B.Sc. PCM-101

MECHANICS AND WAVE MOTION

DIRECTORATE OF DISTANCE EDUCATION

SWAMI VIVEKANAND
SUBHARTI UNIVERSITY
Meerut (National Capital Region Delhi)

MECHANICS AND WAVE MOTION

B Sc PCM-101

Self Learning Material

उत्तिष्ठत जाग्रत प्राप्य वरान्निबोधत

Directorate of Distance Education

SWAMI VIVEKANAND SUBHARTI UNIVERSITY
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PREFACE

In this course, we shall deal with various aspects of Mechanics and Wave Motion

- Newtonian Mechanics and Conservation Law
- Rotational Dynamics
- Motion Under Central Forces : Planets and Satellites
- Simple Harmonic Motion Free, Damped and Forced Vibrations
- Wave Motion

1. Newtonian Mechanics and Conservation Law : Frames of Reference, Frame of Reference, Inertial Frame of Reference, Motion in a Uniformly Accelerated Frame, Non Inertial Frames and Factitious Force, Uniformly Accelerated Frames and Limitations of Newton's Law, Newton's Law of Motion, Dynamics of Particle in Circular Motion, Conservative Force, Conservative Force as Negative Gradient of Potential Energy, Curl of Conservative Force, Conservation of Energy, Linear Momentum, Angular Momentum, Principal of Conservation of Angular Momentum, Angular Momentum of System of Particle about their Central of Mass, Elastic collision in two or three dimensions, Angle of Scattering and cross section.

2. Rotational Dynamics : Rotational Motion : Torque and Angular Momentum, Torque Acting on a Particle, Moment of Inertia, Kinetic Energy of a Rotating Body, Theorems of Moment of Inertia, Moment of Inertia of a Circular Disc, Moment of Inertia of an Circular Disc, Moment of Inertia of a solid Cylinder, Moment of Inertia of Cylinder about its Own Axis, Moment of Inertia of a Thin Spherical Shell, About Diameter, Body rolling down an inclined Plans, Precession, Relation between Elastic Constants, Bending of Beam, The Cantilever, Potential Energy and Oscillations of a Loaded Cantilever, Beam Supported at its ends and Loaded in the Middle. Applications of Bending of Beams, Torsion of a Cylinder, Work Done in twisting a wire or Cylinder.

3. Motion Under Central forces : Planets and Satellites : Central Force, Main Features of Central Force, Conservative Nature of Central Force, The work done around of closed path, For Central Forces (Conservative Forces), the Work Done Around a Closed Path is Zero, Angular Momentum Under a Central Force is Conserved, Motion Under a Central Force Takes Place in a Fixed Plane, Areal Velocity Under Central Force Remains Constant, Reduction of Two Body Problem to One Body Problem : (Motion of a System of Two Particles Under Central Force), Relative and Centre of Mass Coordinates, The Law of Universal Gravitation, Motion Under Inverse Square Law—Kepler's Law of Planetary Motion, Conclusions of Newton From Kepler's Laws, Kepler's Laws From Newton's Law of Gravitation, Motion of Satellites, Geostationary Satellite, Escape Velocity and Orbital Velocity.

4. Simple Harmonic Motion Free, Damped and Forced Vibrations : Simple Harmonic Oscillations, Simple Harmonic Motion, Equation of Motion of a Simple Harmonic Oscillator, Energy of a Particle Executing Simple Harmonic Motion, Time-average of Kinetic Energy and Potential Energy, Period of Oscillation of a Mass Suspended by a Spring, Frequency of Mass Connected With two Springs in Horizontal Position, Simple Pendulum, Component Pendulum, Torsion Pendulum, Damped Harmonic Oscillator, Power Dissipation in the Weak Damping Limit, Relaxation Time, Quality Factor Q, Forced (or Driven) Harmonic Oscillator, Composition of two Perpendicular Simple Harmonic Motions (S.H.M's) : (Lissajous' Figures).

5. Wave Motion : Wave Motion, Progressive Wave, Characteristics of Medium for Mechanics Waves, Types of Mechanical Waves, Some Definition Regarding waves, Relation Between Frequency, Wave-Speed and Wavelength, Equation of a Plane-Progressive Wave, Graphical Representation of Particle-Displacement Against time and Distance in a Progressive Wave, Particle Velocity and Acceleration, Relation Between Particle Velocity and Wave Velocity, Differential Equation of Wave Motion, General Solution of Wave Equation, Energy in a Progressive Wave, Pressure Variation in Longitudinal Waves, Calculation of Velocity of Sound in air, Effect of Various Factors (Pressure, Temperature, Humidity Etc.) on Velocity of Sound, Stationary Waves.

SYLLABUS

UNIT – I

Inertial reference frame, Newton's laws of motion, Dynamics of particle in rectilinear and circular motic
Conservative and Non -conservative forces, Conservation of energy, linear momentum and angular momentu
Collision in one and two dimensions, cross section.

UNIT – II

Rotational energy and rotational inertia for simple bodies, the combined translation and rotational and motion of rigid body on horizontal and inclined planes, Simple treatment of the motions of a top. Relations between elast constants, bending of Beams and Torsion of Cylinder.

UNIT – III

Central forces, Two particle central force problem, reduced mass, relative and centre of mass motion, Law o gravitation, Kepler's laws, motions of planets and satellites, geo-stationary satellites.

UNIT – IV

Simple harmonic motion, differential equation of S. H. M. and its solution, uses of complex notation, damped an forced vibrations, composition of simple harmonic motion.

Differential equation of wave motion, plane progressive waves in fluid media, reflection of waves, phase change o reflection, superposition, stationary waves, pressure and energy distribution, phase and group velocity.

CONTENTS

UNIT 1

NEWTONIAN MECHANICS AND CONSERVATION LAW

STRUCTURE

LEARNING OBJECTIVES

After learning this chapter, you will be able to know ....

- Frame of reference and limitations of Newtons Law
- Conservative force, conservation energy, linear and angular momentum
- Elastic collision two or three dimension
- Angle of scattering and cross section.

• 1.1. FRAMES OF REFERENCE

A system of coordinate axes, relative to which the position and motion of an object is described, is called a frame of reference. Thus a frame of reference is a system of coordinate axes which defines the position of a particle or an event in two or three dimensional space.

The simplest frame of reference is the cartesian coordinate system in which the position of a particle is specified by three space coordinates x, y, z. Then the position *vector of the particle with respect to the origin, is given by*

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

where **i**, **j**, **k** are the unit vector along the three axes X, Y and Z respectively.

Then the velocity v and acceleration a of the particle are given by

$$v = \frac{d\mathbf{r}}{dt} = \mathbf{i}\,\frac{dx}{dt} + \mathbf{j}\,\frac{dy}{dt} + \mathbf{k}\,\frac{dz}{dt}$$

$$a = \frac{dv}{dt} = \frac{d^2 r}{dt^2}$$

For complete identification of an event*, we require its position and time of occurrence. Therefore, in addition to the usual three spatial coordinates x, y and z, we require yet another coordinate, that of time t. A reference frame with such four coordinates (x, y, z, t) is called **space-time frame of reference**.

There is an infinite number of reference frames in the universe. Our earth is itself a frame of reference. The other examples of the frame of reference may be considered to be the walls of the room, the position of stars along plumb line, the sun etc.

Three are two types of frame of reference :

(a) Inertial frame of reference

(b) Non-internal frame of reference

1.2. INERTIAL FRAMES OF REFERENCE

Those frames of reference, **in which Newton's first and second laws hold true, are called inertial frames.** They are also known as Newtonian or Galilean frames. In such frames, if a body is not acted upon by any external force, it continues in its state of rest or moves with constant uniform velocity. It means that in an inertial frame, the acceleration

$$a = \frac{d^2 r}{dt^2} = 0 \quad \text{or} \quad \frac{d^2 x}{dt^2} = 0, \qquad \frac{d^2 y}{dt^2} = 0$$

and $$\frac{d^2 z}{dt^2} = 0 \text{ (because the applied force } F = m\,a = 0)$$

Thus **inertial frames are unaccelerated.** If the frame is accelerated, a body moving with constant velocity will appear accelerated in this frame.

We may, therefore, define **inertial frames as the frames with respect to which an unaccelerated body is unaccelerated** *i.e.*, is at rest or moving with constant linear velocity.

A **frame of reference moving with constant velocity relative to an inertial frame is also inertial**, because acceleration of the body in both the frames is zero. The body, however, has different but uniform velocity in these frames.

Experiments show that a **frame of reference fixed in stars is an inertial frame.** All other frames of reference in uniform motion relative to it also serve as equivalent inertial frames. However, a frame fixed on **earth is not an inertial frame** because the earth itself is rotating.

1.3. MOTION IN A UNIFORMLY ACCELERATED FRAME

Consider two frames of reference S and S' which have their origins O and O' coinciding initially. Let frame S' be moving with acceleration a_0 with respect to frame S. The position and velocity of origin O' of frame S' is determined by position vector r_0 and velocity vector v_0 in frame S.

Consider a point P in space which is denoted by radius vector r in frame S and by r' in frame S'. If point P has velocity v and acceleration a in frame S, then our problem is to find the corresponding values $\vec{v}\,'$ and a' in frame S'.

From fig 1

$$r' = r_0 + r \qquad \ldots (i)$$

If point P suffers a displacement dr in frame S and dr' in frame S' during a time interval dt, then

*Any thing which occurs suddenly in space, is known as event. it involves both, a position and a time of occurence.

$$d\mathbf{r} = d\mathbf{r}_0 + d\mathbf{r}' \qquad \text{... (ii)}$$

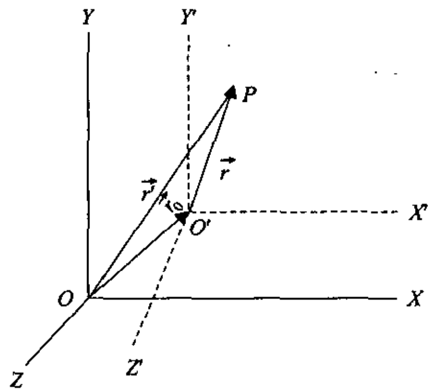

Fig. 1.

Dividing eq. (ii) by dt, we get

$$\mathbf{v} = \mathbf{v} - \mathbf{v}'$$

or
$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0 \qquad \text{... (iii)}$$

Differentiating eq. (iii) with respect to time, we have

$$\mathbf{a}' = \mathbf{a} - \mathbf{a}_0 \qquad \text{... (iv)}$$

If a particle of mass m be placed at point P, then **force on the particle in frame S′**

$$\mathbf{F}' = m\mathbf{a}' = m\,(\mathbf{a} - \mathbf{a}_0) = \mathbf{F} - m\mathbf{a}_0$$
$$= \mathbf{F} - \mathbf{F}_0 \qquad \text{... (v)}$$

where **F** is the force on the particle as seen by an observer in **frame S** and $\mathbf{F}_0$ is the force **due to relative acceleration** $\mathbf{a}_0$ between the two frames.

