $|g(x)| \le k \;\; \forall\, x \in [a, b]$ where k is a positive number then the quotient function f/g is also **R**-integrable on $[a, b]$.*

Proof. Since f and g both are **R**-integrable on $[a, b]$, therefore, they are bounded on $[a, b]$. Also, we know that the quotient of two bounded function is again bounded, therefore f/g is also bounded on $[a, b]$.

Let $\varepsilon > 0$ be given. Since $f \in$ **R** $[a, b]$, therefore, there exists a partition P_1 of $[a, b]$ such that

$$U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2m} k^2. \qquad \ldots(1)$$

Similarly $g \in$ **R** $[a, b]$, therefore, these exists a partition P_2 of $[a, b]$ such that

$$U[P_2, g] - L[P_2, g] < \frac{\varepsilon}{2M} k^2. \qquad \ldots(2)$$

Let $P = P_1 \cup P_2$ be a refinement of P_1 and P_2, then from (1) and (2), we have

$$U[P, f] - L[P, f] < \frac{\varepsilon}{2M} k^2 \qquad \ldots(3)$$

and

$$U[P, g] - L[P, g] < \frac{\varepsilon}{2m} k^2. \qquad \ldots(4)$$

Now, let $m_r, M_r; m_r', M_r'; m_r'', M_r''$ be the supremum and infimum of f, g and f/g respectivley over the subinterval $I_r = [x_{r-1}, x_r]$. Then for all $x, y \in I_r$ we have

$$\begin{aligned}\left|\frac{f}{g}(x) - \frac{f}{g}(y)\right| &= \left|\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right| = \frac{|f(x)\, g(y) - f(y)\, g(x)|}{|g(x)\, g(y)|} \\ &= \frac{|f(x)\, g(y) - f(y)\, g(y) + f(y)\, g(y) - f(y)\, g(x)|}{|g(x)\, g(y)|} \\ &= \frac{|[f(x) - f(y)]\, g(y) + f(y)\, [g(y) - g(x)]|}{|g(x)\, g(y)|} \\ &\le \frac{|g(y)|\, |f(x) - f(y)|}{|g(x)|\, |g(y)|} + \frac{|f(y)|\, |g(x) - g(y)|}{|g(x)|\, |g(y)|} \\ &\le \frac{M}{k^2} |f(x) - f(y)| + \frac{M}{k^2} |g(x) - g(y)|. \qquad \ldots(5)\end{aligned}$$

Now m_r and M_r are the infimum and supremum of f respectively over I_r. Therefore,

$$|f(x) - f(y)| \le M_r - m_r \quad \forall\, x, y \in [a, b]. \qquad \ldots(6)$$

Similarly $\qquad |g(x) - g(y)| \le M_r' - m_r' \quad \forall\, x, y \in [a, b] \qquad \ldots(7)$

which implies

$$\left|\frac{f}{g}(x) - \frac{f}{g}(y)\right| \le \frac{M}{k^2}(M_r - m_r) + \frac{M}{k^2}(M_r' - m_r') \qquad \ldots(8)$$

$$\Rightarrow \qquad M_r'' - m_r'' \le \frac{M}{k^2}(M_r - m_r) + \frac{M}{k^2}(M_r' - m_r'). \qquad \ldots(9)$$

Multiplying (9) by δ_r and adding on the respective sides, we get

$$U[P, f/g] - L[P, f/g] \leq \frac{M}{k^2}[U(P, f) - L(P, f)] + \frac{M}{k^2}[U(P, g) - L(P, g)]$$

$$\leq \frac{M}{k^2}\frac{\varepsilon . k^2}{2M} + \frac{M}{k^2} \cdot \frac{\varepsilon k^2}{2M} = \varepsilon.$$

Hence, $\frac{f}{g}$ is **R**-integrable over $[a, b]$.

Some Important Definitions :

Primitive. *A function $F(x)$ defined on $[a, b]$ called a primitive of a function $f(x)$, if the function $F(x)$ has $f(x)$ as its derivative at each $x \in [a, b]$*

i.e., $$F'(x) = f(x) \quad \forall\, x \in [a, b].$$

Integral Function. *Let $f(x)$ be a* **R***-integrable function on $[a, b]$. Then a function $F(x)$ is called the integral function of the function $f(x)$ if*

$$F(x) = \int_a^b f(t)\, dt, \quad \forall\, x \in [a, b].$$

Theorem 9. *Let $f \in$ **R** $[a, b]$, then the integral function F of f given by*

$$F(x) = \int_a^x f(t)\, dt, \; a \leq x \leq b$$

is continuous on $[a, b]$.

Proof. Let $f \in$ **R** $[a, b]$ is **R**-integrable over $[a, b]$, then obviously it is bounded on $[a, b]$. Therefore, there exists a positive number M such that

$$|f(t)| \leq M \quad \forall\, t \in [a, b].$$

Let $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$. Then, we have

$$|F(x_2) - F(x_1)| = \left|\int_a^{x_2} f(t)\, dt - \int_a^{x_1} f(t)\, dt\right|$$

$$= \left|\int_a^{x_2} f(t)\, dt + \int_{x_1}^{a} f(t)\, dt\right|$$

$$= \left|\int_{x_1}^{x_2} f(t)\, dt\right| \leq M \left|\int_{x_1}^{x_2} dt\right| = M\,|(x_2 - x_1)|.$$

Let $|x_2 - x_1| < \varepsilon/M$ for a given positive number ε. Then, we have

$$|F(x_2) - F(x_1)| < M . \varepsilon/M$$

$\Rightarrow$ $$|F(x_2)| - F(x_1)| < \varepsilon$$

whenever $$|x_2 - x_1| < \delta \quad \forall\, x_1, x_2 \in [a, b]$$

where $$\delta = \frac{\varepsilon}{M}.$$

$\Rightarrow$ F is uniformly continuous on $[a, b]$. Hence it is continuous on $[a, b]$

[$\because$ Every uniformly continuous function is continuous]

Theorem 10. *Let f be a continuous function on $[a, b]$ and let*

$$F(x) = \int_a^x f(t)\, dt; \quad \forall\, x \in [a, b].$$

Then $$F'(x) = f(x); \quad \forall\, x \in [a, b].$$

Proof. Let $x \in [a, b]$. Then choose $h \neq 0$ such that $x + h \in [a, b]$.

Consider $$F(x+h) - F(x) = \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt$$

$$= \int_a^{x+h} f(t)\, dt + \int_a^x f(t)\, dt = \int_a^{x+h} f(t)\, dt \qquad \text{...(1)}$$

Since f is continuous on $[a, b]$, therefore, there exists a number $c \in [x, x+h]$ such that

$$\int_a^{x+h} f(t)\, dt = hf(c). \qquad ...(2)$$

Clearly $c \to x$ as $h \to 0$.

From (1) and (2), we conclude that

$$F(x+h) - F(x) = hf(c)$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} f(c)$$

$$\Rightarrow \quad F'(x) = f(x).$$

Hence, we have

$$F'(x) = f(x) \quad \forall\, x \in [a, b].$$

• 4.7. FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Theorem 15. *Let f be a* **R**-*integrable function on* $[a, b]$ *ant F be a differentiable primitive function on* $[a, b]$ *such that* $F'(x) = f(x)$, $a \le x \le b$, *then*

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Proof. Let f be continuous function on $[a, b]$.

By definition of primitive function, we have

$$F'(x) = f(x); \quad \forall\, x \in [a, b].$$

Also, f is **R**-integrable function on $[a, b]$.

$\Rightarrow$ $F'(x)$ is **R**-integrable function on $[a, b]$.

i.e., for a given positive number ε there exists a partition P of $[a, b]$ such that

$$\left| \sum_{r=1}^{n} F'(t_r)(x_r - x_{r-1}) - \int_a^b F'(x)\, dx \right| < \varepsilon \qquad ...(1)$$

where $t_r \in (x_{r-1}, x_r)$.

By Lagrange's mean value theorem of differential calculus, we find that there exists $t_r \in [x_{r-1}, x_r]$ such that

$$F(x_r) - F(x_{r-1}) = (x_r - x_{r-1})\, F'(t_r)$$

$$\Rightarrow \quad \sum_{r=1}^{n} [(x_r - x_{r-1})\, F'(t_r)] = \sum_{r=1}^{n} [F(x_r) - F(x_{r-1})] = F(b) - F(a).$$

Put this value in (1), we get

$$\left| F(b) - F(a) - \int_a^b F'(x)\, dx \right| < \varepsilon$$

which gives $$F(b) - F(a) = \int_a^b F'(x)\, dx = \int_a^b f(x)\, dx \quad [\because\ F'(x) = f(x)]$$

$$\Rightarrow \quad \int_a^b f(x)\, dx = F(b) - F(a).$$

SOLVED EXAMPLES

Example 1. *Find* $\int_1^2 x^3\, dx$, *using fundamental theorem of integral calculus.*

Solution. Here, we have

$$f(x) = x^3, \quad 1 \le x \le 2$$

Clearly f is continuous on $[1, 2]$

Now, if $$\phi(x) = \frac{x^4}{4} \quad (1 \le x \le 2)$$

Then $$\phi'(x) = x^3 = f(x).$$

Therefore, by fundamental theorem of integral calculus; we have

$$\int_1^2 x^3\,dx = \phi(2) - \phi(1) = \frac{2^4}{4} - \frac{1^4}{4} = \frac{15}{4}.$$

Example 2. *Let f be the function defined on* [0, 1] *by*

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is irrational} \\ 1 & \text{when } x \text{ is rational} \end{cases}$$

Show that f is bounded but not **R**-*integrable.*

Solution. By definition of $f(x)$, we have

$$0 \le f(x) \le 1 \quad \forall\, x \in [0, 1]$$

$\therefore$ $f(x)$ is bounded on $[a, b]$.

Define a partition $P = \{a = x_0, x_1, x_2, \ldots, x_r, \ldots, x_{n-1}, x_n = b\}$ of $[a, b]$.

Let $I_r = [x_{r-1}, x_r]$ be any subinterval of P, with length $\delta_r\,(= x_r - x_{r-1})$. Let M_r and m_r be respectively the supremum and infimum of f in I_r. Then, we have

$$M_r = 1 \text{ and } m_r = 0$$

Now, $$L(P, f) = \sum_{r=1}^{n} m_r\,\delta_r = \sum_{r=1}^{n} 0\,.\,\delta_r = 0$$

and $$U(P, f) = \sum_{r=1}^{n} M_r\,\delta_r = \sum_{r=1}^{n} 1\,.\,\delta_r = \sum_{r=1}^{n} \delta_r = [\delta_1 + \delta_2 + \ldots + \delta_n]$$

$$= [x_1 - x_0] + [x_2 - x_1] + \ldots + [x_n - x_{n-1}] = x_n - x_0 = b - a$$

$\Rightarrow$ $$\underline{\int_a^b} f = \sup\{L(P, f)\} = 0 \qquad \ldots(1)$$

and $$\overline{\int_a^b} f = \inf\{U(P, f)\} = b - a. \qquad \ldots(2)$$

From (1) and (2), we conclude that

$$\underline{\int_a^b} f \neq \overline{\int_a^b} f.$$

Hence, f is not **R**-integrable over $[a, b]$.

Example 3. *If a function f is defined on* $[0, a], a > 0$ *by* $f(x) = x^3$, *then show that f is* **R**-*integrable on* $[0, a]$ *and*

$$\int_0^a f(x)\,dx = \frac{a^4}{4}.$$

Solution. Consider a partition $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots, \frac{(n-1)a}{n}, \frac{na}{n} = a\right\}$ of $[0, a]$.

Let I_r be the r^{th} subinterval of P such that

$$I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$$

with length $\delta_r = \frac{a}{n}$, $r = 1, 2, \ldots, n$.

Now, let M_r and m_r be respectively the supremum and infimum of f in I_r. Also, since $f(x)$ is an increasing function in $[0, a]$, therefore,

$$m_r = \frac{(r-1)^3 a^3}{n^3} \text{ and } M_r = \frac{r^3 a^3}{n^3}, \quad r = 1, 2, \ldots, n.$$

Now $$L(P, f) = \sum_{r=1}^{n} m_r\,\delta_r = \sum_{r=1}^{n} \left[\frac{(r-1)^3 a^3}{n^3}\,\frac{a}{n}\right]$$

$$= \frac{a^4}{n^4} \sum_{r=1}^{n} (r-1)^3 = \frac{a^4}{n^4}[1^3 + 2^3 + \ldots + (n-1)^3]$$

$$= \frac{a^4}{n^4}\left[\frac{(n-1)\,n}{2}\right]^2 = \frac{a^4}{4}\left[1 - \frac{1}{n}\right]^2$$

$$\Rightarrow \quad \underline{\int_0^a} f = \lim_{n\to\infty} L(P,f) = \lim_{n\to\infty} \frac{a^4}{4}\left(1-\frac{1}{n}\right)^2 = \frac{a^4}{4}.$$

Also, $$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n}\left[\frac{r^3a^3}{n^3}\cdot\frac{a}{n}\right]$$

$$= \frac{a^4}{n^4}\sum_{r=1}^{n} r^3 = \frac{a^4}{n^4}[1^3+2^3+\ldots+n^3]$$

$$= \frac{a^4}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = \frac{a^4}{4}\left[1+\frac{1}{n}\right]^2$$

$$\Rightarrow \quad \overline{\int_0^a} f(x)\,dx = \lim_{n\to\infty} U(P,f) = \lim_{n\to\infty}\frac{a^4}{4}\left(1+\frac{1}{n}\right)^2 = \frac{a^4}{4}.$$

Clearly $$\underline{\int_0^a} f = \overline{\int_0^a} f = \frac{a^4}{4}.$$

Hence, f is **R**-integrable over $[0, a]$ and $\int_0^a f(x)\,.\,x = \frac{a^4}{4}$.

Example 4. *Verify first mean value theorem for the function $f(x) = \sin x$ and $g(x) = e^x$ for $x \in [0, \pi]$.*

Solution. Clearly, both the function $f(x)$ and $g(x)$ are continuous on $[0, \pi]$ and $g(x) > 0,\ \forall\, x \in [0, \pi/2]$ [$\because g(x) = e^x$ is an increasing function in $[0, \pi/2]$].

Then, by first mean value theorem

$$\int_0^\pi f(x)\,g(x)\,dx = f(c)\int_0^\pi g(x)\,dx \qquad 0 \le c \le \pi$$

$$\Rightarrow \quad \int_0^\pi \sin x\,.\,e^x\,dx = \sin c\int_0^\pi e^x\,dx \qquad 0 \le c \le \pi$$

$$= (e^\pi - 1)\sin c \qquad 0 \le c \le \pi. \qquad \ldots(1)$$

Now $$\int_0^\pi e^x \sin x\,dx = \left[\frac{1}{\sqrt{2}}e^x \sin\left(x-\frac{\pi}{4}\right)\right]_0^\pi$$

$$= \frac{1}{\sqrt{2}}e^\pi \sin\frac{3\pi}{4} - \frac{1}{\sqrt{2}}e^0\sin\left(0-\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}e^\pi\,.\,\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}(e^\pi + 1). \qquad \ldots(2)$$

From (1) and (2), we conclude that

$$(e^\pi - 1)\sin c = \frac{1}{2}(e^\pi + 1) \quad 0 \le c \le \pi \qquad \ldots(3)$$

$$\Rightarrow \quad \sin c = \frac{1}{2}\left[\frac{e^\pi+1}{e^\pi-1}\right].$$

Now $0 < \frac{1}{2}\left[\frac{e^\pi+1}{e^\pi-1}\right] < 1$, therefore, there exists $c \in \left[0, \frac{\pi}{2}\right] \subset [0, \pi]$ satisfying (3).

Hence, the first mean value theorem is verified.

Example 5. *Using first mean value theorem, show that*

$$\frac{1}{3\sqrt{2}} < \int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx < \frac{1}{3}.$$

Solution. Here, we have

$$f(x) = \frac{1}{\sqrt{1+x}} \text{ and } g(x) = x^2.$$

Clearly $f(x)$ is continuous on [0, 1] and $g(x) > 0$ on [0, 1]. Also, $g(x)$ is continuous on [0, 1]. Therefore, by first mean value theorem, we have

$$\int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx = \frac{1}{\sqrt{1+c}}\int_0^1 x^2\,dx, \qquad 0 \le c \le 1$$

$$= \frac{1}{\sqrt{1+c}}\left[\frac{x^3}{3}\right]_0^1, \qquad 0 \le c \le 1$$

$$= \frac{1}{3\sqrt{1+c}}, \qquad 0 \le c \le 1$$

Now $\quad 0 \le c \le 1 \Rightarrow 1 < (1+c) < 2$

$\Rightarrow$ $$1 > \frac{1}{(1+c)} > \frac{1}{2} \Rightarrow \frac{1}{\sqrt{2}} < \frac{1}{\sqrt{1+c}} < 1$$

Therefore, $$\frac{1}{\sqrt{2}} < 3\int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx < 1$$

or $$\frac{1}{3\sqrt{2}} < \int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx < \frac{1}{3}.$$

• SUMMARY

- Lower Riemann Sum = $L(P, f) = \sum_{r=1}^{n} m_r\,\delta x_r$.
- Upper Riemann Sum = $U(P, f) = \sum_{r=1}^{n} M_r\,\delta x_r$
- $L(P, f) \le U(P), f \ \forall P$
- **Upper Riemann integral :**

$$\int_a^{\overline{b}} f\,dx = \inf_P \{U(P, f)\} = \lim_{\|P\|\to 0} U(P, f) = \lim_{n\to\infty} \sum_{r=1}^{n} M_r\,\delta x_r$$

- **Lower Riemann integral :**

$$\int_{\underline{a}}^{b} f\,dx = \sup_P \{L(P, f)\} = \lim_{n\to\infty} \sum_{r=1}^{n} m_r\,\delta x_r$$

- If $\int_{\underline{a}}^{b} f\,dx = \int_a^{\overline{b}} f\,dx$, then f is R-integrable.
- $m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$.
- **Darboux Theorem :**

$$U(P, f) < \int_a^{\overline{b}} f\,dx + \varepsilon$$

$$L(P, f) > \int_a^{b} f\,dx - \varepsilon$$

- $0 \le U(P, f) - L(P, f) < \varepsilon \ \forall \|P\| < \delta \Leftrightarrow f$ is R-integrable.
- Every continuous function is R-integrable.
- Every monotonic function is R-integrable.
- **First Mean Value Theorem :** $\int_a^b f(x)\,dx = f(c)(b-a), \quad m \le f(c) \le M.$
- **Fundamental Theorem integral calculus :** Let f be a R-integrable function on $[a, b]$ and F be a differentiable primitive function on $[a, b]$ such that $F'(x) = f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

• STUDENT ACTIVITY

1. Let $f(x) = x$, $0 \le x \le 1$ and let $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ be a partion of [0, 1]. Find $U(P, f)$ and $L(P, f)$.