If $\mathbf{F} = 0$, then $\qquad \mathbf{F}' = -\mathbf{F}_0$

i.e., the **particle seems to experience a force** $-\mathbf{F}_0$ **when viewed from S′ even when there is no force on it in frame** S. This force is known as **fictitious force or pseudo force.** The details of fictitious force are, however, discussed in next article.

• 1.4. NON-INERTIAL FRAMES AND FICTITIOUS FORCES

The fundamental physical laws and principles are identical in all inertial frames of reference. If, however, a frame of reference is accelerated relative to an inertial frame, the form of the fundamental physical laws, such as Newton's second law, become completely different. Such *accelerated frames of reference are called non-inertial frames.* Infact, all accelerated and rotating frames are non-inertial frames of reference. Therefore, *non-inertial frames may be defined as those frames of reference in which Newton's first and second laws are not valid.* Thus if the observed acceleration of body of mass m in a non-inertial frame is **a**, then the formula $\mathbf{F} = m\mathbf{a}$ does not hold true in this reference frame.

Consider a non-inertial frame of reference S' moving with an acceleration $\mathbf{a}_0$ relative to an inertial frame S. Then, a particle P, or, in fact, all the particles which are at rest with respect to frame S, will appear to move with an acceleration $-\mathbf{a}_0$ with respect to frame S'. Thus, in frame S', a force $-m\mathbf{a}_0$ will appear to be acting on the particle of mass m. Such a *force, which does not really act, but appears only due to the acceleration of the frame of reference, is called a fictitious, apparent or a pseudo force.* This fictitious force on the particle P of mass m is given by

$$\mathbf{F}_p = -m\mathbf{a}_0 \qquad \text{... (1)}$$

Now if this mass m has an acceleration $\mathbf{a}_i$ in the inertial frame S, then the force which acts on this mass in frame S is

$$\mathbf{F} = m\mathbf{a}_i \qquad \text{... (2)}$$

Hence the total force acting on the particle as observed in frame S' is given by

$$\mathbf{F} = m\mathbf{a}_i - m\mathbf{a}_0$$

$$= m(\mathbf{a}_i - \mathbf{a}_0) = m\mathbf{a} \quad ...(3)$$

where $\mathbf{a}_i - \mathbf{a}_0 = \mathbf{a}$ is the acceleration of the particle as observed in frame S'.

Combining equations (1), (2) and (3)

$$\mathbf{F}' = m\mathbf{a} = m\mathbf{a}_i - m\mathbf{a}_0$$

or
$$\mathbf{F}' = \mathbf{F} + \mathbf{F_P} \quad ...(4)$$

Examples of fictitious force.

(i) Consider a particle of mass m at rest inside a lift falling freely under gravity. Then the fictitious force acting on it

$$F_p = -mg$$

The real force on the particle due to the attraction of the earth

$$F = mg$$

Therefore, the resultant force on the particle, as observed by an observer in the lift (the moving frame)

$$\mathbf{F}' = \mathbf{F} + \mathbf{F}_p = mg - mg = 0$$

i.e., the particle is weightless and thus appears to remain suspended in space. Thus the body is unaccelerated in this frame.

(ii) Consider a point mass m at rest in a non inertial frame (earth) so that in this frame the value of observed acceleration is $\mathbf{a}_0$. The non-inertial frame rotates uniformly with angular velocity ω about an axis fixed with respect to an inertial frame. The acceleration of the point mass with respect to the inertial frame may be written as

$$\mathbf{a}_0 = -\omega^2 \mathbf{r}$$

where $\mathbf{r}$ is the position vector of the particle and is directed outwards to the particle from the axis.

Therefore, the fictitious force on the mass m as observed from the non-inertial (rotating) frame of reference

$$\mathbf{F_p} = -m\mathbf{a}_0 = m\omega^2\mathbf{r}.$$

This fictitious force is called *centrifugal force* and is obviously directed away from the centre. It is on account of these forces that an observer in a rotating frame finds mysterious forces not accounted for by any known origin of force, throwing things outwards towards the wall.

• 1.5. UNIFORMLY ACCELERATED FRAMES AND LIMITATIONS OF NEWTON'S LAWS

If an observer is placed in an accelerated frame of reference, then he himself is *accelerated* and to him the **Newton's first** and **second laws would appear to be violated.**

If a particle is not acted upon by any force, then its acceleration is zero relative to an observer but another observer, moving with an acceleration relative to first, will see the particles to be in acceleration due to the fictitious force acting on the particle relative to second observer. Thus *Newtons 1st and IInd laws are valid only if the observer is situated in an inertial frame.*

Newton's *third law of motion also ceases to hold good for particles of atomic dimensions.* According to third law, the forces exerted by two interacting bodies over each other are equal and opposite provided they are both measured simultaneously. But if two particles are placed at a sufficient distance, then if the first particle produces any change in the second particle, the changed force (or reaction) will reach the first after a finite interval of time (because the forces of actions can not move faster than the speed of light). This means that simultaneous actions and reactions are not equal and hence Newton's third law is not correct when a force is acting at a distance.

Student Activity

(1) Distinguish between inertial and non-inertial frame ?

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(2) Is earth an inertial frame ?

• 1.6. NEWTON'S LAW OF MOTION

The three laws of Newton are stated as follows :

1. First law : A body continues in its state of rest or uniform motion in a straight line in the same direction unless some external force is applied to it. This law is also called *Galilio law or law of inertia.*

2. Second Law : The rate of change of linear momentum ($\mathbf{p} = m\mathbf{v}$) of a body is proportional to the force applied and it takes place in the direction of the force,

i.e. $$\mathbf{F} \propto \frac{d\mathbf{p}}{dt}$$

or $$\mathbf{F} = k\frac{d}{dt}(m\mathbf{v})$$

$$= km\frac{d\mathbf{v}}{dt} \text{ [if mass } m \text{ is constant]}$$

ie. $$\mathbf{F} = km\mathbf{a} \quad \left(\text{since } \mathbf{a} = \frac{d\mathbf{v}}{dt} = \text{acceleration}\right)$$

Defining force in such a way that $k = 1$, then Newton's second law is written as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}. \qquad \dots (1)$$

3. Third law : ***To every action, there is an equal and opposite reaction and action and reaction act on different bodies.***

Out of these, Newton's second law is most general as first and third laws may be derived from second law. Hence equation (1) is known as *general equation of motion.* The inherent limitation of equation of motion $F = ma$ is that it would not hold if mass does not remain constant *e.g.*, in the case of (i) a falling rain drop which may condense more water vapour around it during its fall (ii) a rocket whose mass decreases continuously due to ejected mass in the form of gases during its forward motion and (iii) particles moving with relativistic speeds due to which mass increases with increase of velocity in accordance with relation $m = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$.

• 1.7. DYNAMICS OF PARTICLE IN CIRCULAR MOTION

When a particle moves in a circle with constant speed, it is said to perform a circular motion. In this motion, the velocity of the particles changes continuously in direction but not in magnitude. Therefore, the particle experiences an acceleration.

Consider a particle P moving along the circumference of a plane circle of radius r (Fig. 2) as observed in an inertial frame of reference. Let O be the centre of the circle.

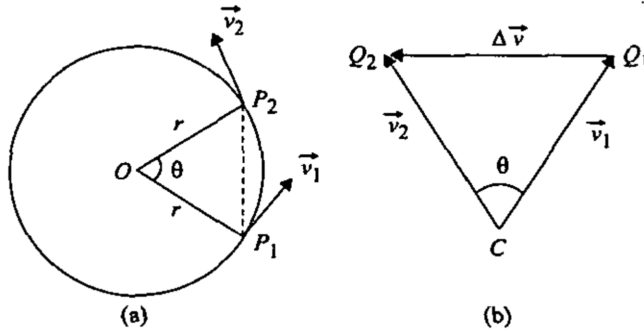

Fig. 2

Let P_1 be the position and $\vec{v_1}$ the velocity of the particle at time t and P_2 be position and $\vec{v_2}$ velocity at time $t + \Delta t$. Then length of the path traversed during the time interval Δt is equal to $\quad \text{arc } P_1P_2 = v\,\Delta t.$
where v is the constant speed of the particle.

To determine the change in velocity $\Delta \vec{v} = \vec{v_2} - \vec{v_1}$ from P_1 to P_2, we draw vectors $\vec{v_1}$ and $\vec{v_2}$ from a common point C (Fig. 2.b). Then triangles OP_1P_2 (Fig. 2a) and CQ_1Q_2 (Fig. 2 b) are similar. Therefore,

$$\frac{\text{chord } P_1P_2}{OP_1} = \frac{Q_1Q_2}{CQ_1}$$

or
$$\frac{v\,.\,\Delta t}{r} = \frac{\Delta v}{v} \qquad (\because \quad |\vec{v_1}| = |\vec{v_2}| = v)$$

$\therefore$
$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

or
$$a = \frac{v^2}{r}$$

where $a = \dfrac{\Delta v}{\Delta t}$ is the magnitude of the instantaneous acceleration $\vec{a}$ of the particle. The direction of $\vec{a}$ is the same as that of $\Delta \vec{v}$, *i.e.*, along the radius pointing towards the centre of the circle. Therefore, it is called '*radial*' or *centripetal* (means seeking centre) *acceleration*.

Since velocity $\vec{v}$ is always tangent to the circle in the direction of motion and acceleration $\vec{a}$ is always directed radially inward, therefore, $\vec{v}$ and $\vec{a}$ are always perpendicular to each other.

Centripetal Force

A particle in uniform circular motion must have a force $\vec{F}$ acting on it which is defined by Newton's second law

$$F = ma = \frac{mv^2}{r}$$

where direction of $\vec{F}$ at any instant is the direction of $\vec{a}$ at that instant, *i.e.*, radially inward. That is why it is called centripetal force.

Student Activity

(1) Explain uniform circular motion and centripetal acceleration.

....

....

• 1.8. CONSERVATIVE FORCE

A force acting on a particle is said to be conservative if the work done by a force in moving a particle from one point to another is independent of the path.

Consider two points P and Q and let C_1 and C_2 be the paths traversed by the particle from P and Q as shown in fig. 3. (a).

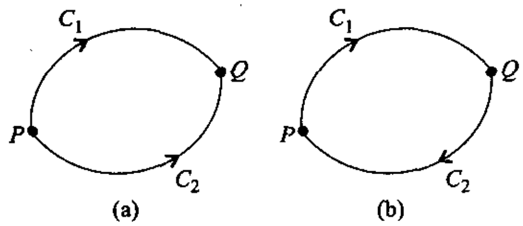

Fig. 3.