2. Prove that every continuous function is R-integrable.

• TEST YOURSELF

1. Let $f(x) = x\ (0 \le x \le 1)$. Let P be the partition $\left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ of [0, 1], compute $U(P, f)$ and $L(P, f)$.
2. Show by definition that $\int_0^1 x^4\,dx = \frac{1}{5}$.
3. Find the value of upper and lower integrals for the function f defined on [0, 2] as follows

$$f(x) = \begin{cases} x^2, & \text{when } x \text{ is rational} \\ x^3, & \text{when } x \text{ is inrational} \end{cases}$$

ANSWERS

1. 2/3, 1/3 **3.** 31/12, 49/12

FILL IN THE BLANKS :

1. Partition of a set is also called ....
2. The value of $x_r - x_{r-1}$ is called .... of the internal $[x_{r-1}, x_r]$
3. Riemann sum is also known as .... sum.
4. The supremum of the set of the lower sums is called the .... integral.
5. The infimum of the set of upper sums is called the .... integral.
6. In computing the integral $\int_a^b f(x)\,dx$, the internal $[a, b]$ is known as .... of the integration.

TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. Every bounded function is **R**-integrable. *(T/F)*
2. Every **R**-integrable function is bounded. *(T/F)*
3. Every monotone function is not necessarily **R**-integrable. *(T/F)*

4. A bounded function f is **R**-integrable in $[a, b]$ if the set of its point of discontinuity is finite. *(T/F)*

MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. If P_1 and P_2 be any two partitions of $[a, b]$, then :
   (a) $U(P_1, f) \geq L(P_2, f)$ (b) $U(P_1, f) = L(P_2, f)$
   (c) $U(P_1, f) \leq L(P_2, f)$ (d) None of these.
2. The value of $\lim\limits_{\|P\| \to 0} L(P, f)$ is :
   (a) $\underline{\int_a^b} f$ (b) $\overline{\int_a^b} f$ (c) $\int_a^b f$ (d) None of these.
3. The value of $\lim\limits_{\|P\| \to 0} U(P, f)$ is :
   (a) $\underline{\int_a^b} f$ (b) $\overline{\int_0^1} f$ (c) $\int_a^b f$ (d) None of these.

ANSWERS

Fill in the Blanks :

1. dissection or net **2.** length **3.** Darboux **4.** lower
5. upper **6.** Range

True or False :

1. F **2.** T **3.** F **4.** T

Multiple Choice Questions :

1. (a) **2.** (a) **3.** (b)

❑❑❑

UNIT

5

CONVERGENCE OF IMPROPER INTEGRALS

STRUCTURE

- Improper Integrals
- Kinds of Improper Integrals
- Convergence of Imroper Integrals
- Convergence Tests : First Kind
- Convergence Tests : Second Kind
- Improper Integrals of Second Kind
  - ❑ Summary
  - ❑ Student Activity
  - ❑ Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What are improper integrals ?
- How to check whether the given integral is convergent or divergent ?

• 5.1. IMPROPER INTEGRALS

Definition. The definite integral $\int_a^b f(x)\,dx$ is called Improper (or Infinite) integral if either any one or both limits are infinite and function $f(x)$ is bounded over the interval or neither the intervals $[a, b]$ is finite nor $f(x)$ is bounded over it.

• 5.2. KINDS OF IMPROPER INTEGRALS

By the definition of Improper Integrals we can divide or categorized it into following three kind.

(1) First kind of improper integrals. First kind of improper integral is in which integrand $f(x)$ is continuous but limits are infinite.

Definition. *A definite integral* $\int_a^b f(x)\,dx$ *in which limits are infinite i.e.,* $b = \infty$, $a = \infty$ *and integrand is continuous is called first kind of improper integrals.*

This first kind of improper integral can be classified into following three categories :

(a) Upper Limit Infinite :

For Example. $\int_0^\infty \frac{1}{1+x^2}\,dx$, here it is first kind of improper integral in which upper limit is infinite and $(1/1+x^2)$ is bounded.

(b) Lower limit infinite :

For Example. $\int_{-\infty}^0 e^x\,dx$.

Here, the lower limit of function is infinite.

(c) Both limit infinite :

For Example. $\int_{-\infty}^\infty \frac{dx}{1+x^2}$.

It is the example in which both upper and lower limits are infinite.

(i) Consider $\int_a^{\infty} f(x)\,dx$. Here $f(x)$ is continuous in $[a, \infty[$. There exists a definite number $b > a$ such that $\int_a^b f(x)\,dx$ as $b \to \infty$. This definite integral becomes the improper integral

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

If limit is finite, then improper integral $\int_a^{\infty} f(x)\,dx$ is convergent, otherwise divergent.

(ii) Consider $\int_{-\infty}^{\infty} f(x)\,dx$, then there exist $a < b$, such that $\int_{-a}^{b} f(x)\,dx$ as $a \to -\infty$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$

If limit is finite, the improper integral is convergent otherwise divergent.

(iii) Consider $\int_{-\infty}^{+\infty} f(x)\,dx$. It is the combination of above 2-procedures so take a constant 'a' between $-\infty$ to $+\infty$ and expressed in the integral in the form of

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{+\infty} f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx + \lim_{b \to \infty} \int_a^b f(x)\,dx$$

If both the limits are finite then $\int_{-\infty}^{\infty} f(x)\,dx$ is convergent otherwise divergent i.e. If anyone or both limits are infinite.

(2) Second kind of improper integral. *Second kind of improper integral is in which limits are finite but integrand is infinite.* The point at which the integrand is infinite is called a singular point.

Second kind of improper integral is classified into following four categories :

(i) Singular point at right end 'b'. If $x = b$ is only singular point of $f(x)$ then there exists $\varepsilon > 0$ (small positive number) such that

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx.$$

Here, $f(x)$ is continuous in $[a, b - \varepsilon]$.

(ii) Singular point at left end 'a'. If $f(x) \to \infty$ as $x \to a$ is only singular point of $f(x)$ then there exists a small positive number $\varepsilon > 0$ such that

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)\,dx.$$

Here, $f(x)$ is continuous in $[a + \varepsilon, b]$.

If $\int f(x)\,dx = F(x) + c$ then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} |f(b) - f(a + \varepsilon)|.$$

So, we can say *the convergence or divergence depend on the limit of $\lim_{\varepsilon \to 0} f(a + \varepsilon)$ respectively.*

(iii) Singular point at 'c'. If $f(x) \to \infty$ as $x \to c$ the singular point of $f(x)$ where $a < c < b$, then $\int_a^b f(x)\,dx$ decomposed into following form :

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon' \to 0} \int_{c+\varepsilon'}^b f(x)\,dx.$$

If one or both integrals in R.H.S. be convergent, then $\int_a^b f(x)\,dx$, $a < c < b$ *is convergent, otherwise divergent.*

(iv) Singular point at both *a* and *b*. If 'a' and 'b' are only singular point of $f(x)$ then there exists c such that $a < c < b$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^c f(x)\,dx + \lim_{\varepsilon' \to 0} \int_c^{b-\varepsilon'} f(x)\,dx.$$

If each integral is convergent then the $\int_a^b f(x)\,dx$ *is convergent.*

(3) Third kind of improper integral. Third kind of improper integral is in which
(i) infinite limits
(ii) infinite integrand.

"*It is the combination of both first kind and second kind of improper integral.*"

Let $\int_a^\infty f(x)\,dx$ is improper integral of third kind when $f(x)$ has a singular point at $x = c$, where $a < c < d$ and $c < d < \infty$ then

$$\int_a^\infty f(x)\,dx = \underset{\text{(I)}}{\int_a^d f(x)\,dx} + \underset{\text{(II)}}{\int_d^\infty f(x)\,dx}. \quad \ldots(1)$$

Here, $\int_a^\infty f(x)\,dx$ is convergent if both integrals are convergent otherwise divergent.

• 5.3. CONVERGENCE OF IMPROPER INTEGRAL

Definition. *The integral* $\int_a^\infty f(x)\,dx$ *is said to converge to the value I, if for any arbitrary chosen positive number* ε, *however small but not zero, there exists a positive number N such that*

$$\left| \int_a^b f(x)\,dx - I \right| < \varepsilon; \textit{ for all values of } b \geq N.$$

If the integral $f(x)$ has a finite limit then improper integral called convergent and if having no finite limit *i.e.*, limits are $+\infty, -\infty$ then it is said to be divergent and when having neither finite value, 0, $+\infty$ nor $-\infty$, the improper integrals is said to be oscillatory.

SOLVED EXAMPLES

Example 1. *Discuss the convergence of the following integral* $\int_1^\infty \frac{dx}{\sqrt{x}}$ *by evaluating them.*

Solution. Since we have

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{x\to\infty}\int_1^x \frac{dx}{\sqrt{x}} = \lim_{x\to\infty}\int_1^x x^{-1/2}\,dx$$

$$= \lim_{x\to\infty}\left(\frac{x^{1/2}}{\frac{1}{2}}\right)_1^x = \lim_{x\to\infty}\left(2x^{1/2}\right)_1^x = \lim_{x\to\infty}(2\sqrt{x}-2) = \infty.$$

$\Rightarrow$ the limit does not exists finitely

$\Rightarrow$the given integral is divergent.

Example 2. *Discuss the convergence of the integral* $\int_1^\infty \frac{dx}{x^{3/2}}$ *by evaluating them.*

Solution. Since we have

$$\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{x\to\infty}\int_1^x x^{-3/2}\,dx = \lim_{x\to\infty}\left[\frac{x^{-1/2}}{-\frac{1}{2}}\right]_1^x$$

$$= \lim_{x\to\infty}\left[-\frac{2}{\sqrt{x}}\right]_1^x = \lim_{x\to\infty}\left[-\frac{2}{\sqrt{x}}+2\right]$$

$$= \frac{-2}{\infty}+2 = 2$$

$\Rightarrow$ the integral exist and finite

$\Rightarrow$ given integral is convergent.

Example 3. *Discuss the convergence of the integral* $\int_0^1 \frac{dx}{\sqrt{1-x}}$ *of evaluating them.*

Solution. Here given integral is $\int_0^1 \frac{dx}{\sqrt{1-x}}$.

It is not bounded at limit $x = 1$.

So
$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{\varepsilon\to 0}\left[-2\sqrt{1-x}\right]_0^{1-\varepsilon} = \lim_{\varepsilon\to 0}[-2\sqrt{\varepsilon}+2]$$

$$= 2$$

which is a finite number.

$\Rightarrow$the given integral is convergent.

Example 4. *Discuss the convergence of the given integral by evaluating* $\int_{-1}^1 \frac{dx}{x^2}$.

Solution. Given integrand is $\int_{-1}^1 \frac{dx}{x^2}$.

It becomes infinite at $x = 0$, $-1 < 0 < 1$.

So
$$\int_{-1}^1 \frac{dx}{x^2} = \lim_{\varepsilon_1\to 0}\int_{-1}^{-\varepsilon_1} \frac{dx}{x^2} + \lim_{\varepsilon_2\to 0}\int_{\varepsilon_2}^1 \frac{dx}{x^2}$$

$$= \lim_{\varepsilon_1\to 0}\left[-\frac{1}{x}\right]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2\to 0}\left[-\frac{1}{x}\right]_{\varepsilon_2}^1$$

$$= \underbrace{\lim_{\varepsilon_1\to 0}\left[\frac{1}{\varepsilon}-1\right]}_{\text{I}} + \underbrace{\lim_{\varepsilon_2\to 0}\left[-1+\frac{1}{\varepsilon_2}\right]}_{\text{II}}$$

Since (I) and (II) do not exist finitely $\Rightarrow$ limit does not exist finitely

Hence given integral is divergent.

Example 5. *If* $\int_0^{2a} \frac{dx}{(x-a)^2}$ *is an integrand then discuses the convergence of given function.*

Solution. The given integral $\int_0^{2a} \frac{dx}{(x-a)^2}$ becomes infinite at $x = a$ and $0 < a < 2a$.

So $$\int_0^{2a} \frac{dx}{(x-a)^2} = \int_0^{a} \frac{dx}{(x-a)^2} + \int_a^{2a} \frac{dx}{(x-a)^2}$$

$$= \lim_{\varepsilon_1 \to 0} \int_0^{a-\varepsilon_1} \frac{dx}{(x-a)^2} + \lim_{\varepsilon_2 \to 0} \int_{a+\varepsilon_2}^{2a} \frac{dx}{(x-a)^2}$$

$$= \lim_{\varepsilon_1 \to 0} \left[\frac{-1}{(x-a)}\right]_0^{a-\varepsilon_1} + \lim_{\varepsilon_2 \to 0} \left[\frac{-1}{(x-a)}\right]_{a+\varepsilon_2}^{2a}$$

$$= \lim_{\varepsilon_1 \to 0} \left[\frac{1}{\varepsilon_1} + \frac{1}{a}\right]_0^{a-\varepsilon_1} + \lim_{\varepsilon_2 \to 0} \left[\frac{1}{\varepsilon_2} - \frac{1}{a}\right]$$

I II

Since the limit of (I) and (II) not exist finitely

$\Rightarrow$ the given integrated is divergent.

Example 6. *Discuss the convergence of integral* $\int_0^1 \frac{dx}{1-x}$.

Solution. We have

$$\int_0^1 \frac{dx}{1-x} = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{dx}{1-x}$$

$$= \lim_{\varepsilon \to 0} \left[-\log(1-x)\right]_0^{1-\varepsilon} = \lim_{\varepsilon \to 0} [-\log \varepsilon + 0].$$

Since $\lim_{\varepsilon \to 0} \log \varepsilon$ is $-\infty$, therefore $\int_0^1 \frac{dx}{1-x}$ is meaningless *i.e.*, limit does not exists. So the integral is said to be divergent.

• TEST YOURSELF

Evaluate the following integral and also discuss their convergence :

1. $\int_1^\infty \frac{dx}{x}$. 2. $\int_0^\infty e^{2x}\, dx$. 3. $\int_{-1}^1 \frac{dx}{x^{2/3}}$. 4. $\int_{-\infty}^\infty e^{-x}\, dx$.

5. $\int_0^1 \frac{dx}{x^3}$. 6. $\int_3^\infty \frac{dx}{(x-2)^2}$.

ANSWERS

1. ∞, divergent **2.** ∞, divergent **3.** 6, convergent **4.** ∞, divergent

5. ∞, divergent **6.** 1, convergent.

• 5.4. CONVERGENCE TESTS : FIRST KIND

Recall that, First kind of improper integal is in which limits are infinite and integrand is continuous.

For Example. $\int_a^\infty f(x)\, dx$ or $\int_{-\infty}^b f(x)\, dx$ is the example of first kind of improper integral which can not be actually integrated. To test its convergence we use the following tests.

(a) Comparison Test :

If $\int_a^{\infty} f(x)\,dx$ *and* $\int_b^{\infty} g(x)\,dx$ *are positive, continuous (bounded) and integrable in the interval* $]a, \infty[$ *and*

(i) $f(x) \le g(x)$, for all x beyond a point $x = c$ and also $\int_b^{\infty} g(x)\,dx$ is convergent, then $\int_a^{\infty} f(x)\,dx$ is convergent.

(ii) If $g(x) \le f(x)$, *for all value of x and* $\int_b^{\infty} g(x)\,dx$ *is divergent, then* $\int_a^{\infty} f(x)\,dx$ *is divergent.*

(b) Limit Form of Comparison Test :

If $\int_a^{\infty} f(x)\,dx$ *and* $\int_b^{\infty} g(x)\,dx$ *are such that the integrands are positive and* $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ *then,*

(i) $\int_a^{\infty} f(x)\,dx$ *is convergent, when* $L = 0$ *and* $\int_b^{\infty} g(x)\,dx$ *is convergent.*

(ii) $\int_a^{\infty} f(x)\,dx$ *is divergent, when* $L = \infty$ *and* $\int_b^{\infty} g(x)\,dx$ *is divergent.*

(iii) both integrals are either convergent or divergent if L exists but non-zero.

Theorem 1. *The integral* $\int_a^{\infty} \frac{dx}{x^n}$, *when* $a > 0$ *is convergent when* $n > 1$, *and divergent when* $n \le 1$.

Proof. We have

$$\int_a^{\infty} \frac{dx}{x^n} = \lim_{x \to \infty} \int_a^x x^{-n}\,dx \qquad \text{(By definition of improper integral)}$$

$$= \lim_{x \to \infty} \left[\frac{x^{1-n}}{1-n}\right]_a^x \quad \text{if } n \ne 1$$

$$= \lim_{x \to \infty} \left[\frac{x^{1-n}}{1-n} - \frac{a^{1-n}}{1-n}\right]. \qquad \ldots(1)$$

Now if $n > 1$ then $(1 - n) < 0 \Rightarrow (n - 1) > 0$ therefore in this case, we have

$$\lim_{x \to \infty} x^{1-n} = \lim_{x \to \infty} \frac{1}{x^{n-1}} = \frac{1}{\infty} = 0.$$

$\therefore$ From (1), we have

$$\int_a^{\infty} \frac{dx}{x^n} = \frac{a^{1-n}}{n-1} \quad \text{if } n > 1.$$

Hence the given integral is convergent when $n > 1$.

Now, If $n < 1$, then $(1 - n) > 0$ and $(n - 1) < 0$

and $$\lim_{x \to \infty} x^{1-n} = \infty.$$

$\therefore$ From (1), $$\int_a^{\infty} \frac{dx}{x^n} = \infty.$$

Therefore, the given integral is divergent when $n < 1$.

If $n = 1$, then $$\int_a^{\infty} \frac{dx}{x^n} = \int_a^{\infty} \frac{dx}{x} = \lim_{x \to \infty} \int_a^x \frac{dx}{x} = \lim_{x \to \infty} \left[\log x\right]_a^x$$

$$= \lim_{x \to \infty} [\log x - \log a] = \infty - \log a = \infty.$$

$\therefore$ The given integral is divergent if $n \le 1$.