The work done by the force along path C_1 in moving the particle from P to Q is

$$W_{P \underset{1}{\to} Q} = \int_{\substack{P \\ C_1}}^{Q} \mathbf{F} \cdot d\mathbf{r}.$$

The work done by the force along path C_2 in moving the particle from P to Q is

$$W_{P \underset{2}{\to} Q} = \int_{\substack{P \\ C_2}}^{Q} \mathbf{F} \cdot d\mathbf{r}.$$

According to definition of conservative force $W_{P \underset{1}{\to} Q} = W_{P \underset{2}{\to} Q}$

i.e.,
$$\int_{\substack{P \\ C_1}}^{Q} \mathbf{F} \cdot d\mathbf{r} = \int_{\substack{P \\ C_2}}^{Q} \mathbf{F} \cdot d\mathbf{r} \qquad \dots (1)$$

Now consider a motion in which the particle goes from a point P to a point Q along path C_1 and back from point Q to point P along path C_2 as shown in fig. 3 (b).

The work done by conservative force in moving the particle from P to Q along C_1 and then from Q to P along C_2 is given by

$$W = \int_{\substack{P \\ C_1}}^{Q} \mathbf{F} \cdot d\mathbf{r} + \int_{\substack{Q \\ C_2}}^{P} \mathbf{F} \cdot d\mathbf{r} = \int_{\substack{P \\ C_1}}^{Q} \mathbf{F} \cdot d\mathbf{r} - \int_{\substack{P \\ C_2}}^{Q} \mathbf{F} \cdot d\mathbf{r}$$

$$= W_{P \underset{1}{\to} Q} - W_{P \underset{2}{\to} Q} = 0 \quad \text{using (1).}$$

Thus we may state that the work done by a conservative force around closed path is zero. Thus alternatively a conservative force is defined as : ***A force acting on a particle is said to be conservative if the kinetic energy of the particle remains unchanged for a complete round trip.***

Examples of the conservative forces are gravitational, elastic, electrostatic and magnetic forces.

The velocity dependent forces like frictional and viscous forces are non-conservative. For example, consider that a block is pushed along a table with friction. We have to do the work against friction in moving the block from P and Q. The force of friction will bring the sliding block to rest and dissipate its kinetic energy. To return the block from Q to P, we have again to do work against friction because the force of friction is always directed opposite to the direction of motion. Thus the total work done around the closed path PQP can never be zero; consequently a frictional force is non-conservative.

However, some velocity dependent forces are conservative. For example, the magnetic force $q\,(\mathbf{v} \times \mathbf{B})$ acting on a particle of charge q placed in a magnetic field of intensity $\mathbf{B}$ and moving with a velocity $\mathbf{v}$ is conservative, because this force is perpendicular to the direction of motion, hence the work done due to this force in going along any path is zero.

• 1.9. CONSERVATIVE FORCE AS NEGATIVE GRADIENT OF POTENTIAL ENERGY

If **F** is conservative force, then according to definition of potential energy

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = V_P - V_Q$$

If dV is the increase in potential energy of a particle when it suffers a displacement $d\mathbf{r}$ under the influence of conservative force, then we have

$$\int_{P.}^Q \mathbf{F} \cdot d\mathbf{r} = -\int_P^Q dV$$

or
$$dV = -\mathbf{F} \,.\, d\mathbf{r}. \qquad \ldots (1)$$

This equation simply indicates that the work done by conservative force is equal to the corresponding decrease in potential energy of the particle.

The displacement $d\mathbf{r}$ may be expressed as

$$d\mathbf{r} = \mathbf{i}\ dx + \mathbf{j}\ dy + \mathbf{k}\ dz$$

Now total increase in potential energy of the particle as a result of displacement $d\mathbf{r}$ can be expressed as

$$dV = \frac{\partial V}{\partial x}\, dx + \frac{\partial V}{\partial y}\, dy + \frac{\partial V}{\partial z}\, dz \qquad \ldots (2)$$

In this equation I, II, III terms represent increase in V on account of increments dx in x, dy in y, dz in z, respectively.

Equation (2) may be written as

$$dV = (\nabla V) \,.\, d\mathbf{r} \qquad \ldots (3)$$

where
$$\nabla \text{ (del)} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Substituting value of dV from (1) in (3), we get

$$\mathbf{F} \cdot d\mathbf{r} = -(\nabla V) \cdot d\mathbf{r} \quad \text{or} \quad (\mathbf{F} + \nabla V) \cdot d\mathbf{r} = 0$$

Since this equation is true for all arbitrary infinitesimal displacements $d\mathbf{r}$, therefore, we get

$$\mathbf{F} + \nabla V = 0$$

or
$$\mathbf{F} = -\nabla V = -\text{grad } V \qquad \ldots (4)$$

This equation represents very important property of conservative forces *viz.*

For every conservative force field we can always define the potential energy V such that the negative gradient of potential energy V at any point gives in magnitude and direction the force experienced by a particle when placed at that point. In other words,

Any conservative force* F *can always be expressed as the negative gradient of potential energy.

• 1.10. CURL OF A CONSERVATIVE FORCE

If **F** is conservative force and V is the potential energy function, then

$$\mathbf{F} = -\text{grad } V$$

$$\therefore \quad \text{curl } \mathbf{F} = \text{curl } (-\text{grad } V) = -\text{curl grad } V = -\nabla \times (\nabla V)$$

$$= -\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) + \left(\hat{\mathbf{i}}\frac{\partial V}{\partial x} + \hat{\mathbf{j}}\frac{\partial V}{\partial y} + \hat{\mathbf{k}}\frac{\partial V}{\partial z}\right)$$

$$= -\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial x} \end{vmatrix}$$

$$= -\left[\hat{\mathbf{i}}\left(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y}\right) - \hat{\mathbf{j}}\left(\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial z \partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x}\right)\right]$$

As V is perfect differential so $\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}$ etc. $\therefore$ curl $\mathbf{F} = 0$.

That is the curl of a conservative force is always zero.

Student Activity

(1) Explain the difference between conservative and non conservative forces.

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(2) Show that the conservative force can be expressed as $\vec{F} = -\,grad\, U$, where U is potential energy.

• 1.11. THE LAW OF CONSERVATION OF ENERGY : THE ENERGY FUNCTION

Let a particle be displaced from a certain point P to a point Q under the influence of a conservative force. Then from the definition of kinetic energy the amount of work done on the particle is

$$W_{PQ} = \int_P^Q \mathbf{F} \cdot d\mathbf{r} = T_Q - T_P = \text{gain in K.E.} \qquad \ldots (1)$$

and according to definition of potential energy (in conservative field)

$$W_{PQ} = \int_P^Q \mathbf{F} \cdot d\mathbf{r} = V_P - V_Q \qquad \ldots (2)$$

From (1) and (2)

$$T_Q - T_P = V_P - V_Q$$

or

$$V_P + T_P = V_Q + T_Q = \ldots = E \text{ (say)} \qquad \ldots (3)$$

This equation indicates another very important property of consevative force *viz.* ***The sum of kinetic and potential energies of a particle in conservative force field remains invariant.*** The energy function E defined by

$$E = T + V$$

is called the total energy of the particle. Summarizing we have

In a conservative force field, the total energy E (= $T + V$) of a particle remains constant.

• 1.12. LINEAR MOMENTUM

The linear momentum of a particle may be defined as the product of its mass and velocity vector. Thus if a particle of mass m is moving with velocity $\mathbf{v}$, its momentum $\mathbf{p}$ is given by

$$\mathbf{p} = m\mathbf{v}. \qquad \ldots (1)$$

The momentum is a vector quantity being the product of a scalar quantity m by a vector $\mathbf{v}$. The unit of linear momentum is kg m/s. or newton-second.

Principle of conservation of linear momentum

(i) For a particle : According to Newton's second law the rate of change of momentum of the particle is proportional to the force impressed on the particle and is in the direction of the force. Therefore in proper unit

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \quad \text{where } \mathbf{p} = m\mathbf{v} \qquad \ldots$$

where **F** is the force impressed on the particle.

If the force acting on the particle is zero, we have

$$\frac{d\mathbf{p}}{dt} = 0$$

or $$\mathbf{p} = m\mathbf{v} = \text{constant.}$$

*i.e., **if the resultant force acting on the particle is zero, the linear momentum** **the particle remains unaltered.*** This is the principle of conservation of line momentum of a particle.

(ii) For a system of particles : Consider a system of particles (or a bo constituted of n particles) having masses $m_1, m_2, m_3, \ldots, m_n$. If M is the total mass the system, we have

$$M = m_1 + m_2 + m_3 + \ldots + m_n$$

Suppose that particles of the system or body interact with each other and are al acted upon by external forces. Every particle will have velocity and hence w contribute to the momentum.

If $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots \mathbf{p}_i, \ldots \mathbf{p}_n$ are the momenta of the particles of mass $m_1, m_2, m_3, \ldots m_i, \ldots, m_n$ respectively, then total momentum **P**, which is the vect sum of momenta of individual particles, is given by

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots \mathbf{p}_i + \ldots + \ldots \mathbf{p}_n. \qquad \ldots ($$

Differentiating with respect to time, we get

$$\frac{d\mathbf{P}}{dt} = \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} + \frac{d\mathbf{p}_3}{dt} + \ldots \frac{d\mathbf{p}_n}{dt}$$

or $$\frac{d\mathbf{P}}{dt} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots + \mathbf{F}_n \qquad \ldots ($$

where $\mathbf{F}_1$ denotes the force acting upon the particle of mass m_1, $\mathbf{F}_2$ the force on particle mass m_2 and so on. [Since force is defined as rate of change of momentum].

In the system of particles the forces include external and internal forces both.

$$\frac{d\mathbf{P}}{dt} = (\mathbf{F}_1^{ext} + \mathbf{F}_1^{int}) + (\mathbf{F}_2^{ext} + \mathbf{F}_2^{int}) + \ldots + (\mathbf{F}_n^{ext} + \mathbf{F}_n^{int})$$

$$= (\mathbf{F}_1^{ext} + \mathbf{F}_2^{ext} + \ldots + \mathbf{F}_n^{ext}) + (\mathbf{F}_1^{int} + \mathbf{F}_2^{int} + \ldots + \mathbf{F}_n^{int}) \qquad \ldots ($$

which superscript ext and int represent external and internal forces respectively.

The individual particles experience internal forces but if we consider the system as a whole, the internal forces contribute nothing to the total force : because they occur in pairs of equal and opposite forces and hence they cancel out as shown in fig. 4. In figure symbol f_{12} represents the internal force due to particle of mass m_2 on particle of mass m_1 and f_{21} the force on particle '2' due to particle 1 ... and so on, *i.e.*,

$$\mathbf{F}^{int} = \mathbf{F}_1^{int} + \mathbf{F}_2^{int} + \ldots + \mathbf{F}_n^{int} = 0.$$

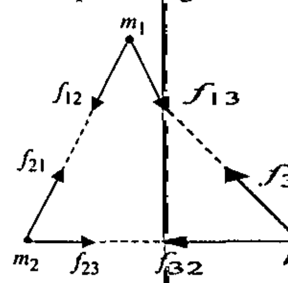


Fig. 4.

Hence equation (5) may be written as

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_1^{ext} + \mathbf{F}_2^{ext} + \mathbf{F}_3^{ext} + \ldots + \mathbf{F}_n^{ext} = \mathbf{F}^{ext} \qquad \ldots ($$

where $\mathbf{F}^{ext}$ is the resultant external force acting on the system. The internal forc cannot change the total momentum of the system, because being equal and opposi they produce equal and opposite changes in the momentum. Therefore, tot momentum of the system can only be changed by changing the external forc impressed upon the system.

Thus if $$\mathbf{F}^{ext} = 0, \quad \mathbf{P} = \text{constant.} \qquad \ldots ($$

i.e., ***if the external impressed force on the system is zero, the total momentum* (P) *of the system is constant.*** This is the principle of conservation of linear momentum for a system of particles.

It is to be noted here that the momenta of the individual particles of the system may change; but their sum or total momentum of the system remains unchanged unless the external force is applied.

• 1.13. ANGULAR MOMENTUM

Angular momentum of a particle about an axis is defined as the moment of *linear moment about that axis. Analytically it is defined as follows.*

Let there be a particle of mass m at a position $\mathbf{r}$ from any arbitrary fixed point O of the co-ordinate system. If $\mathbf{p}$ is the linear momentum of the particle, the angular momentum of the particle with respect to the fixed point O as origin is a vector quantity represented by symbol $\mathbf{J}$ and is defined as

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v})$$

The units of angular momentum are kg m^2 s^{-1} or joule second. The component of $\mathbf{J}$ along any axis passing through the fixed origin is known as angular momentum of the particle about the axis.

Further angular momentum plays the same role in rotational motion as linear momentum in translational motion.

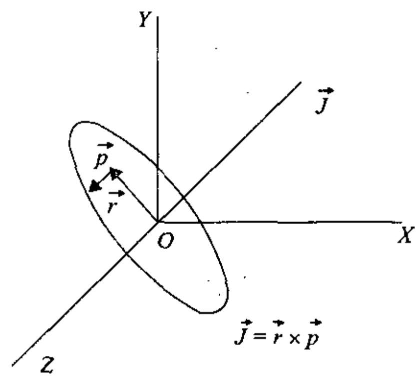


Fig. 5.

For **a system of particles** the angular momentum about any point may be found by adding vectorially the angular momenta of all the individual particles about the point.

The angular momentum of ith particle about any fixed point O may be expressed as

$$\mathbf{J}_i = \mathbf{r}_i \times \mathbf{p}_i = \mathbf{r}_i \times m_i \ \mathbf{v}_i \qquad \ldots (2)$$

where $\mathbf{r}_i$ is the position vector of ith particle relative to O and $\mathbf{p}_i$ is the linear momentum of ith particle.

∴ The net anglar momentum of the system of n particles may be written as

$$\mathbf{J} = \sum_{i=1}^{n} \mathbf{J}_i = \Sigma \, \mathbf{r}_i \times \mathbf{p}_i \qquad \ldots (3)$$

• 1.14. PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM

For a particle, angular momentum

$$\mathbf{J} = \mathbf{r} \times \mathbf{p}. \qquad \ldots (1)$$

Differentiating equation (1) with respect to time, we get

$$\frac{d\mathbf{J}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \left(\mathbf{r} \times \frac{d\mathbf{p}}{dt}\right) + \left(\frac{d\mathbf{r}}{dt} \times \mathbf{p}\right) \qquad \ldots (2)$$

where $\frac{d\mathbf{r}}{dt}$ is the velocity and $\frac{d\mathbf{p}}{dt}$ is the rate of change of linear momentum of the particle at any instant.

Therefore equation (2) can be written as

$$\frac{d\mathbf{J}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \{\mathbf{v} \times (m\mathbf{v})\}.$$

The second term vanishes because cross product of two parallel vectors is zero.

$$\therefore \qquad \frac{d\mathbf{J}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \mathbf{F}$$

Since $\frac{d\mathbf{p}}{dt} = \mathbf{F}$, force acting on the particle.

But $\mathbf{r} \times \mathbf{F} = \tau =$ torque (*i.e.*, moment of force)

$$\tau = \frac{d\mathbf{J}}{dt} \qquad \dots (3)$$

"Therefore rate of change of vector angular momentum of a particle is equal to torque acting on it."

If $\qquad \tau = 0,\ \frac{d\mathbf{J}}{dt} = 0 \qquad$ or $\quad \mathbf{J} =$ constant.

Thus if torque acting on the particle is zero, its angular momentum is conserved. This is the conservation theorem of angular momentum for a particle.

For a system of particles, the angular momentum is

$$\mathbf{J} = \sum_i \mathbf{r}_i \times \mathbf{p}_i \qquad \dots (4)$$

Differentiating above equation with respect to time t, we get

$$\frac{d\mathbf{J}}{dt} = \sum_i \left[\mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} + \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i \right]$$

$$= \sum_i \mathbf{r}_i \times \mathbf{F}_i + \Sigma\, \mathbf{v}_i \times (m_i\ \mathbf{v}_i) \qquad (d\mathbf{r}_i / dt = \mathbf{v}_i \text{ and } \mathbf{p}_i = m_i \mathbf{v}_i)$$

But $\qquad \mathbf{v}_i \times (m_1 \mathbf{v}_i) = m_i\ (\mathbf{v}_i \times \mathbf{v}_i) = 0$

$$\therefore \qquad \frac{d\mathbf{J}}{dt} = \sum_i \mathbf{r}_i \times \mathbf{F}_i \qquad \dots (5)$$

But in a system of particles, the net force acting on a particle is constituted of external as well as internal forces *i.e.*,

$$\mathbf{F}_i = \mathbf{F}_i^{ext} + \mathbf{F}_i^{int}$$

where superscripts ext. and int. respresent external and internal force respectively. Thus equation (5) becomes

$$\frac{d\mathbf{J}}{dt} = \Sigma\, \mathbf{r}_i \times (\mathbf{F}_i^{ext} + \mathbf{F}_i^{int})$$

$$= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{int} \qquad \dots (6)$$

Obviously the total angular momentum of the system changes with time due to torques on particles arising due to external as well as internal forces. But according to Newton's III law the internal forces occur in pairs of equal and opposite forces directed along the line joining the two particles. If we consider the system as a whole, the torques due to internal forcs cancel out, therefore the only source changing the total angular momentum with time about a fixed point is the sum of torque to external forces.

Equation (6) may be expressed as

$$\frac{d\mathbf{J}}{dt} = \Sigma\, \mathbf{r}_i \times \mathbf{F}_i^{ext} = \tau^{ext} \qquad \dots (7)$$

where τ^{ext} represents net external torque.

If the sum of external torque acting on the system is zero, *i.e.*,

If $\tau^{ext} = 0$; then $\frac{d\mathbf{J}}{dt} = 0$ or $\mathbf{J} =$ constant.

i.e., *If the resultant external torque acting on the system is zero, the total angular momentum of the system is constant*. This is the principle of conservation of angular momentum for a system of particles.

If there are n particles in the system and if $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3 \dots \mathbf{J}_n$ are angular momenta of individual particles, the total angular momentum about fixed axis is given by

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 + \dots + \mathbf{J}_n$$

Since total angular momentum $\mathbf{J}$ is constant, therefore the angular momenta of individual particles may change; but sum of angular momentum of the system remains unchanged in the absence of external torque.

1.15. ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES ABOUT THEIR CENTRE OF MASS

The angular momentum of ith particle about the centre of mass is

$$=(\mathbf{r}_i - \mathbf{r}_{cm}) \times m_i \, (\mathbf{v}_i - \mathbf{v}_{cm})$$

where $\mathbf{r}_i$ and $\mathbf{v}_i$ are the position vector and velocity of *ith* particle relative to any arbitrary origin O respectively whereas $\mathbf{r}_{cm}$ and $\mathbf{v}_{cm}$ are the position vector and velocity of centre of mass relative to origin O.

The total angular momentum of a system of particles about their centre of mass (also called **internal angular momentum of the system**) can be expressed as

$$\begin{aligned}\mathbf{J}_{cm} &= \Sigma (\mathbf{r}_i - \mathbf{r}_{cm}) \times m_i \, (\mathbf{v}_i - \mathbf{v}_{cm}) \\ &= \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i - \sum_i \mathbf{r}_i \times m_i \mathbf{v}_{cm} - \sum_i \mathbf{r}_{cm} \times m_i \mathbf{v}_i + \Sigma \mathbf{r}_{cm} \times m_i \mathbf{v}_{cm} \\ &= \sum_i \mathbf{r}_i \times \mathbf{p}_i - (\sum_i m_i \mathbf{r}_i) \times \mathbf{v}_{cm} - \mathbf{r}_{cm} \times (\sum_i m_i \mathbf{v}_i) + \mathbf{r}_{cm} \times \mathbf{v}_{cm} \sum_i m_i \quad \ldots (1)\end{aligned}$$

from definition $\left.\begin{aligned}\sum_i m_i \mathbf{r}_i &= \mathbf{r}_{cm} \sum_i m_i = M\mathbf{r}_{cm} \\ \text{and} \quad \sum_i m_i \mathbf{v}_i &= \mathbf{v}_{cm} \sum_i m_i = M\mathbf{v}_{cm}\end{aligned}\right]$ ... (2)

where $\Sigma m_i = M =$ total mass of the system.

Also $\sum_i \mathbf{r}_i \times \mathbf{p}_i = \mathbf{J}$

$=$ total angular momentum of the system about origin O.... (3)

Using these substitutions equation (1) can be written as

$$\begin{aligned}\mathbf{J}_{cm} &= \mathbf{J} - M\mathbf{r}_{cm} \times \mathbf{v}_{cm} - \mathbf{r}_{cm} \times M\mathbf{v}_{cm} + \mathbf{r}_{cm} \times M\mathbf{v}_{cm} \\ &= \mathbf{J} - M\mathbf{r}_{cm} \times \mathbf{v}_{cm}\end{aligned}$$

i.e.,
$$\mathbf{J} = \mathbf{J}_{cm} + \mathbf{r}_{cm} \times (M\mathbf{v}_{cm}) = \mathbf{J}_{cm} + \mathbf{r}_{cm} \times \mathbf{P} \quad \ldots (4)$$

Equation (4) gives a relation between the total angular momentum of a system of particles about any fixed point O and the internal angular momentum $\mathbf{J}_{cm}$.