Hence $\int_a^\infty \frac{dx}{x^n}$ converges when $n > 1$ and diverges when $n \le 1$.

(c) Dirichlet's Test :

If $f(x)$, $g(x)$ and $g'(x)$ are all continuous in $[a, \infty[$ and $f(x)$, $g(x)$ satisfy the following three conditions

(i) $\lim_{x \to \infty} g(x) = 0$

(ii) $\int_a^\infty |g'(x)|\, dx$ *is convergent and*

(iii) $F(r) = \int_a^r f(x)\, dx$ *is bounded i.e., $|F(r)| \le M$ for some positive constant M.*

Then $\int_a^\infty f(x)\, g(x)\, dx$ *is convergent.*

(d) The μ-Test :

Let $f(x)$ be bounded and integrable in the interval $]a, \infty[$ where $a > 0$.

Then $\int_a^\infty f(x)\, dx$ *is convergent, if there is a number $\mu > 1$, such that* $\lim_{x \to \infty} \mu f(x)$ *exists.*

If there is a number $\mu \le 1$ such that $\lim_{x \to \infty} x^\mu f(x)$ *exists and non-zero, then* $\int_a^\infty f(x)\, dx$ *is dive*

(e) Weierstrass M-test :

If there exists a positive continuous function $M(t)$ such that $|f(x, t)| \le M(t)$, $t \ge a$, $c \le x \le d$, then the improper integral $\int_a^\infty f(x, t)\, dt$ *converges uniformly and absolutely for every x in the interval $[c, d]$ if* $\int_a^\infty M(t)\, dt$ *converges.*

(f) Abel's Test for the Convergence of Integral of Products :

The integral $\int_a^\infty f(x)\, \phi(x)\, dx$ *is convergent, if* $\int_a^\infty f(x)\, dx$ *converges and $\phi(x)$ is bounded and monotonic for $x > a$.*

(g) Absolute Convergence :

If the integral $\int_a^\infty |f(x)|\, dx$ *is convergent then the infinite integral* $\int_a^\infty f(x)\, dx$ *is said to be absolutely convergent.*

• 5.5. CONVERGENCE TEST : SECOND KIND

We test the convergence of a definite integral $\int_a^b f(x)\, dx$ for which limits (intervals) are finite and integrand $f(x)$ is not bounded at one or more points of given integral $[a, b]$.

(a) Comparison Test :

Let $\int_a^b f(x)\, dx$ be the given improper integral, whose limits are finite and $f(x)$ is not bounded only at $x = a$.

Let $x = b$ be a singular point for both $f(x)$ and $g(x)$ in interval $[a, b]$ and

(i) $0 \le f(x) \le g(x)$ *everywhere, except at* $x = b$ *then* $\int_a^b f(x)\,dx$ *is convergent if* $\int_a^b g(x)\,dx$ *is convergent.*

(ii) $f(x) \ge g(x) \ge 0$ *everywhere, except at* $x = a$ *then* $\int_a^b f(x)\,dx$ *is divergent if* $\int_a^b g(x)$ *is* divergent.

(b) Limit Form of Comparison Test :

(i) If $f(x)$ *and* $g(x)$ *are positive and* $\lim\limits_{x \to b} \dfrac{f(x)}{g(x)} = L$, *where L is neither zero nor infinite then* $\int_a^b f(x)\,dx$ *and* $\int_a^b g(x)\,dx$ *either both converge or both diverge at singular point* $x = b$.

(ii) If $L = 0$ *and* $\int_a^b g(x)\,dx$ *converges, then* $\int_a^b f(x)\,dx$ *converges.*

(iii) If $L = \infty$ *and* $\int_a^b g(x)\,dx$ *diverges then* $\int_a^b f(x)\,dx$ *diverges.*

(c) Abel's Test :

If $\phi(x)$ *is bounded and monotonic for* $a \le x \le b$ *and* $\int_a^b f(x)\,dx$ *converges. Then* $\int_a^b f(x)\,\phi(x)\,dx$ *converges.*

(d) Dirichlet's Test :

If $\int_{a+\varepsilon}^b f(x)\,dx$ *is bounded and* $\phi(x)$ *is bounded and monotonic in* $]a, b[$ *converging to zero as* $x \to a$, *then* $\int_a^b f(x)\,\phi(x)\,dx$ *converges.*

(e) Integrand is both +ve and −ve :

Let the integrand be both +ve and −ve in $[a, b]$. Let $x = b$ be a singular point of $f(x)$. Now *if* $\int_a^b f(x)\,dx$ *is convergent then* $\int_a^b f(x)\,dx$ *is absolutely convergent* $\int_a^b f(x)\,dx$ *is convergent but* $\int_a^b |f(x)|\,dx$ *is divergent then* $\int_a^b f(x)\,dx$ *is conditionally convergent.*

(f) The μ-test :

Let $f(x)$ be not bounded at $x = a$ and bounded and integrable in the arbitrary interval $]a + \varepsilon, b[$, where $0 < \varepsilon < b - a$.

If there is a number μ *between* 0 *and* 1 *such that* $\lim\limits_{x \to a+0} (x - a)^{\mu} f(x)$ *exists, then* $\int_a^b f(x)\,dx$ *is convergent.*

If there is a number $\mu \ge 1$ *such that* $\lim\limits_{x \to a+0} (x - a)^{\mu} f(x)$ *exists and is non-zero, then* $\int_a^b f(x)\,dx$ *is divergent and the same is true, if* $\lim\limits_{x \to a+0} (x - a)^{\mu} f(x) = +\infty$ *or* $-\infty$.

SOLVED EXAMPLES

Example 1. *Test the convergence of the integral* $\int_1^{\infty} \dfrac{dx}{\sqrt{x^3 + 1}}$.

Solution. We have $f(x) = \dfrac{1}{\sqrt{x^3+1}} = \dfrac{1}{x^{3/2}\sqrt{1+\dfrac{1}{x^3}}}$.

Let us consider $g(x) = \dfrac{1}{x^{3/2}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\dfrac{1}{x^{3/2}\sqrt{1+\dfrac{1}{x^3}}}}{\dfrac{1}{x^{3/2}}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\left(\dfrac{1}{x^3}\right)}} = 1$$

$\Rightarrow$limit is finite and non-zero.

$\Rightarrow \int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ are either both convergent or divergent.

Now by comparison test

$\int_1^\infty g(x)\,dx = \int_1^\infty \dfrac{dx}{x^{3/2}}$ will be convergent[Since $n > 1$]

$\Rightarrow \int_1^\infty f(x)\,dx$ will be convergent.

Example 2. *Test the convergence of integral* $\int_0^\infty \dfrac{\cos mx}{x^2+a^2}\,dx$.

Solution. Let $f(x) = \dfrac{\cos mx}{x^2+a^2}$, Let $g(x)\ \dfrac{1}{x^2+a^2}$.

Here $f(x)$, $g(x)$ both are positive in interval $]0, \infty[$, and $f(x) < g(x)$ for all $x \geq 0$.

Also, $$I = \int_0^\infty g(x)\,dx = \int_0^\infty \frac{1}{x^2+a^2}\,dx$$

$$= \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+a^2} = \lim_{b\to\infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^b$$

$$= \lim_{b\to\infty} \left[\frac{1}{a}\tan^{-1}\frac{b}{a} - 0\right] = \frac{1}{a}\cdot\frac{\pi}{2},\text{ which is finite.}$$

$\therefore \int_0^\infty \dfrac{dx}{x^2+a^2}$ is convergent.

Hence, $\int_0^\infty \dfrac{\cos mx}{x^2+a^2}\,dx$ is also convergent.

Example 3. *Test the convergence of the following integrals*

(*i*) $\int_1^\infty \dfrac{dx}{\sqrt{x^5+1}}$ (*ii*) $\int_0^\infty \dfrac{x^3\,dx}{(x^2+a^2)^2}$.

Solution. (i) Let $f(x) = \dfrac{1}{\sqrt{x^5+1}}$, and $g(x) = x^{-5/2}$.

So that $$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x^{5/2}}{\sqrt{x^5+1}} = 1,\text{ finite,}$$

and, $$\int_1^\infty g(x)\,dx = \int_1^\infty \frac{dx}{x^{5/2}} \text{ is convergent.}$$

Hence, the given integrals converges.

(ii) $\int_0^\infty \frac{x^3\,dx}{(x^2+a^2)^2} = f(x)$ and let $g(x) = x^{-1}$ so that,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^4}{(x^2+a^2)^2} = \lim_{x\to\infty} \frac{1}{(1+a^2/x^2)^2} = 1.$$

Since $\int_0^\infty g(x)\,dx = \int_0^\infty \frac{1}{x}\,dx$ is divergent therefore given integral also divergent.

Example 4. *Examine the convergence of* $\int_1^\infty \frac{dx}{x^{1/3}(1+x^{1/2})}$.

Solution. Let $$f(x) = \frac{1}{x^{1/3}(1+x^{1/2})} = \frac{1}{x^{1/3}.x^{1/2}(1+1/x^{1/2})}$$
$$= \frac{1}{x^{5/6}.\{1+(1/x^{1/2})\}}$$

$f(x)$ is bounded in the interval $(1,\infty)$ then by μ-test $\mu = \frac{5}{6} - 0 = \frac{5}{6}$.

We have $$\lim_{x\to\infty} x^\mu f(x) = \lim_{x\to\infty} x^{5/6}.\frac{1}{x^{5/6}\{1+1/x^{1/2}\}}$$
$$= \lim_{x\to\infty} \frac{1}{(1+1/x^{1/2})} = 1 \quad \text{(finite and non-zero)}$$

Since $\mu = 5/6 < 1$, so *the given integral is divergent.*

Example 5. *Test the convergence of the integral*

$$\int_a^\infty \frac{\sin x}{\sqrt{x}}\,dx, \text{ where } a > 0.$$

Solution. We have $\int_a^\infty \frac{\sin x}{\sqrt{x}}\,dx$.

Let $$f(x) = \frac{1}{\sqrt{x}} \text{ and } \phi(x) = \sin x$$

$1/\sqrt{x}$ is bounded and monotonically decreasing for all $x \geq a$ and $\lim_{x\to\infty} 1/\sqrt{x} = 0$.

Also, $$\left|\int_a^\infty \phi(x)\,dx\right| = \left|\int_a^\infty \sin x\,dx\right| = |\cos a - \cos\infty| \leq 2.$$

For all finite values of x the value of $\cos x$ lies between -1 and 1.

$\therefore$ $\left|\int_a^\infty \phi(x)\,dx\right|$ is bounded for all finite values of x.

Hence by Dirichlet's test the integral $\int_a^\infty \frac{\sin x}{\sqrt{x}}\,dx$ is convergent.

Example 6. *Show that* $\int_1^\infty \frac{\sin x}{x^4}\,dx$ *is absolutely convergent.*

Solution. If $\int_1^\infty \left|\frac{\sin x}{x^4}\right| dx$ is convergent, then integral $\int_1^\infty \frac{\sin x}{x^4}\,dx$ will be absolutely convergent.

Let $f(x) = \left|\frac{\sin x}{x^4}\right|$ then $f(x)$ is bounded in the inteval $]1,\infty[$.

Now, we have

$$f(x) = \left|\frac{\sin x}{x^4}\right| = \frac{|\sin x|}{x^4} \leq \frac{1}{x^4}, \text{ (since } |\sin x| \leq 1)$$

$\therefore$ By comparison test, if $\int_1^\infty \frac{1}{x^4}\,dx$ is convergent then $\int_1^\infty f(x)\,dx$ is convergent.

But the comparison integral $\int_1^\infty \frac{1}{x^4}\,dx$ is convergent because here $n = 4$ which is greater then 1.

Hence, $\int_1^\infty f(x)\,dx$ is convergent and so the given integral is absolutely convergent.

• TEST YOURSELF

1. Evaluate the following integrals :

   (i) $\int_3^\infty \frac{dx}{(x-2)^2}$. (ii) $\int_0^1 \frac{dx}{1-x}$. (iii) $\int_0^1 \frac{dx}{x^3}$.

2. Test the convergence of the following integrals :

   (i) $\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx$. (ii) $\int_0^\infty \frac{\cos mx}{1+x^2}\,dx$.

3. Test the convergent of the following integrals :

   (i) $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}, a > 0$ (II) $\int_0^\infty \frac{x^3}{(x^2+a^2)^2}\,dx$

4. Show that the integral $\int_0^\infty e^{-x^2}\,dx$ is convergent.

ANSWERS

1. (i) $-\infty$ (ii) ∞
2. (i) convergent (iii) convergent (iv) divergent
3. (i) convergent (ii) divergent

• 5.6. IMPROPER INTEGRALS OF SECOND KIND

We know that an integral $\int_a^b f(x)\,dx$ is said to be of second kind in which the range of integration is finite and the integrand $f(x)$ is unbounded at one or more points of the given interval $[a, b]$. Here, it is sufficient to consider the case when $f(x)$ becomes unbounded at $x = a$ and bounded for all other values of x in the interval $[a, b]$.

$\therefore$ We have $$\int_a^b f(x)\,dx = \lim_{h\to 0} \int_{a+h}^b f(x)\,dx,\ h > 0.$$

Now we use the following test, to test the convergence of the given integral.

(a) Comparison Test :

Let $\int_a^b f(x)\,dx$ be the given improper integral, in which the range of integration $]a, b[$ is finite and $f(x)$ is unbounded only at $x = a$. Let $\phi(x)$ be any positive function in the interval $]a+h, b[$ such that $|f(x)| \leq \phi(x)$.

Then $\int_a^b f(x)\,dx$ is convergent if $\int_a^b \phi(x)\,dx$ is convergent.

Also, if $|f(x)| \geq \phi(x)\ \forall\ x \in\ ]a+h, b[$, then $\int_a^b f(x)\, dx$ is divergent, provided $\int_a^b \phi(x)\, dx$ is divergent.

Theorem 1. *The comparison integral* $\int_a^b \frac{dx}{(x-a)^n}$ *is convergent when* $n < 1$ *and divergent when* $n \geq 1$.

Proof. Consider

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{h \to 0} \int_{a+h}^b \frac{dx}{(x-a)^n} = \lim_{h \to 0} \int_{a+h}^b (x-a)^{-n}\, dx$$

$$= \lim_{h \to 0} \left[\frac{(x-a)^{-n+1}}{1-n}\right]_{a+h}^b, \text{ if } n \neq 1$$

$$= \lim_{h \to 0} \left[\frac{(b-a)^{1-n}}{1-n} - \frac{h^{1-n}}{1-n}\right]. \quad ...(1)$$

Now $\quad n < 1 \Rightarrow 1-n > 0 \Rightarrow \lim_{h \to 0} h^{1-n} = 0.$

Therefore, (1) gives

$$\int_a^b \frac{dx}{(x-a)^n} = \frac{(b-a)^{1-n}}{1-n}, \text{ if } n < 1.$$

$\Rightarrow$ The given integral converges when $n < 1$.

If $n > 1$ then $1 - n < 0 \Rightarrow n - 1 > 0$.

$$\therefore \quad \int_a^b \frac{dx}{(x-a)^n} = \lim_{h \to 0} \left[\frac{(b-a)^{1-n}}{1-n} + \frac{1}{(n-1)\, h^{n-1}}\right] = \infty.$$

$\Rightarrow$ The given integral divergent when $n > 1$.

Now, if $n = 1$, then

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{(x-a)} = \lim_{h \to 0} \int_{a+h}^b \frac{dx}{x-a}$$

$$= \lim_{h \to 0} \left[\log(x-a)\right]_{a+h}^b$$

$$= \lim_{h \to 0} [\log(b-a) - \log h] = \infty.$$

Hence, the given integral diverges when $n = 1$.

SOLVED EXAMPLES

Example 1. *Test the convergence of the integral* $\int_0^1 \frac{dx}{x^3(1+x^2)}$.

Solution. Here, it is clear that the integral

$$f(x) = \frac{1}{x^3(1+x^2)}$$

is unbounded at $x = 0$.

Let $\quad \phi(x) = \frac{1}{x^3}.$

$$\therefore \quad \lim_{x \to 0} \frac{f(x)}{\phi(x)} = \lim_{x \to 0} \frac{1}{1+x^2} = 1, \text{ i.e., finite and non-zero.}$$

Then, by comparison test $\int_0^1 f(x)\, dx$ and $\int_0^1 \phi(x)\, dx$ either both converges or both diverges.

But clearly $\int_0^1 \frac{dx}{x^3}$ is convergent $\quad [\because\ n = 3 > 1]$

Hence, the given integral $\int_0^1 \frac{dx}{x^3(1+x^2)}$ is convergent.

Example 2. *Test the convergence of the integral* $\int_0^{\pi/2} \frac{\cos x}{x^2}\,dx$.

Solution. Here, the integral $f(x) = \frac{\cos x}{x^2}$ is unbounded at $x = 0$.

Let $\phi(x) = \frac{1}{x^2}$

Then
$$\lim_{x \to 0} \frac{f(x)}{\phi(x)} = \lim_{x \to 0} \left\{\frac{\cos x}{x^2} \cdot x^2\right\}$$
$$= \lim_{x \to 0} \cos x = 1, \text{ finite and non-zero.}$$

$\therefore$ by comparison test the integrals $\int_0^{\pi/2} f(x)\,dx$ and $\int_0^{\pi/2} \phi(x)\,dx$, either both converge or both diverge.

But
$$\int_0^{\pi/2} \phi(x)\,dx = \int_0^{\pi/2} \frac{1}{x^2}\,dx = \lim_{h \to 0} \int_h^{\pi/2} \frac{1}{x^2}\,dx$$
$$= \lim_{h \to 0} \left[-\frac{1}{x}\right]_h^{\pi/2} = \lim_{h \to 0} \left[-\frac{2}{\pi} + \frac{1}{h}\right] = \infty.$$

$\therefore \int_0^{\pi/2} \phi(x)\,dx$ is divergent.

Hence, the integral $\int_0^{\pi/2} \frac{\cos x}{x^2}\,dx$ is divergent.