The term $\mathbf{r}_{cm} \times \mathbf{P} = \mathbf{r}_{cm} \times (M\mathbf{v}_{cm})$ represents the angular momentum about the same arbitrary point O of the whole mass of the system concentrated at the centre of mass.

1.16. EXAMPLES OF CONSERVATION OF ANGULAR MOMENTUM

(a) Rutherford scattering of light positively charged particles (*e.g.* α-particles or protons) by heavy nucleus :

Suppose that the nucleus N of charge Ze is very heavy and stationary and a light positively charged particle of charge ne, say (*e.g.* a proton or α-particle) moving along PO approaches it. Both the nucleus and the charged particle are positively charged and there will be a force of repulsion between them given by Coulomb's inverse square law. This force of repulsion will go on increasing as the positively charged particle gets closer to the nucleus.

The charged particle of initial velocity v_0 is repelled by the heavy positive nucleus and changes from straight line to a hyperbola (as shown in fig. 1.23) PAQ having on focus at N. The asymptotes PO and OQ give the initial and final directions of the charged particles.

The perpendicular distance of PQ from $N = MN = p$. This is the shortest distance from nucleus to the initial direction of motion of the charged particle. This distance is called the *"impact parameter"*.

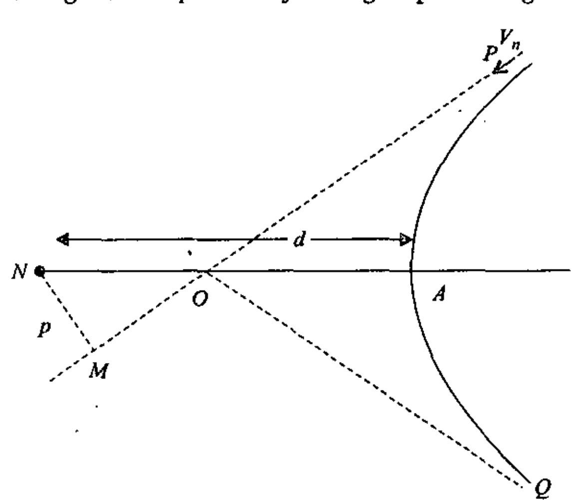


Fig. 6.

If m is the mass of the charged particle, the initial angular momentum of the charged particle about heavy nucleus $N = mv_0\ p$.

At the distance of closest approach *i.e.*, at A the angular momentum $= mv\,(NA) = mvd$ where v is the velocity of the charged particle at point A.

Since here torque $\tau = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{1}{4\pi\varepsilon_0}\ \hat{\mathbf{r}} = 0$, the angular momentum is conserved.

According to principle of conservation of angular momentum.

$$mv_0\ p = mvd.$$

$$\therefore \qquad v = \frac{mv_0\ p}{md} = \frac{v_0\ p}{d}. \qquad \ldots (1)$$

Here we have considered that angular momentum transferred to heavy nucleus is negligible.

The initial energy of the charged particle is wholly kinetic and is equal to $\frac{1}{2}\,mv_0^2$.

The total energy at point $A = \frac{1}{2}\,mv^2 + \frac{1}{4\pi\varepsilon_0} \cdot \frac{(Ze)\,(ne)}{d}$. $\qquad \ldots (2)$

In above equation the first term $\frac{1}{2}\,mv^2$ represents K.E. while the term $\frac{1}{4\pi\varepsilon_0} \cdot \frac{(Ze)\,(ne)}{d}$ represents potential energy. According to law of conservation of energy,

$$\frac{1}{2}\,mv_0^2 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{nZe^2}{d} + \frac{1}{2}\,mv^2$$

or $$\frac{1}{4\pi\varepsilon_0} \cdot \frac{nZe^2}{d} = \frac{1}{2}\,mv_0^2 - \frac{1}{2}\,mv^2 = \frac{1}{2}\,mv_0^2 - \frac{1}{2}\,m\left(\frac{v_0\,p}{d}\right)^2 \text{ from eqn. (1)}$$

$$\therefore \qquad \frac{1}{4\pi\varepsilon_0} \cdot \frac{nZe^2}{d} = \frac{1}{2}\,mv_0^2\left[1 - \left(\frac{p}{d}\right)^2\right] \qquad \ldots (2)$$

which gives the value of d *i.e.*, the distance of closest approach.

For a proton $n = 1$ and for an α-particle $n = 2$; therefore for *proton scattering*

$$\frac{1}{4\pi\varepsilon_0} \cdot \frac{Ze^2}{d} = \frac{1}{2}\,mv_0^2\left[1 - \left(\frac{p}{d}\right)^2\right] \qquad \ldots (3)$$

and for α-particle scattering

$$\frac{1}{4\pi\varepsilon_0} \cdot \frac{2Ze^2}{d} = \frac{1}{2}\,mv_0^2\left[1 - \left(\frac{p}{d}\right)^2\right] \qquad \ldots (4)$$

In a head on collision, the impact parameter $p = 0$, then equation (2) reduces to

$$\frac{1}{4\pi\varepsilon_0} \cdot \frac{nZe^2}{d} = \frac{1}{2}\,mv_0^2 \qquad \ldots (5)$$

From this equation the distance of close approach d is given by

$$d = \frac{1}{4\pi\varepsilon_0} \cdot \frac{n\,.\,2Ze^2}{mv_0^2} \qquad \ldots (6)$$

From *proton scattering* $(n = 1)$

$$d = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2Ze^2}{mv_0^2} \qquad \ldots (7)$$

For α-particle scattering $(n = 2)$

$$d = \frac{1}{4\pi\varepsilon_0} \cdot \frac{4Ze^2}{mv_0^2} \qquad \ldots (8)$$

(b) Angular momentum accompanying contraction. Let a particle of mass m describe circular motion of radius R and velocity V. The particle is connected to a string whose other end passes through a tube. By this arrangement, the radius of the circle *i.e.*, the length of the string can be decreased by pulling the string at point P.

The force acting on the particle due to string is directed along the radius R; the force and the position vector are along the same line. Therefore the torque

$$= |\mathbf{r} \times \mathbf{F}| = rF \sin\theta = 0, \text{ since } \sin\theta = 0.$$

Therefore, the torque is zero as the string is contracted. We know if resultant torque on the particle is zero, the angular momentum is conserved. Therefore in this case, the angular momentum is conserved.

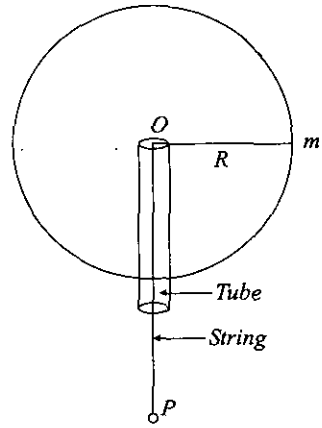


Fig. 7.

The angular momentum of the particle when the length of the string is R

$$= mVR.$$

[since velocity and radius R are perpendicular to each other.

Angular momentum $= [\mathbf{R} \times m\mathbf{V}] = mRV \sin\theta = mRV$ since $\theta = 90°$].

When the string is contracted from R to r, the velocity of particle becomes v, therefore angular momentum $= mvr$.

According to principle of conservation of angular momentum

$$mVR = mvr = K. \qquad \dots (9)$$

The centrifugal force on the particle when it is moving in a circle of radius r is

$$F = \frac{mv^2}{r}$$

The work done against centrifugal force in decreasing the radius from R to r is

$$W = \int_R^r -\frac{mv^2}{r}\,dr = -\int_R^r \frac{m}{r}\cdot\left(\frac{K}{mr}\right)^2 dr \quad \left[\text{from eq. (8) } v = \frac{K}{mr}\right]$$

$$= -\int_R^r \frac{K^2}{mr^3}\,dr = -\frac{K^2}{2m}\left[-\frac{1}{r^2}\right]_R^r$$

$$= -\frac{K^2}{2m}\left[\frac{1}{R^2} - \frac{1}{r^2}\right]$$

$$= \frac{K^2}{2mr^2} - \frac{K^2}{2mR^2} = \frac{1}{2m}(m^2v^2 - m^2V^2).$$

$$\therefore \quad W = \frac{1}{2}mv^2 - \frac{1}{2}mV^2. \qquad \dots (10)$$

From eqn. (9) it is clear that if angular momentum is conserved, velocity v is greater than V since $R > r$, that is, the velocity of particle increases with decrease of radius of the circle. From eqn. (10) it is clear that we have to do extra work in increasing the kinetic energy of the particle as the length of the string or radius of circular motion is decreased.

Student Activity

(1) State Principal of conservation of angular momentum.

----

(2) Prove $\vec{J} = \vec{J}_{cm} + \vec{R} \times \vec{P}$.

• 1.17. ELASTIC AND INELASTIC COLLISIONS

Collision : The literary meaning of collision is the striking of one particle agai another. But in the language of Physics, the bodies may not even touch each other are still said to collide. Actually, when two particles approach each other, a large fo acts on each particle for a short duration and their motion is changed abruptly. Tl the *redistribution of the total momentum of the particles takes place and they are said collide.*

There are two limiting cases of a collision :

(i) Perfectly Elastic Collision :

If the forces of interaction between the colliding bodies are conservative, the kine energy of the system does not change and the collisions are called elastic collisio Thus in elastic collisions, the *conservation of momentum and conservation of ene principles are applicable, i.e.*

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

and also, $$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

where m_1 and m_2 are the masses of two particles and u_1, u_2 and v_1, v_2 their velocit before and after the collision.

Collisions between the molecules of a gas, atoms, nuclei and fundamental partic are the examples of elastic collisions.

(ii) Perfectly Inelastic Collisions

When *two particles stick together after collision, the kinetic energy is not conserv* and the collision is said to be inelastic. In inelastic collision, *only the law of conservati of momentum is applicable*. The example of inelastic collision is a bullet hitting a targ and remained embedded in the target.

(b) Elastic Collision in One Dimensions : Let us consider an elast one-dimensional (head-on) collision between two bodies. (The relative motion aft collision is along the same line as the relative motion before collision). Let the masses the bodies and the initial and final velocities be as shown in Fig. 1.14.

By the law of conservation of momentum, we obtain

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

or $$m_1(u_1 - v_1) = m_2(u_2 - u_2). \quad \ldots ($$

Since the collision is elastic, the kinetic energy is also conserved. This gives

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

or $$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2). \quad \ldots ($$

Dividing eq. (2) by eq. (1), we get

$$\frac{(u_1^2 - v_1^2)}{(u_1 - v_1)} = \frac{(v_2^2 - u_2^2)}{(v_2 - u_2)}$$

or $$(u_1 + v_1) = v_2 + u_2$$

or $$u_1 - u_2 = v_2 - v_1 \quad \ldots ($$

Thus ***in an elastic one-dimensional collision, the relative velocity "approach before collision is equal to the relative velocity of separation" afte collision.***

We can find out final velocities of the bodies from the above equation. Putting th value of v_2 $(= u_1 + v_1 - u_2)$ from eq. (3) in eq. (1), we get

$$m_1(u_1 - v_1) = m_2(u_1 + v_1 - 2u_2)$$

or $$-v_1(m_1 + m_2) = u_1(m_2 - m_1) - 2m_2u_2$$

or $$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

Likewise, inserting $v_1 = v_2 + u_2 - u_1$ from eq. (3) in eq. (1), we get

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right) u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) u_2.$$

Special Cases :

(1) When $m_1 = m_2$, eq. (1) gives

$$u_1 - v_1 = v_2 - u_2.$$

This, on comparing with eq. (3), gives

$$v_1 = u_2 \qquad \text{and} \qquad v_2 = u_1$$

That is, **in a one-dimensional elastic collision of two bodies of equal masses, the bodies simply exchange velocities as a result of collision.**

(2) When $u_2 = 0$ (*i.e.*, the body m_2 is initially at rest), the final velocities are given by

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1, \qquad v_2 = \left(\frac{2m_1}{m_1 + m_2}\right) u_1.$$

Now in this case we may consider three situations :

(a) If $m_1 = m_2$, then $v_1 = 0$ and $v_2 = v_1$ as expected. The first body is "stopped cold" and the second one "takes off" with the velocity the first one originally had. Both the momentum and the kinetic energy of the first are completely transferred to the second.

(b) If $m_2 >> m_1$, then $v_1 \approx -u_1$ and $v_2 \approx 0$. That is, when a light body collides with a much heavier body at rest, the velocity of the light body is approximately reversed and the heavier body remains approximately at rest. A ball dropped on earth rebounds with reversed velocity attaining approximately the same height from which it falls (provided the collision is elastic).

(c) If $m_2 << m_1$, then $v_1 \approx u_1$ and $v_2 \approx 2u_2$. That is, when a heavy body collides with a much lighter body at rest, the velocity of the heavy body remains practically unchanged but the light body rebounds with approximately twice the velocity of the heavy body.

The above considerations show that in order to "slow down" the fast neutrons in a *nuclear reactor* they must be made to collide with the stationary targets (nuclei) of nearly the same mass as the neutrons themselves ($m_1 = m_2$). That is why paraffin (which is rich in hydrogen whose nucleus 'proton' has nearly the same mass as a neutron) is a very good moderator.

Maximum energy transfer in a head-on elastic collision. Suppose a ball of mass m_1 moving with velocity u_1 collides a ball m_2 at rest. Let v_1 be the final velocity of the first ball. Then the initial kinetic energy of the ball is $K_1 = \frac{1}{2} m_1 u_1^2$ and the final kinetic energy $K_1' = \frac{1}{2} m_1 v_1^2$. The fractional decrease in kinetic energy is

$$\frac{K_1 - K_1'}{K_1} = \frac{u_1^2 - v_1^2}{u_1^2} = 1 - \frac{v_1^2}{u_1^2}$$

But, for such a collision, we have

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1$$

or

$$\frac{v_1^2}{u_1^2} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2$$

Therefore,

$$\frac{K_1 - K_1'}{K_1} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 = \frac{4m_1 m_2}{(m_1 + m_2)^2}.$$

let us put $m_1 = m$ and $m_2 = nm$. Then

$$\frac{K_1 - K_1'}{K_1} = \frac{4n}{(1 + n)^2}.$$

The transfer is maximum when $K_1' = 0$, so that

$$\frac{4n}{(1+n)^2} = 1$$

or $$n = 1.$$

Thus, when the mass ratio is unity, the 'whole' of the kinetic energy of the moving ball is transfered to the ball initially at rest.

• 1.18. ELASTIC COLLISION IN TWO OR THREE DIMENSIONS

To determine the velocities of the particles after collision in two or three dimensional elastic collision only the law of conservation of momentum and energy are not enough because they provide only four relations (three for conservation of momentum for each of the three dimensions and one for conservation of energy) while the velocity of each particle has three components. Therefore, to be able to determine the motion after collision, we need at least the angle of deflection of one of the particles. It may be seen evidently from the following two cases :

(a) In the Laboratory Frame of Reference.

Referring to Fig. 6., let a particle of mass m_1 moving with velocity $\mathbf{u}_1$ collide with a particle of mass m_2 at rest (*i.e.*, $\mathbf{u}_2 = 0$). Let after collision, the particle of mass m_1 be deflected (or scattered) at an angle θ_1 with its original direction and move with velocity v_1 while particle of mass m_2 move making an angle θ_2 with the original direction with velocity v_2. Then law of conservation of angular momentum along X-axis provides

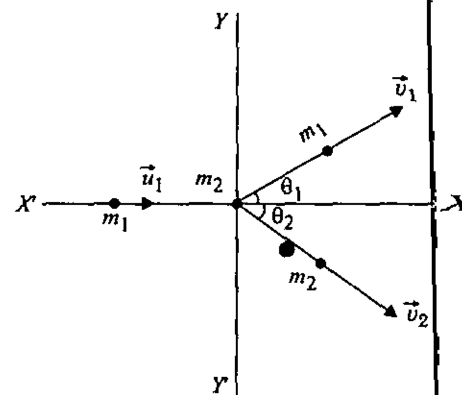


Fig. 8.

$$m_1u_1 + 0 = m_1v_1 \cos\theta_1 + m_2v_2 \cos\theta_2 \qquad ...(1)$$

Applying law of conservation of angular momentum along Y-axis, we get

$$0 + 0 = m_1v_1 \sin\theta_1 - m_2v_2 \sin\theta_2 \qquad ...(2)$$

The law of conservation of energy gives

$$\frac{1}{2} m_1u_1^2 + 0 = \frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2 \qquad ...(3)$$

In order to determine the four unknown quantities v_1, v_2, θ_1 and θ_2, these three equations are insufficient and we must be given the value of one more quantity (say θ_1). Here we shall obtain the values of v_1, v_2 and θ_2 for a particular case when the two masses are equal *i.e.*, $m_1 = m_2$. In that case, equation (1), gives

$$u_1 = v_1 \cos\theta_1 + v_2 \cos\theta_2$$

or $$u_1 - v_1 \cos\theta_1 = v_2 \cos\theta_2 \qquad ...(4)$$

For $m_1 = m_2$, equation (2) becomes

$$v_1 \sin\theta_1 = v_2 \sin\theta_2 \qquad ...(5)$$

Squaring both sides of equation (4), we get

$$u_1^2 + v_1^2 \cos\theta_1 - 2u_1\, v_1 \cos\theta_1 = v_2^2 \cos^2\theta_2 \qquad ...(6)$$

And squaring both sides of equation (5), we get

$$v_1^2 \sin^2\theta_1 = v_2^2 \sin^2\theta_2 \qquad ...(7)$$

Adding equations (6) and (7), we have

$$u_1^2 + v_1^2 - 2u_1\, v_1 \cos\theta_1 = v_2^2 \qquad ...(8)$$

For $m_1 = m_2$, equation (3) gives

$$u_1^2 = v_1^2 + v_2^2$$

or $$u_1^2 - v_1^2 = v_2^2 \quad \text{... (9)}$$

Subtracting equation (9) from equation (8), we get

$$2v_1^2 - 2u_1v_1 \cos\theta_1 = 0$$

or $$v_1 = u_1 \cos\theta_1 \quad \text{... (10)}$$

Thus v_1 can be calculated knowing the initial conditions (u_1) and θ_1 (given).

Putting $v_1 = u_1 \cos\theta_1$ in equation (9), we have

$$u_1^2 - u_1^2 \cos^2\theta_1 = v_2^2$$

or $$v_2^2 = u_1^2 (1 - \cos^2\theta_1) = u_1^2 \sin^2\theta_1$$

or $$v_2 = u_1 \sin\theta_1 \quad \text{... (11)}$$

Thus v_2 can also be obtained by knowing initial condition (u_1) and θ_1 (given).

It is obvious from equations (10) and (11) that v_1 and v_2 are perpendicular components of u_1

or $$\theta_1 + \theta_2 = 90°$$

Thus in a perfectly two dimensional elastic collision between two particles of equal masses, when one particle is initially at rest, the two particles move in directions perpendicular to each other after collision.

Further, from equation

$$\sin\theta_2 = \frac{v_1}{v_2} \sin\theta_1 \quad \text{... (12)}$$

Thus knowing v_1 (from 10), v_2 (from 11) and θ_1 (given), we can also determine θ_2.

(b) In the centre of mass frame of reference.

The centre of mass frame of reference moves with a velocity v_{cm} relative to the laboratory frame, where v_{cm} is given by

$$m_1u_1 + m_2 \times 0 = (m_1 + m_2)\, v_{cm}$$

or $$v_{cm} = \frac{m_1u_1}{m_1 + m_2}$$

The centre of mass remains at rest in this frame. Therefore, the initial velocities u'_1 and u'_2 of the two particles of masses m_1 and m_2 in this frame of reference are given by

$$u'_1 = u_1 - v_{cm}$$

and $$u'_2 = u_2 - v_{cm} = -v_{cm} \qquad (\because\ u_2 = 0)$$

where u_1 and u_2 are the velocities before collision in laboratory frame of reference.

Similarly, the final velocities v'_1 and v'_2 of the particles after collision are given by

$$v'_1 = v_1 - v_{cm} \quad \text{and} \quad v'_2 = v_2 - v_{cm}$$

where v_1 and v_2 are the velocities after collision in laboratory frame.