(b) The μ-test :

Let the function $f(x)$ be unbounded at $x = a$ and integrable in the interval $]a+h, b[$, $0 < h < b - a$. *If there is a number* μ *between* 0 *and* 1 *such that* $\lim_{x \to a+0} (x-a)^{\mu} f(x)$, *exists, then* $\int_a^b f(x)\,dx$ *is convergent and if there is a number* $\mu \geq 1$ *such that* $\lim_{x \to a+0} (x-a)^{\mu} f(x)$ *exists and non-zero, then* $\int_a^b f(x)\,dx$ *is divergent and if*

$$\lim_{x \to a+0} (x-a)^{\mu} f(x) = +\infty \text{ or } -\infty, \text{ then } \int_a^b f(x)\,dx$$

is also divergent.

(c) Abel's Test :

If $\int_a^b f(x)\,dx$ *converges and* $\phi(x)$ *is bounded and monotonic for* $a \leq x \leq b$, *then* $\int_a^b f(x)\,\phi(x)\,dx$ *converges.*

(d) Dirichlet's Test :

If $\int_{a+h}^b f(x)\,dx$ *be bounded and* $\phi(x)$ *be bounded and monotonic on the interval* $a \leq x \leq b$ *converging to zero as x tends to a, then* $\int_a^b f(x)\,\phi(x)\,dx$ *converges.*

SOLVED EXAMPLES

Example 1. *Show that the integral* $\int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}}$ *converges.*

Solution. Here $f(x) = \frac{1}{\sqrt{\{x(1-x)\}}}$ is unbounded at $x = 0$ and 1.

Let a be any number such that $0 < a < 1$.

Then $\int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}} = \int_0^a \frac{dx}{\sqrt{\{x(1-x)\}}} + \int_a^1 \frac{dx}{\sqrt{\{x(1-x)\}}} = I_1 + I_2.$

In the integral I_1, the integrand $f(x)$ is unbounded at lower limit of integration $x = 0$ and in integration I_2, the integrand $f(x)$ is unbounded at the upper limit of integration $x = 1$.

To test the convergence of I_1, taking $\mu = \frac{1}{2}$, we have

$$\lim_{x\to 0} x^{\mu} f(x) = \lim_{x\to 0} \frac{x^{1/2}}{\sqrt{\{x(1-x)\}}} = \lim_{x\to 0} \frac{1}{\sqrt{1-x}} = 1.$$

So, the above limit exists.

Since, $0 < \mu < \frac{1}{2}$, so I_1 is convergent by μ-test.

To test the convergence of I_2 taking $\mu = \frac{1}{2}$, we have

$$\lim_{x\to 1-0} (1-x)^{\mu} . f(x) = \lim_{x\to 1-0} (1-x)^{1/2} . \frac{1}{\sqrt{\{x(1-x)\}}}$$

$$= \lim_{x\to 1-0} \frac{1}{\sqrt{x}} = \lim_{h\to 0} \frac{1}{\sqrt{1-h}} = 1.$$

Since $0 < \mu < 1$, so I_2 is convergent by μ-test.

Thus, the given integral is the sum of two convergent integrals. Hence, the given integral is convergent.

Example 2. *Test the convergence of the integral* $\int_0^1 x^{n-1} \log x \, dx$.

Solution. Since $\lim_{x\to 0} x^r \log x = 0$ where $r > 0$, the integral is a proper integral if $n > 1$.

If $n = 1$, then we have

$$\int_0^1 \log x \, dx = \lim_{h\to 0} \int_h^1 \log x \, dx = \lim_{h\to 0} [x \log x - x]_{\varepsilon}^{1}$$

$$= \lim_{h\to 0} [-1 - h \log h + h] = -1.$$

So the given integral is convergent if $n = 1$.

If $n < 1$ and $f(x) = x^{n-1} \log x$ then, we have

$$\lim_{x\to 0} x^{\mu} f(x) = \lim_{x\to 0} x^{\mu+n-1} . \log x = 0 \text{ if } \mu > 1-n \quad \text{...(i)}$$

$$= -\infty \text{ if } \mu \leq 1-n. \quad \text{...(ii)}$$

Hence, if $0 < n < 1$, then we can take μ between 0 and 1 and satisfying (i).

Then if $0 < n < 1$ then the integral is convergent by μ-test.

Again if $n \leq 0$ then, we can take $\mu = 1$ and satisfying (ii).

Hence if $n \leq 0$ then the integral is divergent by μ-test.

So by the above discussion we get, the given integral is convergent if $n > 0$ and divergent if $n \leq 0$.

Example 3. *Discuss the convergence of the given integral*

$$\int_0^{\infty} x^{n-1} e^{-x} dx, \text{ if } n > 0.$$

Solution. Here given that

$$I = \int_0^{\infty} x^{n-1} e^{-x} dx$$

$$I = \int_0^1 x^{n-1} e^{-x}\, dx + \int_1^{\infty} x^{n-1} e^{-x}\, dx.$$

Let $$I_1 = \int_0^1 x^{n-1} e^{-x}\, dx$$

$$I_2 = \int_1^{\infty} x^{n-1} e^{-x}.$$

Here for discuss the convergence of given integral, we use μ-test in I_2 and comparison test in I_1.

$$I_1 = \int_0^1 x^{n-1} e^{-x}\, dx$$

$f(x) = x^{n-1} e^{-x}$ at $x = 0$, it will be unbounded.

Let $g(x) = x^{n-1}$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} e^{-x} = 1.$$

By comparison test if $g(x)$ is convergent then $f(x)$ will also be convergent or if divergent then $f(x)$ will be divergent

$$\int_0^1 g(x)\, dx = \int_0^1 x^{n-1}\, dx = \lim_{\varepsilon\to 0} \int_{\varepsilon}^1 x^{n-1}\, dx$$

$$= \lim_{\varepsilon\to 0} \left[\frac{x^n}{n}\right]_{\varepsilon}^1 = \lim_{\varepsilon\to 0} \left[\frac{1}{n} - \frac{\varepsilon^n}{n}\right]$$

$= \dfrac{1}{n}$, which is a finite real number.

$\Rightarrow \int_0^1 g(x)\, dx$ is convergent

$\Rightarrow f(x)$ will be convergent.

Now $$I_2 = \int_1^{\infty} x^{n-1} e^{-x}\, dx.$$

Here $f(x) = x^{n-1} e^{-x}$. It is bounded in the interval $(1, \infty)$

$$\lim_{x\to\infty} x^{\mu} f(x) = \lim_{x\to\infty} \frac{x^{\mu} . x^{n-1}}{e^x} = \lim_{x\to\infty} \frac{x^{\mu+n-1}}{1 + x + \frac{x^2}{2!} + \ldots} = 0$$

For $\mu > 1$, we have $\int_1^{\infty} x^{n-1} e^{-x}\, dx$ is convergent.

From the above result we can say I will be convergent because I_1 and I_2 both are convergent.

• SUMMARY

- First kind of improper integrals :

$$\int_a^{\infty} f(x)\, dx, \int_{-\infty}^{a} f(x)\, dx, \int_{-\infty}^{\infty} f(x)\, dx.$$

- Second kind of improper integrals :

$$\int_a^{\infty} \frac{dx}{x-a)^n} \text{ or } \int_a^{\infty} \frac{dx}{(x-b)^n}$$

- When $a > 0$, then $\int_a^{\infty} \frac{dx}{x^n}$ is

(i) convergent if $n > 1$
(ii) divergent if $n \leq 1$

- The integral $\int_a^b \frac{dx}{(x-a)^n}$ is :

(i) Convergent if $n < 1$.
(ii) Divergent if $n \geq 1$.

• STUDENT ACTIVITY

1. Discuss the convergence of the integral $\int_0^1 \frac{dx}{\sqrt{1-x}}$.

2. Test the convergence of the integral $\int_0^1 x^{n-1} \log x \, dx$.

• TEST YOURSELF

1. Show that the integral $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is convergent.
2. Test the convergence of the integral $\int_1^2 \frac{dx}{\sqrt{x^4-1}}$.
3. Test the convergence of the integral $\int_0^1 \frac{dx}{(x+1)\sqrt{(1-x^2)}}$.
4. Show that $\int_0^1 x^{n-1} e^{-x} \, dx$ is convergent if $n > 0$.

ANSWERS

2. convergent 3. convergent

FILL IN THE BLANKS :

1. The definite integral $\int_a^b f(x) \, dx$ is called .... integral if either any one or both limits are finite and function is bounded over the range of integration.
2. A definite integral $\int_a^b f(x) \, dx$ in which limits are infinite and integrand is continuous is called .... kind of improper integral.
3. If improper integral having finite value, then it is called ....
4. The point at which the integrand is infinite is called .... point.

5. The integral $\int_0^1 \frac{dx}{1-x}$ is ....

TRUE OR FALSE :

Write 'T' for true and 'F' for false statement :

1. The integral $\int_0^\infty \frac{dx}{(1+x)^{2/3}}$ is convergent. *(T/F)*
2. The comparison integral $\int_0^\infty \frac{dx}{x^n}$, when $a > 0$ is convergent when $n > 1$ and divergent when $n \leq 1$. *(T/F)*
3. In μ-test the value of μ is usually taken to be equal to the highest power of x in the denominator of the integrand minus the highest power of x in the numerator of the integrand. *(T/F)*

MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The integral $\int_0^\infty x^{n-1} e^{-x}\, dx$ is convergent if :

   (a) $n > 0$ (b) $n = 0$ (c) $n < 0$ (d) None of these.

2. The integral $\int_0^\infty \frac{\sin x}{x}\, dx$ converges :

   (a) uniformly (b) conditionally (c) absolutely (d) None of these.

ANSWERS

Fill in the Blanks :

1. Improper 2. First 3. Convergent 4. Singular 5. Divergent

True or False :

1. F 2. T 3. T

Multiple Choice Questions :

1. (a) 2. (b)

❑❑❑

UNIT 6

FUNCTIONS OF A COMPLEX VARIABLE

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is a complex Number and how to represent it ?
- How to find the equation of a straight line and a circle in complex form.
- What are analytic functions ?
- What are harmonic functions ?

6.1. COMPLEX NUMBER

The concept of numbers, as we now is gradually extended from natural numbers to integers. Integers to rational numbers and from rational numbers to real numbers. We know that the square of every real number is non-negative, therefore, there exist no real number whose square equal to -1.

For example, there is no solution in real number of the equation $x^2+1=0$ and $x^2-2x+3=0$. Euler (1707–1783) was first to introduce the symbol i for the square root of -1

i.e., $i=\sqrt{-1}$ and $i^2=-1$. So $i^3=i^2 \, . \, i=(-1)\,i=-i$

$i^4=(i)^2=1$ and so on.

Gauss (1777–1855) first proved in a satisfactory manner that every algebraic equation with real coefficient has complex roots of the form $x+iy$, the real roots being a particular case of complex numbers for which the coefficient of i is zero. **Hamilton** (1805–1865) also made a great contribution to the development of the theory of complex numbers.

Imaginary Numbers :

Definition. *Square root of a negative number is called as imaginary number.*

For example : $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}$ etc.

Complex Numbers :

Definition. *A complex number may be defined as an ordered pair $x + iy$, of real numbers and may be denoted by the symbol (x, y).*

If we write $\mathbf{z} = (x, y)$ *i.e.* $x + iy$, then x is called the real part and y is the imaginary part of the complex number $\mathbf{z}$ and may be denoted by $R(z)$ and $I(z)$ respectively.

For example : $5 + 2i, 3 + 6i, 2 - i, 0 + i$ etc. all are complex numbers.

Equality of Two Complex Numbers :

'Two complex number are said to be equal if and only if their real as well as imaginary parts are equal : if $x_1 + iy_1$ and $x_2 + iy_2$ are two complex numbers, then

$$x_1 + iy_1 = x_2 + iy_2 \Leftrightarrow x_1 = x_2, \text{and } y_1 = y_2$$

$$\because \quad x_1 + iy_1 = x_2 + iy_2$$
$$\Rightarrow \quad (x_1 - x_2) + i(y_1 - y_2) = 0$$
$$\Rightarrow \quad (x_1 - x_2) = i(y_2 - y_1)$$
$$\Rightarrow \quad (x_1 - x_2)^2 = (-1)(y_2 - y_1)^2$$
$$\Rightarrow \quad (x_1 - x_2)^2 + (y_2 - y_1)^2 = 0$$
$$\Rightarrow \quad x_1 - x_2 = 0 \text{ and } y_2 - y_1 = 0 \Rightarrow x_1 = x_2 \text{ and } y_1 = y_2$$

or we can say $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$.

Important Results :

(i) If x and y are two positive real numbers then

$$\sqrt{-x} \times \sqrt{-y} = -\sqrt{xy}.$$

(ii) For any two real numbers $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$ is true only when at least one of x and y is either positive or zero.

i.e., $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$ is not valid, if both x and y are negative.

(iii) For any positive real number x, we have

$$\sqrt{-x} = \sqrt{-1 \times x} = \sqrt{-1} \times \sqrt{x} = i\sqrt{x}.$$

• 6.2. ALGEBRA OF COMPLEX NUMBERS

(A) Addition of complex numbers. *Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, then their sum $z_1 + z_2$ is the number $(x_1 + x_2) + i(y_1 + y_2)$.*

From the definition, it is clear that the sum of $(z_1 + z_2)$ is

real $(z_1 + z_2) + i$ imag $(z_1 + z_2)$

where Re $(z_1 + z_2)$ = Re (z_1) + Re (z_2)

andimag $(z_1 + z_2)$ = imag (z_1) + imag (z_2).

For example : Let $z_1 = 5 + 3i$ and $z_2 = 3 + 6i$ be any two complex numbers then, we have

$$z_1 + z_2 = (5 + 3) + i(3 + 6) = 8 + 9i.$$

Properties of the Addition of Complex Numbers :

(i) Commutativity. If z_1 and z_2 are two complex numbers, then

$z_1 + z_2 = z_2 + z_1$.

(ii) Associativity. For three complex numbers z_1, z_2 and z_3, we have

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(iii) Additive identity. The complex number $0 = 0 + i0$ is the identity element for addition *i.e.*,$z + 0 = 0 + z = z$ for all $z \in \mathbf{C}$.

(iv) Additive inverse. Corresponding to every non-zero complex number $z = x + iy$, there exist a complex number

$$-z = -(x + iy) = -x - iy \text{ such that}$$
$$z + (-z) = 0 = (-z) + z.$$

Here, $-z$ is called the additive inverse of z.

(B) Substraction of complex numbers. *If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, then their difference $z_1 - z_2$ is the number $z_1 + (-z_2)$.*

Symbolically :

The difference of two complex number z_1 and z_2 can be written as

$$z_1 - z_2 = z_1 + (-z_2) = (x + iy_1) + (-x_2 - iy_2) = (x_1 - x_2) + i(y_1 - y_2).$$

For example : Let $z_1 = 3 + 7i$ and $z_2 = 1 + 5i$ are any two complex numbers, then

$$z_1 - z_2 = (3 - 1) + i\,(7 - 5) = 2 + 2i.$$

(C) Multiplication of complex numbers. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers then the product of z_1 and z_2, given by

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 + ix_1y_2 + i\,x_2y_1 + y_1\,y_2 i^2 \;(i^2 = -1) \\ &= (x_1x_2 - y_1\,y_2) + i\,(x_1y_2 + x_2y_1) \\ &= \{\text{Re}\,(z_1) \,.\, \text{Re}\,(z_2) - \text{im.}\,(z_1) \,.\, \text{im.}\,(z_2)\} + i\,\{\text{Re}\,(z_1) \,.\, \text{im.}\,(z_2) + \text{Re}\,(z_2) \,.\, \text{im}\,(z_1)\}. \end{aligned}$$

For example : Le $z_1 = 3 + 2i$ and $z_2 = 5 + 3i$ be two complex numbers, then

$z_1 \cdot z_2 = (3 + 2i) \,.\, (5 + 3i) = (15 - 6) + i\,(9 + 10) = 9 + 19i.$

Properties of Multiplication of Complex Numbers :

(i) Commutativity. For any two complex number z_1 and z_2, we have

$$z_1z_2 = z_2z_1.$$

(ii) Associativity. For any three complex number z_1, z_2 and z_3, we have

$$(z_1 \,.\, z_2) \,.\, z_3 = z_1 \,.\, (z_2 \,.\, z_3).$$

(iii) Multiplicative identity. The complex number $1 = 1 + i \,.\, 0$ is the identity element for multiplication

i.e., $$z \,.\, 1 = z = 1 \,.\, z \text{ for all } z \in \mathbf{C}.$$

(iv) Multiplicative inverse. Corresponding to every non-zero complex number $z = x + iy$, there exists a complex number $z_1 = x_1 + iy_1$, such that

$$z \,.\, z_1 = 1 = z_1 \,.\, z.$$

Here, z_1 is called the multiplicative inverse of z.

(v) Distributivity. For any three complex numbers z_1, z_2, and z_3

$$z_1\,(z_2 + z_3) = z_1z_2 + z_1z_3 \qquad \text{(Left distributive law)}$$

and $$(z_2 + z_3)\,z_1 = z_2z_1 + z_3z_1 \qquad \text{(Right distributive law)}$$

(vi) Cancellation law for multiplication. If z_1, z_2 and z_3 are three complex numbers and $z_3 \neq 0$ then,

$$z_1z_3 = z_2z_3 \;\Rightarrow\; z_1 = z_2.$$

(D) Conjugate of a complex number. If $z = x + iy$ is a complex number, then conjugate of z, denoted by $\bar{z}$ given by $x - iy$, which is obtain by replacing $-i$ for i in z.

• 6.3. PROPERTIES OF CONJUGATE OF A COMPLEX NUMBER

(i) $\overline{(\bar{z})} = z.$

(ii) $z + \bar{z} = 2\,\text{Re}\,(z).$

(iii) $z - \bar{z} = 2i\,\text{Im}\,(z).$

(iv) $z = \bar{z} \Leftrightarrow z$ is purely real.

(v) $z + \bar{z} = 0 \Rightarrow z$ is purely imaginary.

(vi) $z\bar{z} = \{\text{Re}\,(z)\}^2 + \{\text{Im}\,(z)\}^2.$

(vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2.$

(viii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2.$

(ix) $\overline{z_1z_2} = \bar{z}_1 \,.\, \bar{z}_2.$

(x) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ (provided $z_2 \neq 0$).