Since the centre of mass remains at rest both before and after collision in centre of mass reference frame, the total linear momentum is always zero and hence must be zero before and after collision. Thus

$$m_1v'_1 + m_2u'_2 = 0 \quad \text{... (13)}$$

and $$m_1v'_1 + m_2v'_2 = 0 \quad \text{... (14)}$$

From equation (13), $\frac{m_2}{m_1} = -\frac{u'_1}{u'_2}$ or $u'_2 = -u'_1 \frac{m_1}{m_2}$

From equation (14), $\frac{m_2}{m_1} = -\frac{v'_1}{v'_2}$ or $v'_2 = -v'_1 \frac{m_1}{m_2}$

$$\therefore \quad \frac{m_2}{m_1} = -\frac{u'_1}{u'_2} = -\frac{v'_1}{v'_2} \quad \text{... (15)}$$

The negative sign in equation (15) shows that u'_1 and u'_2 or v'_1 and v'_2 of the two particles are oppositely directed. It may be inferred that *the velocities of two particles after collision, are inversely proportional to their masses, are along the same straight line but in opposite directions inclined at the same angle to the initial direction of motion of the particles. It has been shown in Fig. 7.*

Now according to the law of conservation of energy, we have

$$\frac{1}{2} m_1 u'^2_1 + \frac{1}{2} m_2 u'^2_2 = \frac{1}{2} m_1 v'^2_1 + \frac{1}{2} m_2 v'^2_2 \quad \dots (16)$$

But from above, $u'_1 = u_1 - v_{cm} = u_1 - \dfrac{m_1 u_1}{(m_1 + m_2)} = \dfrac{m_2 u_1}{m_1 + m_2}$

and
$$u'_2 = - v_{cm} = - \frac{m_1 u_1}{m_1 + m_2}$$

Substituting the values of u'_2 and v'_2 from equation (15) and equation (16), we get

$$\frac{1}{2} m_1 u'^2_1 + \frac{1}{2} m_2 \left(\frac{m_1^2 u'^2_1}{m_2^2} \right) = \frac{1}{2} m_1 v'^2_1 + \frac{1}{2} m_2 \left(\frac{m_1^2 v'^2_1}{m_2^2} \right)$$

or
$$\frac{1}{2} m_1 u'^2_1 + \frac{m_1^2 u'^2_1}{m_2} = m_1 v'^2_1 + \frac{m_1^2 v'^2_1}{m_2}$$

or
$$m_1 u'^2_1 + \left(1 + \frac{m_1}{m_2}\right) = m_1 v'^2_1 + \left(1 + \frac{m_1}{m_2}\right)$$

which gives

$$v'_1 = u'_1 = \frac{m_2 u_1}{m_1 + m_2} \quad \dots (17)$$

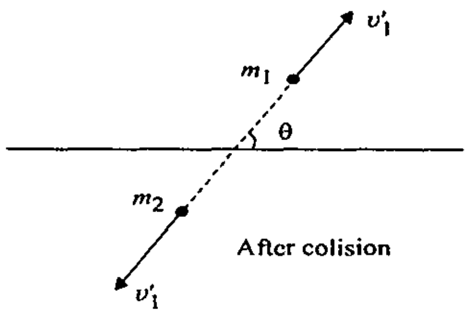


Fig. 9.

Similarly, it can be shown that

$$v'_1 = u'_2 = - \frac{m_1 u_1}{m_1 + m_2} = - v_{cm} \quad \dots (18)$$

Thus in centre of mass frame of reference, the magnitudes of velocities of the particles remain unchanged in elastic collision, although their is a change of direction.

• 1.19. ANGLE OF SCATTERING AND SCATTERING CROSS-SECTION

Angle of Scattering : ***When an incoming particle experiences interacting force due to other particle, the direction of incoming particle gets changed. The angle between its initial and final directions is called angle of scattering.***

Consider that a particle of mass m_1 moving with velocity $\mathbf{u}_1$ experiences elastic interaction due to a particle of mass m_2 at rest. Let θ_c be angle of scattering of mass m_1 in *C*-system and θ_1 in *L*-system. The angle θ_1 is unrestricted while conservation principles restrict this angle in *L*-system.

The velocity of centre of mass is given by

$$\mathbf{v}_{cm} = \frac{m_1 \mathbf{u}_1}{m_1 + m_2} \quad \dots (1)$$

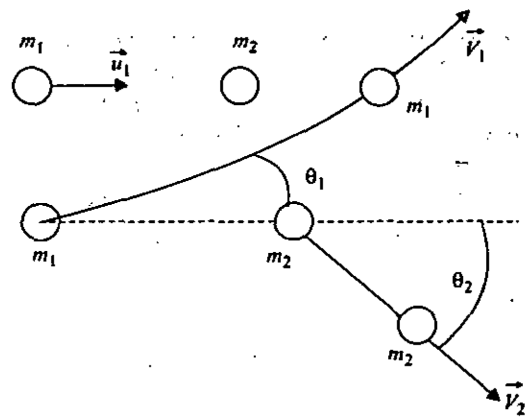


Fig. 10.

The initial velocity in C-system are

$$\mathbf{u}_{1c} = \frac{m_2 \mathbf{u}_1}{m_1 + m_2} \quad \text{and} \quad \mathbf{u}_{2c} = -\frac{m_1 \mathbf{u}_1}{m_1 + m_2} \qquad \dots (2)$$

From the velocity-diagram, we see that the laboratory scattering angle (θ_1) of incident particle is given by

$$\tan \theta_{1c} = \frac{v_{1c} \sin \theta_c}{v_{cm} + v_{1c} \cos \theta_c}$$

As the scattering is elastic $v_{1c} = u_{1c}$. Hence

$$\tan \theta_1 = \frac{u_{1c} \sin \theta_c}{v_{cm} + u_{1c} \cos \theta_c} = \frac{\sin \theta_c}{\frac{v_{cm}}{u_{1c}} + \cos \theta_c}$$

But from (1) and (2)

$$\frac{v_{cm}}{u_{1c}} = \frac{m_1}{m_2}$$

$$\therefore \qquad \tan \theta_1 = \frac{\sin \theta_c}{\frac{m_1}{m_2} + \cos \theta_c} \qquad \dots (3)$$

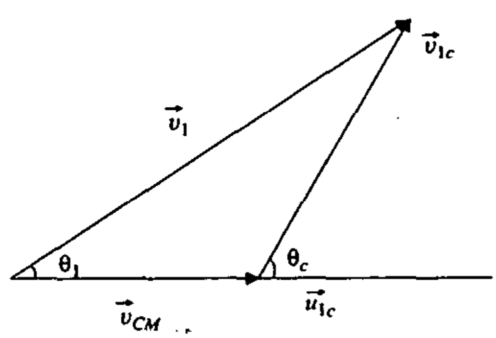


Fig. 11.

The centre of mass scattering angle θ_c depends on the details of interaction, but in general it can take any value [fig. (a)]. If $m_1 < m_2$ then θ_1 is unrestricted. But if $m_1 > m_2$, then θ_1 is never greater than a certain value $(\theta_1)_{max}$ (fig. b). The maximum value of θ_1 occurs when v_1 and v_{1c} are both perpendicular. In this case

$$\sin (\theta_1)_{max} = \frac{u_{1c}}{v_{cm}} = \frac{m_2}{m_1}. \text{ If } m_1 >> m_2, (\theta_1)_{max} \to 0$$

Physically it means that a light particle at rest cannot deflect a massive particle.

18. In an elastic collision :
    (a) Kinetic energy remains constant
    (b) Kinetic energy decreases
    (c) Kinetic energy increases
    (d) Kinetic energy may decrease or increase
19. If the torque acting on a system of particles is zero; what is conserved :
    (a) linear momentum (b) angular momentum
    (c) energy (d) all of above
20. A central force is an example of :
    (a) conservative force (b) non-conservative force
    (c) fictitious force (d) frictional force
21. The unit of angular momentum is :
    (a) kg ms^{-1} (b) kg ms^{-2}
    (c) kg m^2s^{-1} (d) Js^{-1}
22. The earth is revolving about the sun under gravitational force. What is conserved for the system?
    (a) linear momentum (b) angular momentum
    (c) both above (d) neither (a) nor (b)
23. The unit of torque is :
    (a) newton × metre (b) newton/metre
    (c) newton/metre^2 (d) newton-metre^2
24. The reference frames where fundamental laws of physics are invariant are called
    (a) rotational frames (b) inertial frames
    (c) accelerated frames (d) frames attached to earth
25. In a head on elastic collision between two particles, the transference of energy is maximum when their mass ratio is
    (a) zero (b) infinity
    (c) half (d) unity

ANSWERS

12. (a) **13.** (d) **14.** (b) **15.** (c) **16.** (a) **17.** (b) **18.** (a) **19.** (b) **20.** (a) **21.** (c)
22. (b) **23.** (a) **24.** (b) **25.** (d)

UNIT 2

ROTATIONAL DYNAMICS

STRUCTURE

- Rotational Motion : Torque and Angular Momentum
- Torque Acting on a Particle
- Moment of Inertia
- Kinetic Energy of a Rotating Body
- Theorems of Moment of Inertia
- Moment of Inertia of a Circular Disc
- Moment of Inertia of an Annular Disc
- Moment of Inertia of a Solid Cylinder
- Moment of Inertia of Cylinder about its Own Axis
- Moment of Inertia of a Thin Spherical Shell
- Moment of Inertia of Solid Sphere about Diameter
- Body Rolling down an Inclined Plane
- Precession
- Relation between the Elastic Constants
- Bending of Beams
- Cantilever
- Potential Energy and Oscillations of a Loaded Cantilever
- Beam Supported at its ends and Loaded in the Middle
- Applications of Bending of Beams
- Torsion of a Cylinder
- Work Done in Twisting a wire or Cylinder
  - Summary
  - Test yourself
  - Answers

LEARNING OBJECTIVES

After learning this chapter, you will be able to know ....

- Rotational Motion in detail along with the torque and Angular Momentum acting on the rotating body.
- What is Moment of Inertia and its application on masses of different shape.
- Study of Elastic Constants
- Study of Bending of Beam and its Application
- Basic Study of Torsion of Cylinder

• 2.1. ROTATIONAL MOTION : TORQUE AND ANGULAR MOMENTUM

(i) Angular Displacement (θ) : Angular displacement is the angle described by the position vector $\vec{r}$ about the axis of rotation. It is denoted by θ and is measured in radian or degree.

If θ is positive then rotation will be anticlockwise and if θ is negative then rotation will be clockwise.

(ii) Angular velocity (ω) : The rate of change of angular displacement known as angular velocity. It is denoted by ω and is defined as

$$\boxed{\omega = \frac{d\theta}{dt}}$$

It is measured in radian/sec. In rigid body the radial lines from all the particles of th body are perpendicular to the axis of rotation. These particles sweep out equal angles : equal time intervals so the angular velocity ω is same for every particle of the rig body.

Angular velocity depends upon the point about which the rotation is considered.

(iii) Angular acceleration (α) : The rate of change of angular velocity of body about the axis of rotation is known as angular acceleration. It is denote by α and is defined as

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) \qquad \left[\because\ \omega = \frac{d\theta}{dt}\right]$$

$$\boxed{\alpha = \frac{d^2\theta}{dt^2}}$$

It is measured in radian/sec^2.

Angular acceleration is same for all particles of the rigid body.