(E) Division of complex number. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, then division of z_1 and z_2, denoted by $\dfrac{z_1}{z_2}$ is given by

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} = (x_1 + iy_1)\,(x_2 + iy_2)^{-1} \\ &= (x_1 + iy_1)\left\{\frac{x_2}{x_2^2 + y_2^2} - i\,\frac{y_2}{x_2^2 + y_2^2}\right\} \end{aligned}$$

$$= \left\{ \left(\frac{x_1 x_2}{x_2^2 + y_2^2} + \frac{y_1 y_2}{x_2^2 + y_2^2} \right) + i \left(\frac{-x_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1}{x_2^2 + y_2^2} \right) \right\}$$

$$= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{y_1 x_2 - y_2 x_1}{x_2^2 + y_2^2} \right), \text{ provide } x_2^2 + y_2^2 \neq 0.$$

For example : If $z_1 = 1 + 4i$ and $z_2 = 2 - 3i$, then

$$\frac{z_1}{z_2} = \frac{1+4i}{2-3i} = \frac{1+4i}{2-3i} \times \frac{2+3i}{2+3i}$$

$$= \frac{(1+4i)(2+3i)}{2^2 - (3i)^2} = \frac{2+3i+8i-12}{9-9i^2} = \frac{11i-10}{4-(-9)} = \frac{11i-10}{13} = \left(\frac{-10}{13} \right) + i \left(\frac{11}{13} \right).$$

Dot and cross product of complex number. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers (vectors). Then dot product (scalar product) of z_1 and z_2 is defined by

$$z_1 \circ z_2 = |z_1| \,.\, |z_2| \cos\theta = x_1 x_2 + y_1 y_2$$

$$= \text{Re}\,(\bar{z}_1 z_2) = \frac{1}{2} (\bar{z}_1 z_2 + z_1 \bar{z}_2) \qquad \text{...(1)}$$

where θ is the angle between z_1 and z_2 which lies between 0 and π and cross product of z_1 and z_2 is

$$z_1 \times z_2 = |z_1| |z_2| \sin\theta = x_1 y_2 - y_1 x_2$$

$$= \text{Im}\,(\bar{z}_1 z_2) = \frac{1}{2i} (\bar{z}_1 z_2 - z_1 \bar{z}_2). \qquad \text{...(2)}$$

By (1) and (2), it is clear that

$$\bar{z}_1 z_2 = (z_1 \circ z_2) + i (z_1 \times z_2) = |z_1| |z_2| e^{i\theta}. \qquad \text{...(3)}$$

If both z_1 and z_2 are non zero, then

(i) *z_1 and z_2 is perpendicular if and only if $z_1 \circ z_2 = 0$.*

(ii) *z_1 and z_2 is parallel if and only if $z_1 \times z_2 = 0$.*

(iii) *The magnitude of the projection of z_1 on z_2 is* $\dfrac{|z_1 \circ z_2|}{|z_2|}$.

(iv) *The area of a parallelogram whose side z_1 and z_2, is $|z_1 \times z_2|$.*

• 6.4. MODULUS AND ARGUMENT OF A COMPLEX NUMBER

Let $z = x + iy$ be any complex number. Let $x = r\cos\theta$, $y = r\sin\theta$, then $r = +\sqrt{x^2 + y^2}$ is called the ***modulus*** *of the complex number z written as $|z|$ and $\theta = \tan^{-1}\frac{y}{x}$ is called the argument or amplitude of z, written as $\arg z$.*

Thus, $\quad r = |z| = \sqrt{x^2 + y^2}$

$\Rightarrow \quad |z|^2 = x^2 + y^2 = z \,.\, \bar{z}$

$\Rightarrow \quad z \,.\, \dfrac{\bar{z}}{|z|^2} = 1, \text{ if } z \neq 0.$

• 6.5. SOME PROPERTIES OF MODULI

Theorem 1. *The modulus of the product of two complex numbers is the product of their moduli i.e., $|z_1 \,.\, z_2| = |z_1| \,.\, |z_2|$.*

Proof. We have, $\quad |z_1 \,.\, z_2|^2 = (z_1 \,.\, z_2)\,\overline{(z_1 \,.\, z_2)}$

$$= z_1 \,.\, z_2 \,.\, \bar{z}_1 \,.\, \bar{z}_2 = (z_1 \,.\, \bar{z}_1) \,.\, (z_2 \,.\, \bar{z}_2) = |z_1|^2 \,.\, |z_2|^2$$

$\Rightarrow \quad |z_1 \,.\, z_2|^2 = |z_1|^2 \,.\, |z_2|^2$

$\Rightarrow \quad |z_1 \,.\, z_2| = |z_1| \,.\, |z_2|.$

Theorem 2. *The modulus of the sum of two complex numbers is less than or equal to the sum of their moduli*

i.e., $\quad |z_1 + z_2| \leq |z_1| + |z_2|.$

Proof. To show $|z_1 + z_2| \leq |z_1| + |z_2|$.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 + z_2 = r_1 e^{i\theta_1} + r_2 e^{i\theta_2} = r_1 (\cos \theta_1 + i \sin \theta_1) + r_2 (\cos \theta_2 + i \sin \theta_2)$$
$$= (r_1 \cos \theta_1 + r_2 \cos \theta_2) + i (r_1 \sin \theta_1 + r_2 \sin \theta_2)$$

$\therefore$ $$| z_1 + z_2 | = \sqrt{(r_1 \cos \theta_1 + r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 + r_2 \sin \theta_2)^2}$$
$$= \sqrt{[r_1^2 + r_2^2 + 2r_1 r_2 \cos (\theta_1 - \theta_2)]}$$
$$\leq \sqrt{[r_1^2 + r_2^2 + 2r_1 r_2]} \qquad [\text{For } \cos (\theta_1 - \theta_2) \leq 1]$$
$$= r_1 + r_2 = | z_1 | + | z_2 |.$$

Hence, $| z_1 + z_2 | \leq | z_1 | + | z_2 |$.

Theorem 3. *The modulus of the difference of two complex numbers is greater than or equal to the difference of their moduli*

i.e., $$| z_1 - z_2 | \geq | z_1 | - | z_2 |.$$

Proof. To show $| z_1 - z_2 | \geq | z_1 | - | z_2 |$.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

then $| z_1 | = r_1$ and $| z_2 | = r_2$ $\qquad (\because \ | e^{i\theta} | = 1)$

$$z_1 - z_2 = r_1 e^{i\theta_1} - r_2 e^{i\theta_2} = r_1 (\cos \theta_1 + i \sin \theta_1) - r_2 (\cos \theta_2 + i \sin \theta_2)$$
$$\Rightarrow \quad z_1 - z_2 = (r_1 \cos \theta_1 - r_2 \cos \theta_2) + i (r_1 \sin \theta_1 - r_2 \sin \theta_2).$$

Now $$| z_1 - z_2 | = \sqrt{[(r_1 \cos \theta_1 - r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 - r_2 \sin \theta_2)^2]}$$
$$= \sqrt{[r_1^2 + r_2^2 - 2r_1 r_2 \cos (\theta_1 - \theta_2)]}$$
$$\geq \sqrt{[r_1^2 + r_2^2 - 2r_1 r_2]} \qquad [\because \ -\cos (\theta_1 - \theta_2) \geq -1]$$
$$= r_1 - r_2 = | z_1 | - | z_2 |.$$

Hence, $| z_1 - z_2 | \geq | z_1 | - | z_2 |$.

Cor. *Prove that* $| z_1 - z_2 | \leq | z_1 | + | z_2 |$.

Proof. We have,

$$| z_1 - z_2 | = | z_1 + (-z_2) | \leq | z_1 | + | (-z_2) | = | z_1 | + | z_2 |.$$

Hence, $| z_1 - z_2 | \leq | z_1 | + | z_2 |$.

So, by above results, we get

$$| z_1 | - | z_2 | \leq | z_1 - z_2 | \leq | z_1 | + | z_2 |.$$

Theorem 4. *Prove that* $| z_1 + z_2 | \geq | z_1 | - | z_2 |$.

Proof. To show $| z_1 + z_2 | \geq | z_1 | - | z_2 |$.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $| z_1 | = r_1$ and $| z_2 | = r_2$

$$z_1 + z_2 = r_1 (\cos \theta_1 + i \sin \theta_1) + r_2 (\cos \theta_2 + i \sin \theta_2)$$
$$= (r_1 \cos \theta_1 + r_2 \cos \theta_2) + i (r_1 \sin \theta_1 + r_2 \sin \theta_2).$$

Now, $$| z_1 + z_2 | = \sqrt{(r_1 \cos \theta_1 + r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 + r_2 \sin \theta_2)^2}$$
$$= \sqrt{[r_1^2 + r_2^2 + 2r_1 r_2 \cos (\theta_1 - \theta_2)]}$$
$$\geq \sqrt{[r_1^2 + r_2^2 - 2r_1 r_2]} \qquad [\text{Since } \cos (\theta_1 - \theta_2) \geq -1]$$
$$= r_1 - r_2 = | z_1 | - | z_2 |.$$

Hence, $| z_1 + z_2 | \geq | z_1 | - | z_2 |$.

Theorem 5. *Prove that*

$$| z_1 + z_2 |^2 + | z_1 - z_2 |^2 = 2 [| z_1 |^2 + | z_2 |^2].$$

Proof. Let $z_1 = r_1 e^{i\theta_1}$, and $z_2 = r_2 e^{i\theta_2}$ then $| z_1 | = r_1$ and $| z_2 | = r_2$

$$z_1 + z_2 = r_1 (\cos \theta_1 + i \sin \theta_1) + r_2 (\cos \theta_2 + i \sin \theta_2)$$

or $$z_1 + z_2 = (r_1 \cos \theta_1 + r_2 \cos \theta_2) + i (r_1 \sin \theta_1 + r_2 \sin \theta_2)$$

and $$z_1 - z_2 = (r_1 \cos \theta_1 - r_2 \cos \theta_2) + i (r_1 \sin \theta_1 - r_2 \sin \theta_2)$$

Now $$| z_1 + z_2 |^2 = (r_1 \cos \theta_1 + r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 + r_2 \sin \theta_2)^2$$

and $$| z_1 - z_2 |^2 = (r_1 \cos \theta_1 - r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 - r_2 \sin \theta_2)^2.$$

Taking $$| z_1 + z_2 |^2 + | z_1 - z_2 |^2 = [(r_1 \cos \theta_1 + r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 + r_2 \sin \theta_2)^2]$$
$$+ [(r_1 \cos \theta_1 - r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 - r_2 \sin \theta_2)^2]$$
$$= [r_1^2 + r_2^2 + 2r_1 r_2 \cos (\theta_1 - \theta_2)] + [r_1^2 + r_2^2 - 2r_1 r_2 \cos (\theta_1 - \theta_1)]$$
$$= 2 [r_1^2 + r_2^2] = 2 [| z_1 |^2 + | z_2 |^2].$$

Hence, $| z_1 + z_2 |^2 + | z_1 - z_2 |^2 = 2 [| z_1 |^2 + | z_2 |^2]$.

• 6.6. PROPERTIES OF ARGUMENTS

Theorem 1. *The argument of the product of two complex numbers is equal to the sum of their arguments*

i.e., $$arg.\,(z_1, z_2) = arg.\,(z_1) + arg.\,(z_2).$$

Proof. To show $\arg.\,(z_1, z_2) = \arg.\,(z_1) + \arg.\,(z_2)$.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then $\arg.\,(z_1) = \theta_1$ and $\arg.\,(z_2) = \theta_2$.

Taking $$z_1 \cdot z_2 = r_1 \cdot r_2\, e^{i(\theta_1 + \theta_2)}$$

$\Rightarrow$ $\text{Arg.}\,(z_1 \cdot z_2) = \theta_1 + \theta_2 = \arg.\,(z_1) + \arg.\,(z_2)$.

Hence, $\text{Arg.}\,(z_1 \cdot z_2) = \arg.\,(z_1) + \arg.\,(z_2)$.

Theorem 2. *The argument of the quotient of two complex numbers is equal to the difference of their arguments*

i.e., $$arg.\left(\frac{z_1}{z_2}\right) = arg.\,(z_1) - arg.\,(z_2).$$

Proof. To show $\arg.\left(\frac{z_1}{z_2}\right) = \arg.\,(z_1) - \arg.\,(z_2)$

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$\arg.\,(z_1) = \theta_1 \text{ and } \arg.\,(z_2) = \theta_2.$$

Taking $$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$\Rightarrow \quad \text{Arg}\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg\,(z_1) - \arg\,(z_2).$$

Hence, $$\arg.\left(\frac{z_1}{z_2}\right) = \arg.\,(z_1) - \arg.\,(z_2).$$

SOLVED EXAMPLES

Example 1. *Express* $\frac{1+7i}{(2-i)^2}$ *in the modulus amplitude form.*

Solution. Here, $$\frac{1+7i}{(2-i)^2} = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{3-4i} = \frac{(1-7i)}{3-4i}\frac{(3+4i)}{3+4i}$$

$$= \frac{3+4i+21i+28i^2}{9-16i^2} = \frac{-25+25i}{25} = -1+i.$$

Now let $-1 + i = r(\cos\theta + i\sin\theta)$.

On comparing real and imaginary part, we have

$$r\cos\theta = -1 \quad \ldots(1)$$

$$r\sin\theta = 1. \quad \ldots(2)$$

Squaring (1) and (2), and adding,

$$r^2 = 1 + 1 = 2 \qquad \therefore r = \sqrt{2}.$$

Now putting $r = \sqrt{2}$ in (i) and (ii), we have

$$\cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}}, \text{ giving } \theta = \frac{3\pi}{4}.$$

Hence $$\frac{1+7i}{(2-i)^2} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

• 6.7. GEOMETRICAL REPRESENTATION OF COMPLEX NUMBER

A complex number $z = x + iy$ can be represented by a point P in the cartesian plane. The co-ordinate of P are (x, y) referred to rectangular axis OX and OY, where OX is called real axis and OY is called imaginary axis.

The complex number $0 + i \cdot 0$ corresponds to the origin, the real number $x = x + i0$ and imaginary number $iy = 0 + iy$ correspond to the points on X-axis and Y-axis respectively.

Obviously, the polar co-ordinate of P are (r, θ) where $r = OP = \sqrt{x^2 + y^2}$ is the modulus and the angle $\theta = \tan^{-1}\frac{y}{x}$ is the argument of $z = x + iy$.

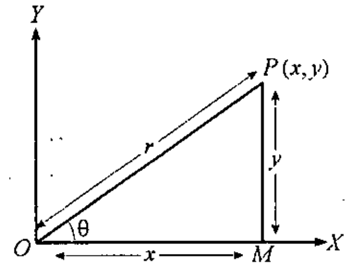


Fig. 1

To each complex number there exists one and only one point in the X-Y plane, and to each point in the X-Y plane there exist one and only one complex number, by this fact, the complex number $z = x + iy$ is referred to the point z in this plane. This plane is called complex plane or Gaussian plane or Argand plane. The representation of complex number is called Argand diagram. The distance between the points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is given by

$$|z_1 - z_2| = \sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]}.$$

Some Geometrical Interpretations :

(i) $z_1 + z_2$. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, represented by the image P and Q. Complete the parallelogram $OPRQ$.

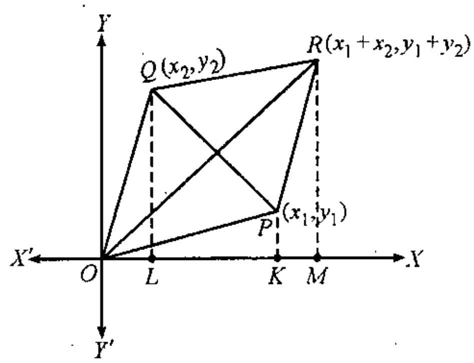


Fig. 2

Let PK, QL and RM represents the perpendicular from P, Q and R respectively on X-axis.

Since the diagonal of a parallelogram bisect each other, therefore, co-ordinates of the mid point of PQ and also that of OR is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Therefore, the co-ordinate of R are $(x_1 + x_2, y_1 + y_2)$.

Hence, R represents the complex number

$$(x_1 + x_2) + i\,(y_1 + y_2)$$
$$= (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2.$$

(ii) $z_1 - z_2$. Let P and Q be two points represents the two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

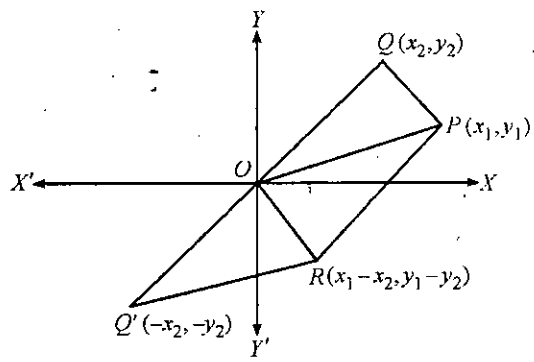


Fig. 3

Since, the sum of z_1 and $-z_2$ is represented by the extremely R of the diagonal OR of parallelogram $OPRQ'$.

Hence, R represents the complex number

$$(x_1 - x_2) + i\,(y_1 - y_2)$$
$$= (x_1 + iy_1) - (x_2 + iy_2)$$
$$= z_1 - z_2.$$

• 6.8. POLAR FORM OF A COMPLEX NUMBER

Consider a point P in the Argand plane (or complex plane) corresponding to a complex number $z = x + iy$.

put $\quad x = r\cos\theta,\ y = r\sin\theta$

then $\quad r = \sqrt{x^2 + y^2} = |x + iy| = |z|$

and $\quad \theta = \tan^{-1}\frac{y}{x}.$

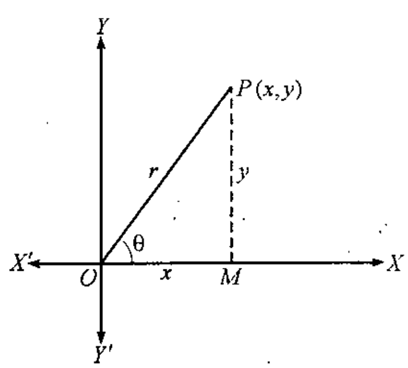


Fig. 4

It follows that

$$z = x + iy = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta) = re^{i\theta}$$
$$z = re^{i\theta} \qquad (\because\ e^{i\theta} = \cos\theta + i\sin\theta)$$

which is called the polar form of the complex number z.

r and θ called polar co-ordinate of z. r is the modulus or absolute value of z and θ is the argument or amplitude of z.

It is also written as $\theta = \text{arg.}(z)$ or $\theta = \text{amp.}(z)$.

SOLVED EXAMPLES

Example 1. *Find the moduli and arguments of the following complex numbers*

(*i*) $\dfrac{1-i}{1+i}$ (*ii*) $\dfrac{3-i}{2+i}+\dfrac{3+i}{2-i}$ (*iii*) $\left(\dfrac{2+i}{3-i}\right)^2$.

Solution. (i) Here, we have

$$\frac{1-i}{1+i}=\frac{1-i}{1+i}\cdot\frac{1-i}{1-i}=\frac{(1-i)^2}{1-i^2}=\frac{-2i}{2}=-i$$

$$\therefore \qquad \left|\frac{1-i}{1+i}\right| = |-i| = \sqrt{0+(-1)^2}=1$$

and $$\text{arg.}\left(\frac{1-i}{1+i}\right)=\text{arg.}(-i)=-\frac{\pi}{2}$$

because $$-i=0-i=\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right).$$

(ii) Here, we have

$$\frac{3-i}{2+i}+\frac{3+i}{2-i}=\frac{(3-i)(2-i)+(3+i)(2+i)}{(2+i)(2-i)}$$

$$=\frac{6-3i-2i-1+6+3i+2i-1}{4-i^2}=\frac{10}{5}=2.$$

So $$\left|\frac{3-i}{2+i}+\frac{3+i}{2-i}\right|=2$$

and $$\text{arg.}\left(\frac{3-i}{2+i}+\frac{3+i}{2-i}\right)=\text{arg.}(2)=0 \qquad (\because \text{ argument of a positive real number is } 0)$$

(iii) Here, we have

$$\left(\frac{2+i}{3-i}\right)^2=\frac{(2+i)^2}{(3-i)^2}=\frac{3+4i}{8-6i}=\frac{3+4i}{8-6i}\cdot\frac{8+6i}{8+6i}=\frac{50i}{100}=\frac{1}{2}i.$$

$$\therefore \qquad \left|\left(\frac{2+i}{3-i}\right)^2\right|=r=\frac{1}{2}.$$

Now, let $$\frac{1}{2}i=r(\cos\theta+i\sin\theta),$$

then, $$r\cos\theta=0,\quad r\sin\theta=\frac{1}{2}.$$

Squaring and adding above relation, we get

$$r^2=\frac{1}{4} \Rightarrow r=\frac{1}{2}.$$

Putting $r=\dfrac{1}{2}$, we have $\cos\theta=0$, $\sin\theta=1$.

The value of θ lying between $-\pi$ and π, which satisfies both these equation is $\dfrac{\pi}{2}$.

Hence, principal value of arg. $\left(\dfrac{2+i}{3-i}\right)^2=\dfrac{\pi}{2}$.

Example 2. *The real numbers A and B if*

(*i*) $A+iB=\dfrac{3-2i}{7+4i}$

(*ii*) $A+iB=\dfrac{1}{(1-2i)(2+3i)}$.

Solution. (i) Here, we have

$$A+iB=\frac{3-2i}{7+4i}=\frac{(3-2i)(7-4i)}{(7+4i)(7-4i)}=\frac{13-26i}{65}=\frac{13}{65}-\frac{26}{65}i=\frac{1}{5}-\frac{2}{5}i.$$

Equating real and imaginary parts of both the sides, we get

$$A = \frac{1}{5},\ B = -\frac{2}{5}.$$

(ii) Here, we have

$$A + iB = \frac{1}{(1-2i)(2+3i)} = \frac{1}{(2+3i-4i-6i^2)} = \frac{1}{8-i}$$

$$= \frac{8-i}{(8-i)(8+i)} = \frac{8+i}{64-i^2} = \frac{8+i}{64+1} = \frac{8+i}{65} = \frac{8}{65} + \frac{1}{65}i.$$

Equating real and imaginary parts of both the sides, we get

$$A = \frac{8}{65},\ B = \frac{1}{65}.$$

Example 3. *Show that arg. z + arg. $\bar{z} = 2n\pi$, where n is any integer.*

Solution. Let $z = x + iy$, then $\bar{z} = x - iy$, where x and y are real.

Now we have

$$\text{arg. } z + \text{arg. } \bar{z} = \text{arg. } (z\,.\,\bar{z}) = \text{arg. } \{(x+iy)(x-iy)\} = \text{arg. } (x^2+y^2).$$

Now $x^2 + y^2$ is a positive real number, say c. Since c is a positive real number, so the representative point of c in the argand plane will lie on the positive side of the real axis. So the principal value of arg. c is 0 and the general values is $2n\pi$, where n is any integer.

Hence, arg. z + arg. $\bar{z} = 2n\pi$.

Example 4. *Prove that* $|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ *interpret the result geometrically and deduce that*

$$|\alpha + \sqrt{\alpha^2-\beta^2}| + |\alpha - \sqrt{\alpha^2-\beta^2}| = |\alpha+\beta| + |\alpha-\beta|$$

all the numbers involved being complex.

Solution. We have,

$$|z_1+z_2|^2 + |z_1-z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)} + (z_1-z_2)\overline{(z_1-z_2)}$$

$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2) + (z_1-z_2)(\bar{z}_1-\bar{z}_2)$$

$$= 2z_1\bar{z}_1 + 2z_2\bar{z}_2 = 2|z_1|^2 + 2|z_2|^2 \qquad ...(i)$$

Geometrical interpretation. Let A and B be the points of affix z_1 and z_2 respectively. Complete the parallelogram $OABC$.

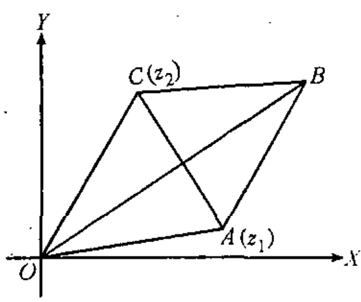


Fig. 5

Then, we have

$$OA = |z_1|,\ OC = |z_2|$$

$$OB = |z_1+z_2|,\ AC = |z_1-z_2|.$$

Now, from the property of parallelogram

$$OB^2 + CA^2 = 2OA^2 + 2OC^2$$

or $\quad |z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2.$

Deduction. Le $z_1 = \alpha + \sqrt{\alpha^2-\beta^2}$ and $z_2 = \alpha - \sqrt{\alpha^2-\beta^2}$, then we have

$$\frac{1}{2}|z_1+z_2|^2 + \frac{1}{2}|z_1-z_2|^2 = |z_1|^2 + |z_2|^2 \qquad \text{[from (i)]}$$

or $$\frac{1}{2}|2\alpha|^2 + \frac{1}{2}|2\sqrt{(\alpha^2+\beta^2)}|^2 = |z_1|^2 + |z_2|^2$$

or $$2|\alpha|^2 + 2|(\alpha^2-\beta^2)| = |z_1|^2 + |z_2|^2$$

and so $$[|z_1|+|z_2|]^2 = |z_1|^2 + |z_2|^2 + 2|z_1 z_2| = 2|\alpha|^2 + 2|\alpha^2-\beta^2| + 2|\beta|^2$$

$$= |\alpha+\beta|^2 + |\alpha-\beta|^2 + 2|\alpha^2-\beta^2| \qquad \text{[using (i)]}$$

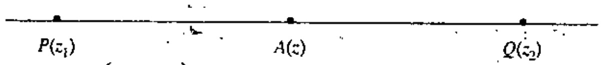
$$= [|\alpha+\beta| + |\alpha-\beta|]^2.$$

So $\quad |z_1| + |z_2| = |\alpha+\beta| + |\alpha-\beta|.$

Hence, $\quad |\alpha + \sqrt{\alpha^2-\beta^2}| + |\alpha - \sqrt{\alpha^2-\beta^2}| = |\alpha+\beta| + |\alpha-\beta|.$

• 6.9. EQUATION OF STRAIGHT LINE IN COMPLEX FORM

Equation of straight line passing through two given complex number. If z_1 and z_2 be any two points (complex numbers) in argand plane and $A(z)$ be the any current point. To find the equation of a straight line passing through the point $P(z_1)$ and $Q(z_2)$. Consider the following figure :



Evidently, $\arg.\left(\dfrac{z-z_1}{z_1-z_2}\right)=0$ or π.

Consequenly, $\left(\dfrac{z-z_1}{z_1-z_2}\right)$ is purely real. So, we have

$$\frac{z-z_1}{z_1-z_2}=\overline{\left(\frac{z-z_1}{z_1-z_2}\right)}=\frac{\bar{z}-\bar{z}_1}{\bar{z}_1-\bar{z}_2}$$

or $$(z-z_1)(\bar{z}_1-\bar{z}_2)=(z_1-z_2)(\bar{z}-\bar{z}_1)$$

or $$z(\bar{z}_1-\bar{z}_2)-z_1(\bar{z}_1-\bar{z}_2)=\bar{z}(z_1-z_2)-\bar{z}_1(z_1-z_2)$$

or $$z(\bar{z}_1-\bar{z}_2)-z_1\bar{z}_1+z_1\bar{z}_2=\bar{z}(z_1-z_2)-z_1\bar{z}_1+\bar{z}_1z_2$$

or $$z(\bar{z}_1-\bar{z}_2)-\bar{z}(z_1-z_2)+(z_1\bar{z}_2-z_2\bar{z}_1)=0 \qquad ...(1)$$

which is the required equation of straight line in Argand plane.

Now multiplying (1) by i, we get

$$iz(\bar{z}_1-\bar{z}_2)-i\bar{z}(z_1-z_2)+i(z_1\bar{z}_2-z_2\bar{z}_1)=0. \qquad ...(2)$$

Now we take the coefficient of $\bar{z}$ is α and the coefficient of z, which is the conjugate of that of z is α. Agian $\bar{z}_1z_2$ is the complex conjugate of $z_1\bar{z}_2$. So, the number $z_1\bar{z}_2-z_2\bar{z}_1$ is imaginary and the number $i(z_1\bar{z}_2-z_2\bar{z}_1)$ is purely real. So we have

$$i(z_1\bar{z}_2-\bar{z}_1z_2)=k, \text{ where } k \text{ is real.}$$

Now from (2), we get

or $$\bar{\alpha}z+\alpha\bar{z}+\boldsymbol{k}=\mathbf{0},\ \alpha\neq 0 \text{ and } k \text{ is real.}$$

where α and k are constant.

Which is the general equation of a straight line.

Some Important Theorems :

Theorem 1. *The equation of any straight line passing through the origin and making an angle α with the real axis is $z=re^{i\alpha}$ where r is any real parameter.*

Proof. Consider a point $z=x+iy$ on the straight line passing through the origin and making an angle α with real axis. Then, we have

For $x=r\cos\alpha,\quad y=r\sin\alpha$

$$z=x+iy=r\cos\alpha+ir\sin\alpha=r(\cos\alpha+i\sin\alpha)=r(\cos\alpha+i\sin\alpha)....(4)$$

$z=re^{i\alpha}$, which is the required equation.

Theorem 2. *The equation of any straight line passing through the point z_1 and making an angle α with the real axis is $z=z_1+re^{i\alpha}$ where r is any real parameter.*

Proof. Let $z=x+iy$ be any point on the straight line passing through the point z_1 and making an angle α with the real axis. Then

$$x-x_1=r\cos\alpha$$

which implies $(x-x_1)+i(y-y_1)=r\cos\alpha+ir\sin\alpha$

$\Rightarrow$ $(x+iy)-(x_1+iy_1)=r\cos\alpha+ir\sin\alpha$

$\Rightarrow$ $z-z_1=r(\cos\alpha+i\sin\alpha)$, where $z_1=x_1+iy_1$.

Hence $z-z_1=re^{i\alpha}$, which is the required equation.

Theorem 3. *The equation of the straight line joining the point z_1 and z_2 is $z=tz_1+(1-t)z_2$, where t is any real parameter.*

Proof. Suppose that z be the affix of any point on the straight line joining the points z_1 and z_2. Again, suppose that the point z divides the join of z_1 and z_2 in the ratio $\lambda:1$, where λ is any real number not equal to -1.

We have $$z=\frac{z_1+\lambda z_2}{1+\lambda} \text{ or } z=\left(\frac{1}{1+\lambda}\right)z_1+\left(\frac{\lambda}{1+\lambda}\right)z_1. \qquad ...(1)$$

Put $\dfrac{1}{1+\lambda}=t$ *i.e.*, $1-t=\dfrac{\lambda}{1+\lambda}$ in (1), we get

$$z=tz_1+(1-t)z_2, \text{ which is the required equation.}$$

• 6.10. EQUATION OF A CIRCLE IN ARGAND PLANE

Show that the equation of a circle in the Argand plane is of the form

$$z\bar{z} + c\bar{z} + \bar{c}z + k = 0$$

where k is real and c is a complex constant.

Proof. Consider a circle, whose centre is $c(b)$ where b is any complex number and r be the radius of the circle and let $A(z)$ be any point on the circle.

Then, the line CA = radius of the circle

$$|z - b| = \rho.$$

On squaring, we get

$$|z - b|^2 = \rho^2$$

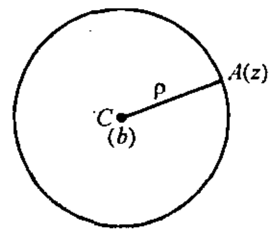


Fig. 6

$$\Rightarrow \quad (z - b)\,\overline{(z - b)} = \rho^2 \;\Rightarrow\; (z - b)(\bar{z} - \bar{b}) = \rho^2$$

$$\Rightarrow \quad z\bar{z} - \bar{b}z + b\bar{b} - b\bar{z} = \rho^2$$

$$\Rightarrow \quad z\bar{z} - \bar{b}z - b\bar{z} + (|b|^2 - \rho^2) = 0.$$

Taking $-b = c$ and $(|b|^2 - \rho^2) = k$ = real number.

Then, we get $\quad z\bar{z} + c\bar{z} + \bar{c}z + k = 0$

where k is real and c be a complex constant.

Which is the required equation of a circle.

General equation of a circle. We know that the equation of a circle is given by

$$z\bar{z} + c\bar{z} + \bar{c}z + k = 0 \qquad \text{(where } k \text{ is real)} \quad ...(1)$$

The above equation can be written as

$$(z + c)(\bar{z} + \bar{c}) = c\bar{c} - k \;\Rightarrow\; |z + c|^2 = c\bar{c} - k.$$

Here, equation (1) represents a circle if k is real and

$$c\bar{c} - k \geq 0.$$

Thus, the general equation of the circle is of the form

$$\boldsymbol{z\bar{z} + \bar{c}z + c\bar{z} + k = 0}, \; k \text{ is a real and } c\bar{c} - k \geq 0.$$

• 6.12. CONDITION FOR FOUR POINTS TO BE CONCYCLIC

Let $P(z_1)$, $Q(z_2)$, $R(z_3)$, $S(z_4)$ be the four points (complex numbers). Then the given four points P, Q, R, S are concyclic if $\angle PRQ$, $\angle PSQ$ are either equal or differ by π

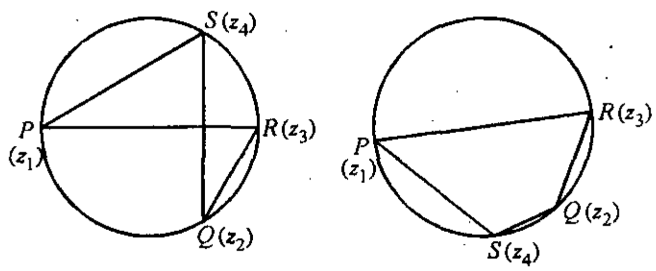


Fig. 7

$$\Rightarrow \quad \arg.\frac{z_3 - z_1}{z_3 - z_2},\ \arg.\frac{z_4 - z_1}{z_4 - z_2} \text{ are either equal or differ by } \pi$$

$$\Rightarrow \quad \arg.\left[\frac{z_3 - z_1}{z_3 - z_2} \Big/ \frac{z_4 - z_1}{z_4 - z_2}\right] = 0 \text{ or } \pi$$

$$\Rightarrow \quad \frac{z_3 - z_1}{z_3 - z_2} \Big/ \frac{z_4 - z_1}{z_4 - z_2} \text{ is real}$$

$$\Rightarrow \quad \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \text{ is purely real.}$$

Hence, four points z_1, z_2, z_3, z_4 are concyclic if $\dfrac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}$ is purely real.

Cor. Equation of a circle passing through three points.

Let z_1, z_2, z_3 be any three points (complex numbers) on a circle and let z be any point on the circle, then we know that the four points z_1, z_2, z_3, z are concyclic if $\frac{(z_2 - z_1)(z - z_3)}{(z - z_1)(z_2 - z_3)}$ is purely real

$$\Rightarrow \quad \frac{(z_2 - z_1)(z - z_3)}{(z - z_1)(z_2 - z_3)} = \frac{\overline{(z_2 - z_1)}\,\overline{(z - z_3)}}{\overline{(z - z_1)}\,\overline{(z_2 - z_3)}} \qquad (\because \ z \text{ is real if } z = \bar{z})$$

$$\Rightarrow \quad \frac{(z_2 - z_1)(z - z_3)}{(z - z_1)(z_2 - z_3)} = \frac{(\bar{z}_2 - \bar{z}_1)(\bar{z} - \bar{z}_3)}{(\bar{z} - z_1)(\bar{z}_2 - \bar{z}_3)}$$

which is the required equation of the circle passing through three points.

Example 1. *Find the region of the Argand plane for which*

$$|z - 1| + |z + 1| \le 3.$$

Solution. We have $z = x + iy$, then

$$|z - 1| + |z + 1| = |x + iy - 1| + |x + iy + 1| = |(x - 1) + iy| + |(x + 1) + iy|$$
$$= \sqrt{[(x-1)^2 + y^2]} + \sqrt{[(x+1)^2 + y^2]}.$$

But it is given $|z - 1| + |z + 1| \le 3$. So

$$\sqrt{[(x-1)^2 + y^2]} + \sqrt{[(x+1)^2 + y^2]} \le 3$$

or
$$\sqrt{[(x-1)^2 + y^2]} \le 3 - \sqrt{[(x+1)^2 + y^2]}.$$

Squaring both side, we get

$$(x-1)^2 + y^2 \le 9 + (x+1)^2 + y^2 - 6\sqrt{[(x+1)^2 + y^2]}$$

or
$$9 + 4x - 6\sqrt{[(x+1)^2 + y^2]} \ge 0$$

or
$$6\sqrt{[(x+1)^2 + y^2]} \le 4x + 9.$$

Again squaring, we get

$$36[(x+1)^2 + y^2] \le 16x^2 + 81 + 72x \ \text{ or } \ 36x^2 + 36y^2 + 36 \le 16x^2 + 81$$

or
$$20x^2 + 36y^2 \le 45 \ \text{ or } \ \frac{x^2}{9/4} + \frac{y^2}{5/4} \le 1.$$

Hence, the region of the Argand plane is boundary and interior of the ellipse $\frac{x^2}{9/4} + \frac{y^2}{5/4} \le 1.$

Example 2. *If P, Q, R are points of affix $z_1, z_2, z_1 + z_2$ respectively then prove that OPRQ is a parallelogram.*

Solution. Let $z_1, z_2, (z_1 + z_2)$ be three points such that

$$z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2,\ z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

Thus the co-ordinate of O, P, Q, R are $(0, 0), (x_1, y_1), (x_2, y_2), (x_1 + x_2, y_1 + y_2)$ respectively.

Now, mid point of $\quad PQ = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

and the mid point of $\quad OR = \left(\frac{0 + x_1 + x_2}{2}, \frac{0 + y_1 + y_2}{2}\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$

Hence, $OPQR$ is a parallelogram.

Example 3. *Show that* $\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1$ *if* $|z_1| < 1$ *and* $|z_2| < 1$.

Solution. The given inequality $\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1$ will hold if

$$|z_1 - z_2| < |1 - \bar{z}_1 z_2| \ \text{ or } \ |z_1 - z_2|^2 < |1 - \bar{z}_1 z_2|^2$$

$$\Rightarrow \quad (z_1 - z_2)\overline{(z_1 - z_2)} < (1 - \bar{z}_1 z_2)\overline{(1 - \bar{z}_1 z_2)} \qquad (\because \ |z|^2 = z\bar{z})$$
$$\Rightarrow \quad (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) < (1 - \bar{z}_1 z_2)(1 - z_1 \bar{z}_2)$$
$$\Rightarrow \quad z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2 < 1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_1\bar{z}_1 \cdot z_2\bar{z}_2$$
$$\Rightarrow \quad |z_1|^2 + |z_2|^2 < 1 + |z_1|^2 |z_2|^2$$
$$\Rightarrow \quad |z_1|^2 + |z_2|^2 - 1 - |z_1|^2 |z_2|^2 < 0$$
$$\Rightarrow \quad (|z_1|^2 - 1)(1 - |z_2|^2) < 0.$$

Now, the above inequality will hold if $|z_1| < 1$ and $|z_2| < 1$.

Hence, $\left|\dfrac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1$ if $|z_1| < 1$ and $|z_2| < 1$.

Example 4. *Determine the region of the argand plane, for which* $\left|\dfrac{z-c}{z-\bar{c}}\right| < 1, = 1$ *or* > 1 *where real part of c is positive.*

Solution. Here, we have $\left|\dfrac{z-c}{z-\bar{c}}\right| < 1, = 1$ or > 1

$$\Rightarrow \quad |z-c|^2 <=> |z+\bar{c}|^2$$
$$\Rightarrow \quad (z-c)\,\overline{(z-c)} <=> (z+\bar{c})\,\overline{(z+\bar{c})}$$
$$\Rightarrow \quad (z-c)\,(\bar{z}-\bar{c}) <=> (z+\bar{c})\,(\bar{z}+c)$$
$$\Rightarrow \quad (z\bar{z} - z\bar{c} - c\bar{z} + c\bar{c}) <=> (z\bar{z} + zc + \bar{c}\,\bar{z} + \bar{c}\,c)$$
$$\Rightarrow \quad z\bar{z} - (z\bar{c} + c\bar{z}) <=> (z\bar{z} + zc + \bar{c}\,\bar{z})$$
$$\Rightarrow \quad -(z\bar{c} + c\bar{z}) - (zc + \bar{c}\,\bar{z}) <=> 0$$
$$\Rightarrow \quad (z\bar{c} + c\bar{z}) + (zc + \bar{c}\,\bar{z}) >=< 0$$
$$\Rightarrow \quad z\,(c+\bar{c}) + \bar{z}\,(c+\bar{c}) >=< 0$$
$$\Rightarrow \quad (z+\bar{z})\,(c+\bar{c}) >=< 0 \;\Rightarrow\; 2x\,2\,\mathrm{Re}\,(c) >=< 0$$
$$\Rightarrow \quad x >=< 0 \qquad (\because \ \mathrm{Re}\,(z) \text{ is positive})$$

Hence the required region is the right half of the Argand plane, imaginary axis and left half of the Argand plane respectively.

Example 5. *Show that the radius and centre of the circle*

$$\left|\frac{z-i}{z+i}\right| = 5.$$

Solution. We have $\quad \left|\dfrac{z-i}{z+i}\right| = 5 \qquad \text{...(1)}$

or $\quad |z-i| = 5\,|z+i| \qquad \left[\because \ \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right]$

or $\quad |x+iy-i| = 5\,|x+iy+i|.$

Squaring both sides, we get

or $\quad |x+i\,(y-1)|^2 = 25\,|x+i\,(y+1)|^2$

or $\quad x^2 + (y-1)^2 = 25\,[x^2 + (y+1)^2]$

or $\quad x^2 + y^2 - 2y + 1 = 25\,[x^2 + y^2 + 2y + 1]$

or $\quad 24\,(x^2 + y^2) + 52y + 24 = 0$

or $\quad x^2 + y^2 + \dfrac{13}{6}y + 1 = 0 \qquad \text{...(2)}$

which is the equation of a circle.

Therefore the locus of the points on the Argand plane which satisfy the condition (1) is a circle.

The co-ordinates of the centre of the circle equation (1) are $\left(0, -\dfrac{13}{12}\right)$ and its radius is

$$\sqrt{\left[(0)^2 + \left(-\frac{13}{2}\right)^2 - 1\right]} = \sqrt{\frac{25}{144}} = \frac{5}{12}.$$

Hence the locus of the given circle is the point whose affix is

$$z = 0 + (-13/12)\,i \ \ i.e., \ (-13/12)\,i$$

and its radius is 5/12,

• TEST YOURSELF

1. Find the modulus and arguments of the following :

   (i) $\dfrac{1+2i}{1-(1-i)^2}$ (ii) $\dfrac{2+i}{4i+(1+i)^2}$.

2. For two complex number z_1, z_2 prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ if and only if, $z_1\,\bar{z}_2$ is purely imaginary.

• 6.12. ANALYTIC FUNCTION

Some Important Definitions

Single and multiple valued function. *If we get only one value of w corresponding to one value of z, then we say that w is a single valued function of z or f(z) is a single valued function.*

For example. If $w = z^2$. Here, corresponds to one value of z we get only one value of w. Hence, $w = z^2$ is a single valued function of z.

On, the other hand if we get one or more value of w, corresponding to each value of z, then we say that w is a multiple valued function (or many valued function).

For example. If $w = \sqrt{z}$, then we get two value of w, corresponding to each value of z. Hence, w is a multiple valued function of z.

A multiple valued function can be considred as a collection of single valued functions, whose every member is called a branch of the function. And a particular member is called a principal branch of the multiple valued function and the value of the funcion according to his branch is known as ***principal value.***

Limits and continuity of a complex function. *Let f(z) be a single valued function defined in a bounded and closed domain D. Then a number l is said to be the limit of f(z) at $z = z_0$, if for any positive number ε (however small) we can find a positive number δ such that*

$$|f(z) - l| < \varepsilon \;\; \forall\, z \text{ for which } 0 < |z - z_0| < \delta.$$

The limit must be independent of the manner in which $z \to z_0$.

Symbolically, we write $\lim\limits_{z \to z_0} (z) = l$.

Some important results on limits. If $\lim\limits_{z \to z_0} f(z) = l$ and $\lim\limits_{z \to z_0} g(z) = m$, then

(i) $\lim\limits_{z \to z_0} [f(z) \pm g(z)] = \lim\limits_{z \to z_0} f(z) \pm \lim\limits_{z \to z_0} g(z) = l \pm m$

(ii) $\lim\limits_{z \to z_0} [f(z) \,.\, g(z)] = \lim\limits_{z \to z_0} f(z) \,.\, \lim\limits_{z \to z_0} g(z) = l \,.\, m$

(iii) $\lim\limits_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\lim\limits_{z \to z_0} f(z)}{\lim\limits_{z \to z_0} g(z)} = \dfrac{l}{m}$ if $m \neq 0$.

Continuity. *Let f(z) be a single valued function of z defined in the closed and bounded domain D. Then f(z) is said to be continuous at a point z_0 in D iff, for any positive number (however small) we can find a positive number δ such that*

$$|f(z) - f(z_0)| < \varepsilon \;\; \textit{whenever} \;\; |z - z_0| < \delta.$$

From the definition of limit and continuity we can say that $f(z)$ is continuous at $z = z_0$ if and only if $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

Note. If $f(z)$ is continuous at $z = z_0$ then this implies three conditions.

(i) $\lim\limits_{z \to z_0} f(z) = l$ must exists.

(ii) $f(z_0)$ must exists.

(iii) $f(z_0) = l$.

For example. If $f(z) = z^2$, $\forall\, z$ then $f(z)$ is continuous at $z = i$ because

$$\lim_{z \to i} f(z) = \lim_{z \to i} z^2 = i = -1.$$

Discontinuity. At any point z_0, at which $f(z)$ is not continuous then $f(z)$ is said to be discontinuous at z_0. If $\lim\limits_{z \to z_0} f(z)$ exist but not equal to $f(z_0)$, then this type of discontinuity is called removable discontinuity.

Continuiy in real and imaginary part of $f(z)$. If $f(z) = u(x, y) + iv\,(x, y)$ is a continuous function of z, then $u(x, y)$ and $v(x, y)$ are also continuous function of x, y and if $u(x, y)$ and $v\,(x, y)$ are continuous function of x, y then $f(z)$ is also a continuous function.

Uniform continuity. Let A function $f(z)$ defined in a domain D, then $f(z)$ is said to be uniform continuous in D if for any $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(z_1) - f(z_2)| < \varepsilon \text{ whenever } 0 < |z_1 - z_2| < \delta, \text{ where } z_1, z_2 \in D.$$

Differentiability. Let $f(z)$ be a single valued function of z defined in a domain D, then $f(z)$ is said to be differentiable at point $z = z_0$ of D iff

$$f'(z) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

provided that the limit exists and does not depend upon path which $h \to 0$ or we can say

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Theorem 1. *Continuity is a necessary but not a sufficient condition for the existance of a finite derivative.*

Proof. Let $f(z)$ be a differentiable function at $z = z_0$ then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exist.}$$

Now, we can take

$$f(z) - f(z_0) = (z - z_0) \frac{f(z) - f(z_0)}{z - z_0} \text{ if } z \neq z_0.$$

Taking limit of both sides,

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \left[(z - z_0) \frac{f(z) - f(z_0)}{z - z_0} \right]$$

$$= \lim_{z \to z_0} (z - z_0) \cdot \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= 0 \cdot f'(z_0) = 0 \qquad \left(\text{Since } \lim_{z \to z_0} (z - z_0) = z_0 - z_0 = 0\right)$$

or
$$\lim_{z \to z_0} [f(z) - f(z_0)] = 0 \Rightarrow \lim_{z \to z_0} f(z) - \lim_{z \to z_0} f(z_0) = 0$$

$$\Rightarrow \quad \lim_{z \to z_0} f(z) = f(z_0).$$

Hence, $f(z)$ is continuous at $z = z_0$ thus continuity is a necessary condition for differentiability.

Now we shall show that continuity is not a sufficient condition for differentiability. It is clear from the following example. Consider the function $f(z) = |z|^2$, where $z = x + iy$.

The function $|z|^2 = x^2 + y^2$ is continuous at every point.

Now $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\bar{z}_0 + \Delta \bar{z}) - z_0 \bar{z}_0}{\Delta z} = \lim_{\Delta z \to 0} \left[\frac{z_0 \Delta \bar{z} + \Delta z \cdot \bar{z}_0 + \Delta z \cdot \Delta \bar{z}}{\Delta z} \right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{\Delta \bar{z}}{\Delta z} z_0 + \bar{z} + \Delta \bar{z} \right] = \lim_{\Delta z \to 0} \left[\frac{\Delta \bar{z}}{\Delta z} z_0 + \bar{z}_0 \right] \qquad (\because \Delta z \to 0 \Rightarrow \Delta \bar{z} \to 0)$$

So at $z_0 = 0$, $\bar{z}_0 = 0$ so that $f'(z_0) = 0$.

Again at $z_0 \neq 0$. Now let $\Delta z = r(\cos\theta + i \sin\theta)$

then, $$\Delta \bar{z} = r(\cos\theta - i \sin\theta) \Rightarrow \frac{\Delta \bar{z}}{\Delta z} = \frac{\cos\theta - i \sin\theta}{\cos\theta + i \sin\theta} = \cos 2\theta - i \sin 2\theta$$

which does not tend to a unique limit. Since this limit depends upon arg. Δz.

Thus the function $f(z)$ is continuous everywhere but not differentiable for any non zero value of z.

Analytic function. *Consider a single valued funcion $f(z)$ defined in a domain D, then the function $f(z)$ is said to be analytic at $z = z_0$ of D, if it is differentiable not only at z_0 but also in some neighbourhood of z_0.*

Or

A function $f(z)$ is said to be analytic in a domain D. If $f(z)$ is differentiable at every point of a domain D.

Singular point. *A point $z = z_0$ at which $f'(z_0)$ does not exists, is said to be singular point of $f(z)$.*

If a function $f(z)$ is analytic in every neighbourhood of a point z_0 except z_0. Then z_0 is known as **isolated singularity** of $f(z)$.

If $f(z)$ is not analytic at $z = z_0$ but it can be made analytic by taking a suitable value to $f(z)$ at point z_0, then $f(z)$ is said to have an **removable singularity** at a point z_0 of D.

A function $f(z)$ is analytic in some deleted neighbourhood of z_0 and has a removable singularity at z_0. Then the function $f(z)$ is said to be **regular** at z_0.

• 6.13. CAUCHY-RIEMANN EQUATIONS

A necessary condition that $w = f(z)$, where $f(z) = u(x, y) + i\, v(x, y)$ be analytic in a domain D,s $u(x, y)$ and $v(x, y)$ satisfy the equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \qquad ...(1)$$

The above equation (1) is known as the Cauchy-Riemann equation.

If partial derivative in (1) are continuous then it is the sufficient condition for a function $f(z)$ to be analytic in D.

• 6.14. THE NECESSARY AND SUFFICIENT CONDITION FOR A FUNCION $f(z)$ TO BE ANALYTIC

(i) Necessary condition for $f(z)$ to be analytic.

Theorem 2. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = x + iy$ in a domain D, then the partial derivative u_x, v_x, u_y, v_y should exist and satisfy the equations $u_x = v_y$ and $u_y = -v_x$.*

Proof. Since $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$ then

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad ...(1)$$

must exists and unique as $\Delta z \to 0$ in any manner.

If $\quad z = x + iy$ and $\Delta z = \Delta x + i\,\Delta y$.

Now, using the above relations, equation (1) can be written as

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \left[\frac{u\,(x + \Delta x, y + \Delta y) - u\,(x, y)}{\Delta x + i\,\Delta y} + i\,\frac{v(x + \Delta x, y + \Delta y) - v(x, y)}{\Delta x + i\,\Delta y}\right]. \qquad ...(2)$$

Taking Δz to be wholly real (along real axis) so that $\Delta y = 0$ then, quation (2) gives

$$= \lim_{\Delta x \to 0} \left[\frac{u\,(x + \Delta x, y) - u\,(x, y)}{\Delta x} + i\,\frac{v\,(x + \Delta x, y) - v(x, y)}{\Delta x}\right]. \qquad ...(3)$$

Now, since $f(z)$ is differentiable, then the partial derivative $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ must exist and the limit is

$$\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = u_x + iv_x. \qquad ...(4)$$

Again taking Δz to be wholly imaginary (along imaginary axis) so that $\Delta x = 0$, then equation (2) gives

$$\lim_{\Delta y \to 0} \left[\frac{u(x, y + \Delta y) - u(x, y)}{i\,\Delta y} + i\,\frac{v(x, y + \Delta y) - v(x, y)}{i\,\Delta y}\right]. \qquad ...(5)$$

Since, $f(z)$ is differentiable, then the partial derivative $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ must exist and the limit is

$$\frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = v_y - iu_y. \qquad ...(6)$$

Since, the limit given by $\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$ is unique. So the limits given in (4) and (6) must be identical. Now equating the real and imaginary parts, we get

$$u_x = v_y \text{ and } u_y = -v_x$$

or

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Above two equation is known as Cauchy-Riemann partial differential equations.

(ii) Sufficient condition for a function $f(z)$ to be analytic.

Theorem 3. *A single valued continuous function*

$$f(z) = u(x, y) + iv(x, y)$$

is analytic in a domain D. If the four partial derivative u_x, v_x, u_y, v_y exist, are continuous and satisfy Cauchy-Riemann partial differential equations at every point of D.

Proof. Let $w = f(z) = u(x, y) + iv(x, y)$ be a single valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of a region D and satisfying the equation *i.e.*, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we shall show that $f(z)$ is analytic, *i.e.*, $f'(z)$ exists at every points of the region D.

By Taylor's theorem for functions of two variables, we have, on omitting second and higher degree terms of δx and δy

$$\begin{aligned}
f(z + \delta z) &= u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y) \\
&= \left[u(x, y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right)\right] + i\left[v(x, y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\delta y}\delta y\right)\right] \\
&= [u(x, y) + iv(x, y)] + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y \\
&= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y
\end{aligned}$$

$$\begin{aligned}
f(z + \delta z) - f(z) &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)i\,\delta y \qquad (\because\ i^2 = -1) \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\,\delta y) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta z \qquad (\because\ \delta z = \delta x + i\,\delta y)
\end{aligned}$$

$$\Rightarrow \quad \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Thus $f'(z)$ exists, because $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ exist. Hence $f(z)$ is analytic.

Cauchy-Riemann Equation in Polar Form :

Here, we have

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

So $\quad r^2 = x^2 + y^2 \qquad \text{...(1)}$

and $\quad \theta = \tan^{-1}\frac{y}{x}. \qquad \text{...(2)}$

Now, differentiating (1) and (2) partially w.r. to x and y, we get

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \text{ and } \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

and

$$\frac{\partial \theta}{\partial y} = \frac{1}{\left(1 + \frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \qquad (\because\ r^2 = x^2 + y^2)$$

Taking

$$\left.\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\cdot\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}
\end{aligned}\right\} \qquad \text{...(3)}$$

and
$$\left.\begin{aligned}\frac{\partial v}{\partial x}&=\frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\frac{\partial v}{\partial r}\cos\theta-\frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}\\ \frac{\partial v}{\partial y}&=\frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}=\frac{\partial v}{\partial r}\sin\theta+\frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}\end{aligned}\right\}. \quad ...(4)$$

Now by Cauchy-Riemann equation, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$. ...(5)

Using (3) and (4), (5) becomes

$$\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}=\frac{\partial v}{\partial r}\sin\theta+\frac{\partial v}{\partial \theta}\cos\theta \quad ...(6)$$

and
$$\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}=-\frac{\partial v}{\partial r}\cos\theta+\frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}. \quad ...(7)$$

Now multiplying (6) by cos θ, and (7) by sin θ and adding, we get

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}. \quad ...(8)$$

Again multiplying (6) by sin θ and (7) by cos θ, and substracting, we get

$$\frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial v}{\partial r}. \quad ...(9)$$

Equation (8) and (9) be the required Cauchy Riemann equation in polar form.

Conjugate function. If $f(z)=u+iv$ is a analytic function. If u and v satisfy the Laplace's equation, then u and v are called conjugate Harmonic function or conjugate function.

Harmonic function. If u is a function of x and y and u has continuous partial derivaive of first and second order and satisfies the Laplace's equaion then u is called a Harmonic function.

Orthogonal system. If $u(x,y)=c_1$ and $v(x,y)=c_2$ be the two families of curves then these two families are said to form an orthogonal system if they intersect at right angles at each of their points of intersection.

Firstly, differentiaing $u(x,y)=c_1$, we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}=m_1 \text{ (say)}.$$

Now differentiating $v(x,y)=c_2$, we get

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\cdot\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}=m_2 \text{ (say)}.$$

Now two families of curves intersect orthogonally if $m_1m_2=-1$

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}=0.$$

SOLVED EXAMPLES

Example 1. *Show that the function $f(z)=z^n$ is an analytic function, where n is a positive integer.*

Solution. Here, we have $f(z)=z^n$,

then, $$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{(z+\Delta z)^n-z^n}{\Delta z}.$$

Now $f'(z)$ exists if the above limits exists and does not depend on the manner in which $\Delta z\to 0$. By Binomial theorem, we have

$$f'(z)=\lim_{\Delta z\to 0}\left[nz^{n-1}+\frac{\Delta z}{2}(n-1)z^{n-2}+\ldots+\Delta^{n-1}z\right]=nz^{n-1}.$$

Therefore, $f'(z)$ exists for all finite values of z.

Hence, $f(z)$ is an analytic function.

Example 2. *Show that the function $f(z)=|z|^2$ is continuous everywhere but nowhere differentiable except at the origin.*

Solution. Here, the function $f(z)=|z|^2$ is continuous everywhere. Since x^2+y^2 is continuous every where.

Now $$f'(z_0)=\lim_{\Delta z\to 0}\frac{|z_0+\Delta z|^2-|z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\bar{z}_0 + \Delta\bar{z}) - z_0\bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\bar{z}_0 + \Delta\bar{z} + z_0 \frac{\Delta\bar{z}}{\Delta z}\right] = \lim_{\Delta z \to 0} \left[\bar{z}_0 + z_0 \frac{\Delta\bar{z}}{\Delta z}\right] \qquad (\because \ \Delta z \to 0 \Rightarrow \Delta\bar{z} = 0)$$

So at $z_0 = 0, f'(0) = 0$.

When $z_0 \neq 0$, let $\Delta z = r(\cos\phi + i\sin\phi)$, then $\Delta\bar{z} = r(\cos\phi - i\sin\phi)$ so that

$$\frac{\Delta\bar{z}}{\Delta z} = \frac{\cos\phi - i\sin\phi}{\cos\phi + i\sin\phi} = \cos 2\phi - i\sin 2\phi$$

which does not tend to a unique limit, since limit depends upon arg. Δz. Hence, the function $|z|^2$ is not differentiable for any non-zero value of z.

Example 3. *If $f(z) = u + iv$ is an analytic function of $z = x + iy$, then prove that the curves u = constant and v = constant on the z plane intersect at right angles.*

Solution. Let $f(z) = u + iv$ be an analytic function of z, then Cauchy- Riemann equation is $u_x = v_y$ and $u_y = -v_x$ satisfied.

Now, let slop of tangent to the curve $u = c_1$ is m_1

and slop of tangent to the curve $v = c_2$ is m_2.

To show that both the curve $u = c_1$ and $v = c_2$ is orthogonal we shall show that $m_1m_2 = -1$.

Taking differential of $u = c_1$ and $v = c_2$, we get

$$du = 0 \text{ and } dv = 0$$

or
$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \text{ and } \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$

or
$$m_1 = \frac{dy}{dx} = -\frac{u_x}{u_y} \text{ and } m_2 = \frac{dy}{dx} = -\frac{v_x}{v_y}.$$

So
$$m_1m_2 = \left(\frac{-u_x}{u_y}\right) - \frac{v_x}{v_y} = \frac{u_xv_x}{u_yv_y} = \frac{u_xv_x}{(-v_x)(u_x)} \qquad \text{(by C-R equation)}$$

$\Rightarrow$
$$m_1m_2 = -1.$$

Hence, both the curve intersect at right angle on z-plane.

• 6.15. CONSTRUCTION OF ANALYTIC FUNCTION

Milne's Thomson's method. We have $z = x + iy$ so that $x = \dfrac{z+\bar{z}}{2}$

and
$$y = \frac{z - \bar{z}}{2i}.$$

Now
$$w = f(z) = u + iv = u(x, y) + iv(x, y)$$

or
$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

The above relation is formal identity in two independent variable z and $\bar{z}$.

Taking $x = z$, $y = 0$ so that $z = \bar{z}$, we get

$$f(z) = u(z, 0) + iv(z, 0). \qquad \text{...(1)}$$

We know that

$$f'(z) = \frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}. \qquad \text{(by C-R equations)}$$

Now taking
$$\frac{\partial u}{\partial x} = \phi_1(x, y) = \phi_1(z, 0)$$

$$\frac{\partial u}{\partial y} = \phi_2(x, y) = \phi_2(z, 0)$$

we get
$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0).$$

On integration, we get

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz + c$$

where c is a constant. Now we can obtain $f(z)$ if u is known.

Similarly, if $v(x, y)$ is given, then

$$f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]\, dz + c'$$

where $\quad \psi_1 = \dfrac{\partial v}{\partial y}$ and $\psi_2 = \dfrac{\partial v}{\partial x}$.

Example 1. *Obtain the analytic function $f(z) = u + iv$, whose real part u is $e^x (x \cos y - y \sin y)$.*

Solution. Here, we have $\quad u = e^x (x \cos y - y \sin y)$

then
$$\frac{\partial u}{\partial x} = e^x (x \cos y - y \sin y) + e^x \cos y$$

and
$$\frac{\partial u}{\partial y} = e^x [- x \sin y - \sin y - y \cos y]$$

$$\left(\frac{\partial u}{\partial x}\right)_{y=0} = xe^x + e^x = e^x (x + 1)$$

and
$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = e^x \,.\, 0 = 0.$$

Now
$$\phi_1 (x, 0) = \left(\frac{\partial u}{\partial x}\right)_{y=0} = e^x (x + 1)$$

$$\phi_2 (x, 0) = \left(\frac{\partial u}{\partial y}\right)_{y=0} = 0.$$

Now, by Milne's thomson's method, we have
$$\begin{aligned} f(z) &= \int [\phi_1 (z, 0) - i\, \phi_2 (z, 0)]\, dz \\ &= \int [e^z (z + 1) - i \,.\, 0]\, dz + c = \int (ze^z + e^z)\, dz + c \\ &= e^z (z - 1) + e^z + c = ze^z + c. \end{aligned}$$

Hence, $\quad f(z) = ze^z + c.$

Example 2. *If $f(z) = u + iv$ and $u - v = e^x (\cos y - \sin y)$, find $f(z)$.*

Solution. Here, we have $u - v = e^x (\cos y - \sin y)$

then
$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x (\cos y - \sin y) \qquad ...(1)$$

and
$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^x (- \sin y - \cos y)$$

or
$$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = - e^x (\sin y + \cos y) \qquad \text{(By C-R equations)}$$

or
$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = e^x (\sin y + \cos y). \qquad ...(2)$$

Now, from (1) and (2)
$$\frac{\partial u}{\partial x} = e^x \cos y = \phi_1 (x, y) \quad \text{and} \quad \frac{\partial v}{\partial x} = e^x \sin y = \phi_2 (x, y).$$

Now
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \phi_1 (z, 0) + i\phi_2 (z, 0)$$

$\therefore$
$$\begin{aligned} f(z) &= \int [\phi_1 (z, 0) + i\phi_2 (z, 0)]\, dz + c \\ &= \int (e^z \cos 0 + ie^z \sin 0)\, dz + c = \int e^z\, dz + c \end{aligned}$$

or
$$f(z) = e^z + c.$$

Example 3. *If $f(z) = \dfrac{x^3 y (y - ix)}{x^6 + y^2}$, $z \neq 0$ and $f(0) = 0$ prove that $\dfrac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not $z \to 0$ in any manner i.e., $f(z)$ is not differentiable at $z = 0$.*

Solution. Here, we have
$$\frac{f(z) - f(0)}{z} = \frac{f(z) - 0}{z} = \frac{f(z)}{z} = \frac{x^3 y (y - ix)}{(x^6 + y^2)\, z} = \frac{- ix^3 y (x + iy)}{(x^6 + y^2)\, z} = \frac{- ix^3 y}{x^6 + y^2}.$$

Now we take the path $y = mx$ (radius vector)
$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{- ix^3\, mx}{x^6 + m^2 x^2} = \lim_{z \to 0} \frac{- imx^2}{m^2 + x^4} \neq 0.$$

Also, along the path $y = x^3$
$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{- ix^3 \,.\, x^3}{x^6 + x^6} = \frac{-i}{2} \neq 0.$$

Hence, $\lim_{z \to 0} \frac{f(z) - f(0)}{z} \neq 0$ along any path except radius vector.

Example 4. *Prove that an analytic function with constant modulus is constant.*

Or

Show that an analytic function cannot have a constant modulus without reducing to a constant.

Solution. Let $f(z) = u + iv$ be the given analytic function then u and v satisfy the equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad \text{...(i)}$$

We have, $|f(z)| = \text{constant} = c$.

Then $$u^2 + v^2 = c^2. \quad \text{...(ii)}$$

Now, differentiating (ii) partially w.r. to x and y, we get

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \text{ and } u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \quad u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \text{ and } u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0, \quad \text{[using (i)]}$$

Now eliminating $\frac{\partial u}{\partial y}$ in above equation, we get

$$(u^2 + v^2)\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0, \text{ provided } u + iv \neq 0.$$

Similarly, we have $$\frac{du}{\partial y} = 0 = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}.$$

Now, since $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are zero. So u and v are constant.

Hence, $f(z) = u + iv$ is a constant function.

Example 5. *If $f(z) = u + iv$ is an analytic function, then show that u and v are both Harmonic functions.*

Solution. Let $f(z) = u + iv$ is an analytic function then Cauchy-Riemann equation satisfied. *i.e.*,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad \text{...(1)}$$

Now, since u and v are the real and imaginary part of $f(z)$. So partial derivative of u and v exist and continuous function of x and y.

Now from equation (1), we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \, \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \, \partial x}.$$

Adding both the equations, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \, \partial y} - \frac{\partial^2 v}{\partial y \, \partial x} = 0.$$

Hence $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly, we can easily shown that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$

So the function u and v satisfy the Laplace equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$

Hence, u and v both are the Harmonic functions.

Example 6. *Show that the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ is continuous everywhere and analytic everywhere.*

Solution. Here, we have

$$f(z) = \sin x \cosh y + i \cos x \sinh y.$$

Now$u(x, y) = \sin x \cosh y$ and $v(x, y) = \cos x + \sinh y$.

Since, u and v both are the rational functions of x and y with non-zero denominators for all value of x and y. So u and v are boh continuous everywhere.

Now to show $f(z)$ is analytic everywhere, we have

$$\frac{\partial u}{\partial x} = \cos x \cosh y, \qquad \frac{\partial u}{\partial y} = \sin x \sinh y$$

and $\quad \dfrac{\partial v}{\partial x} = -\sin x \sinh y, \qquad \dfrac{\partial v}{\partial y} = \cos x \cosh y.$

So, by above relations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

So, u and v satisfying the Cauchy-Riemann equations.

Hence, $f(z)$ is analytic everywhere.

Example 7. *Show that the function $u(x, y) = e^x \cos y$ is harmonic. Determine its harmonic conjugate $v(x, y)$ and the analytic function $f(z) = u + iv$.*

Solution. We have $\quad u = e^x \cos y$

$\Rightarrow \quad \dfrac{\partial u}{\partial x} = e^x \cos y, \qquad \dfrac{\partial u}{\partial y} = -e^x \sin y$

and $\quad \dfrac{\partial^2 u}{\partial x^2} = e^x \cos y$ and $\dfrac{\partial^2 u}{\partial y^2} = -e^x \cos y$

which implies $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

Also, first and second order partial derivatives of u are continuous.

$\therefore$ u is a harmonic function.

Now, let v be the harmonic conjugate of u, therefore

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \text{ (By C-R equations)}$$

$$= e^x \sin y \, dx + e^x \cos y \, dy.$$

On integrating, we get

$$v = e^x \sin y + C.$$

Therefore, $\quad f(z) = u + iv = e^x \cos y + i\,(e^x \sin y + c)$

$$= e^x (\cos y + i \sin y) + ic = e^x . e^{iy} + ic$$

$$= e^{x+iy} + ic = e^z + d, \text{ where } d = ic, \text{ a complex constant.}$$

• SUMMARY

- Complex number $= \{z = x + iy : x, y \in R\}$.
- If $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$, $\arg(z) = \tan^{-1}\left(\dfrac{y}{x}\right)$.
- $z + \bar{z} = 2\text{Re}(z)$
- $z - \bar{z} = 2i \text{ Im}(z)$
- $z = \bar{z} \Leftrightarrow z$ is purely real.
- $z + \bar{z} = 0 \Rightarrow z$ is purely imaginary
- $\overline{z_1 \pm z_2} = \bar{z}_1 + \bar{z}_2$
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[|z_1|^2 + |z_2|^2]$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
- $z = |z|\, e^{i\theta}$
- Equation of a straight line is $\bar{\alpha} z + \alpha \bar{z} + k = 0$, where $\alpha \neq 0$ and k is real.
- Equation of a circle is $z\bar{z} + c\bar{z} + \bar{c} z + k = 0$ where k is real and c is a complex number.
- $C\text{–}R$ equations are $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$
- $u = f(z)$ is harmonic if $\nabla^2 u = 0$.

• STUDENT ACTIVITY

1. Prove that $|Z_1 + Z_2|^2 + |Z_1 - Z_2|^2 = 2\,[|Z_1|^2 + |Z_2|^2]$

2. Obtain the analytic functions $f(z) = u + iv$ whose real part u is $e^x (x \cos y - y \sin y)$.

• TEST YOURSELF

1. Show that the following function are Harmonic and find their Harmonic conjugate :
   (i)$u = \frac{1}{2} \log (x^2 + y^2)(ii)u = \cos x \cosh y$.
2. Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin, although Cauchy-Riemann equations are satisfied at the origin.
3. If $f(z) = \frac{xy^2 (x + iy)}{x^2 + y^4}$, $z \neq 0, f(0) = 0$, then prove that $\frac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.
4. Show that the following function are Harmonic and find the corresponding analytic function $u + iv$
   (i) $u = \sin x \cosh y + 2 \cos x . \sinh y + x^2 - y^2 + 4xy$ (ii) $u = e^x \cos y$.

ANSWERS

1. (i) $\tan^{-1} \frac{y}{x} + c$ (ii) $-\sin x \sinh y + c$
4. (i) $\sin z + z^2 - 2i (\sin z + z^2) + c$ (ii) $e^z + c$

FILL IN THE BLANKS :

1. A complex number is defined as an ordered pair (x, y) of .... numbers.
2. Two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are said to be equal if $x_1 = x_2$ and ....
3. Every non-zero element having multiplicative ....
4. Two complex numbers are said to be equal iff their conjugate are ....

TRUE OR FALSE :

Write 'T' for true and 'F' for flase statement :

1. Two complex numbers are said to be equal iff their conjugate are equal. *(T/F)*
2. A function, which is analytic is also called Holomorphic function. *(T/F)*
3. Continuity is a necessary but not a sufficient condition for differentiability. *(T/F)*
4. Argument of a complex number is unique. *(T/F)*
5. Conjugate of a complex number can be obtained by replacing i by $-i$ in the given complex number. *(T/F)*
6. A complex number is purely real if $z - \bar{z} = 0$. *(T/F)*

MULTIPLE CHOICE QUESTIONS :

Choose the most appropriate one :

1. The conjugate of $\frac{1}{2+i}$ is :
   (a) $\frac{2+i}{5}$ (b) $\frac{1}{2-i}$ (c) $\frac{2-i}{5}$ (d) $\frac{5}{2+i}$.
2. $\text{Arg } z + \text{Arg } \bar{z}$ $(z \neq 0)$ is :
   (a) 0 (b) π (c) $\pi/2$ (d) 12π.

ANSWERS

Fill in the Blanks :

1. Real 2. $y_1 = y_2$ 3. Inverse 4. Equal

True or False :

1. T 2. T 3. T 4. F 5. T 6. T

Multiple Choice Questions :

1. (a) 2. (a)