(iv) Equations of rotational motion : There are three relations between rotational kinematic variables. These are :

(1) $\omega = \omega_0 = \alpha t$ (2) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ (3) $\omega^2 = \omega_0^2 + 2\alpha\theta$

where ω_0 = initial angular velocity

ω_0 = final angular velocity

α = angular acceleration

θ = angular displacement

t = time

These equations are known s **equations of rotational motion.**

When the motion is linear then above equations reduce to :

(1) $v = u + at$ (2) $s = ut + \frac{1}{2}at^2$ (3) $v^2 = u^2 + 2as$

Now we have to prove the equations of rotational motion.

Proof : $\boxed{\omega = \omega_0 + \alpha t}$

Let a rigid body be rotating about an axis with a uniform angular acceleration α, then we know

$$\alpha = \frac{d\omega}{dt}$$

$$\therefore \quad d\alpha = \alpha\ dt \qquad \text{... (1)}$$

Let at $t = 0$ $\omega = \omega_0$

and at $t = t$ $\omega = \omega$

On integrating the eq. (1) between above limits, *i.e.*,

$$\int_{\omega_0}^{\omega} = \int_0^t \alpha\ dt$$

$$[\omega]_{\omega_0}^{\omega} = \alpha[t]_0^t$$

$$\therefore \quad \omega - \omega_0 = \alpha\,(t - 0)$$

$$\boxed{\omega = \omega_0 + \alpha t} \qquad \text{... (2)}$$

This proves the relation first.

Proof : $\boxed{\theta = \omega_0 t + \frac{1}{2}\alpha r^2}$

Let ω be the angular velocity of the rigid body at any time 't' then we know that

$$\omega = \frac{d\theta}{dt}$$

$$\therefore \qquad d\theta = \omega\, dt \qquad \dots (3)$$

Let at $t = 0$ $\qquad \theta = 0$

and at $t = t$ $\qquad \theta = 0$

So on integrating equation (3) we get

$$\int_0^\theta d\theta = \int_0^t \omega\, dt$$

$$[\theta]_0^\theta = \int_0^t (\omega_0 + \alpha t)\, dt \qquad \text{[by 2)]}$$

$$\theta - 0 = \left[\omega_0 t + \frac{\alpha t^2}{2}\right]_0^t$$

$$\boxed{\theta = \omega_0 t + \frac{1}{2}\alpha t^2}$$

This proves the relation second.

Proof : $\qquad \omega^2 = \omega_0^2 + 2\alpha\theta \qquad \dots (4)$

We know that $\qquad \omega = \frac{d\theta}{dt}$

and $\qquad \alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\left(\frac{d\theta}{dt}\right)$

$$\alpha = \left(\frac{d\omega}{d\theta}\right)\omega$$

$$\omega d\omega = \alpha\, d\theta \qquad \dots (5)$$

where $\theta = 0$, $\omega = \omega_0$ initial angular velocity

and when $\theta = \theta_0$, $\omega = \omega$ final angular velocity.

$\therefore$ On integrating (5) we get

$$\int_{\omega_0}^{\omega} \omega\, d\omega = \int_0^\theta \alpha\, d\theta$$

$$\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \alpha[\theta]_0^\theta$$

$$\frac{\omega^2 - \omega_0^2}{2} = \alpha(\theta - 0)$$

$$\boxed{\omega^2 = \omega_0^2 + 2\alpha\theta} \qquad \dots (6)$$

This proves the relation third.

• 2.2. TORQUE ACTING ON A PARTICLE

(a) Torque Acting on a Particle : The torque acting on a particle can be explained easily by the following example. When we switch on a fan then the centre of the fan remains unmoved while the fan rotates with an equal acceleration. As the centre of mass of the fun remains at rest then the vector sum of external force acting on the fan must be zero. **This means that an angular acceleration is produced even w** **resultant external force is zero.** And we also know that we can not produce ang acceleration without applying an external force. Hence here is question arises or v is the reason for producing angular acceleration ?

The answer is **torque** due to the force. When an external force acts on a body the has a tendency to rotate the body about a fixed axis. In this position the force acting the body is known as torque on the body.

The torque acting on the body is equal to the product of the magnitude of force perpendicular distance of the line of action of force from the axis of rotation. I denoted by τ.

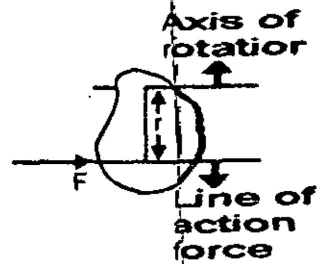


Fig. 1.

Thus torque = force × perpendicular distance

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

or $$\tau = Fr \sin \theta$$

where F = magnitude of force

r = perpendicular distance

Its unit is N-m in S.I. system.

Angular momentum of a particle : Let us consider a particle of mass m who position vector is $\vec{r}$ from the origin O as shown in the fig. 2. The linear momentum of th particle is given by

$$\boxed{\vec{P} = m\,\vec{v}} \qquad \dots (1)$$

where $\vec{v}$ = linear velocity of the particle.

The angular momentum of the particle about origin O is equal to the vector product of $\vec{r}$ and $\vec{p}$ *i.e.*,

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

In magnitude

$$L = rp \sin \theta$$

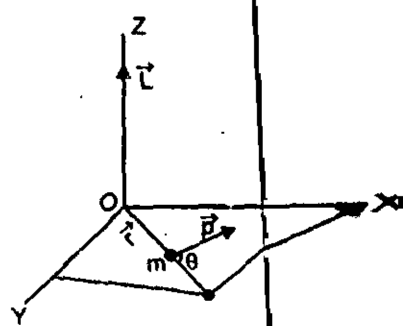


Fig. 2.

where θ is the angle between $\vec{r}$ and $\vec{p}$.

From above it is clear that the angular momentum about O is zero when the line of action of $\vec{P}$ passes through O. In this position $\theta = 0$.

Relation between torque and angular momentum :

From eqn. (1)

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times p)$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{r}}{dt}$$

$$= \vec{v} \times m\vec{v} + \vec{r} \times \frac{d\vec{p}}{dt} = 0 \times \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\therefore \quad \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} \qquad \dots (2)$$

But according to Newton's 2^{nd} law

We have force $$\vec{F} = \frac{d\vec{p}}{dt}$$

$\therefore$ by (2) $$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \qquad \text{... (3)}$$

But $$\vec{\tau} = \vec{r} \times \vec{F}$$

so, by (3) $$\frac{d\vec{L}}{dt} = \vec{\tau}$$

Thus, **the rate of change of angular momentum of a particle is equal to the torque acting on the particle.**

(d) Angular momentum of a particle moving with constant velocity : From eqn. (2), we have

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{r} \times m\,\frac{d\vec{v}}{dt}$$

If $\vec{v}$ is constant, *i.e.*, $\frac{d\vec{v}}{dt} = 0$

Then $$\frac{d\vec{L}}{dt} = 0$$

$\therefore$ $$\boxed{\vec{L} = \text{constant}}$$

Hence the angular momentum about any point of a single particle moving with constant velocity remains constant throughout the motion.

Student Activity

1. If a body is rotating, is it necessarily being acted upon by an external torque ?

2. Torque and work are both defined as force times distance. Explain, how do they differ ?

• 2.3. MOMENT OF INERTIA

(a) Moment of inertia : "The moment of inertia of a rigid body about a given axis of rotation is the sum of the products of the masses of the various particles and squares of their perpendicular distance from the axis of rotation."

Consider a body rotating about an axis OZ and let m_1, m_2, m_3 ... be the masses of the particles of the body and r_1, r_2, r_3 be the distances from the axis of rotation OZ. Then moment of inertia of the body about axis of rotation is given by

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots$$

$$\boxed{I = \Sigma\, m r^2}$$

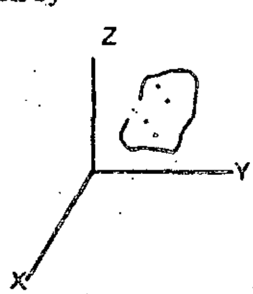


Fig. 3.

The unit of moment of inertia is cgs system is g cm^2 and **kg m^2** in S.I. system. Its dimensional formula is $[ML^2T^0]$. It is **tensor** quantity.

Moment of inertia depends upon the following factors :

1. Mass of the body.

2. Distribution of the mass of the body.

3. Distance from the axis of rotation.

(b) Physical Significance :

We know that

K.E. of translation of body $= \frac{1}{2} mv^2$

K.E. of rotation of body $= \frac{1}{2} I \omega^2$

On comparing it is clear that u is similar to ω therefore m is similar to I. The **moment of inertia (I) plays the same role in rotational motion as mass plays in linear motion.**

This is the physical significance of moment of inertia.

(c) Radius of Gyration : The distance from the axis of rotation for every body may always be found, at which if whole mass of the body is concentrated then the moment of inertia of the body about that axis remains same. This distance from the axis of rotation is called radius of gyration about the axis.

Let M by the mass of the body which is concentrated at distance K from the axis of rotation, then moment of inertia is given by

$$I = MK^2$$

so

$$\boxed{K = \sqrt{\frac{I}{M}}}$$

Hence it may be defined as "the perpendicular distance from the axis of rotation, the square of which when multiplied with total mass of the body, gives the moment of inertia of the body about that axis."

Unit in M.K.S. system is meter and dimensional formula is [L]

The radius of gyration is not a constant quantity.

• 2.4. KINETIC ENERGY OF A ROTATING BODY

Let us consider a body of mass M rotating with an angular velocity ω about an axis whose kinetic energy is to be determined. Let $m_1, m_2, m_3, \ldots$ be the masses of particles of the body and $r_1, r_2, r_3 \ldots$ be the distances of these particles from the axis of rotation. When the body rotates then all the particles of the body rotate with the same angular velocity ω but move with different linear velocities. Let $v_1, v_2, v_3 \ldots$ be the linear velocities of the different particles.

In this position

Kinetic energy of the particle of mass m_1

$$K_1 = \frac{1}{2} m_1 v_1^2$$

or

$$K_1 = \frac{1}{2} m_1 r_1^2 \omega^2$$

Similarly, kinetic energy of the particle of mass m_2

$$K_2 = \frac{1}{2} m_2\, r_2^2 \omega^2$$

therefore kinetic energy of the body

$$K = K_1 = K_2 + K_3 \ldots$$

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + \ldots$$

$$K = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + \ldots$$

$$K = \frac{1}{2}\omega^2 \Sigma mr^2$$

$$K = \frac{1}{2} I \omega^2 \qquad [\because\ I = \Sigma\, mr^2]$$

This is the required expression for the kinetic energy of the body in terms of moment of inertia.

From above

$$I = \frac{2K}{\omega^2}$$

If $\omega = 1$ Radian

then $\boxed{I = 2K}$

Hence, "The moment of inertia of a body about a given axis is equal to double of the kinetic energy of the body rotating with unit angular velocity about the given axis."

(b) Angular momentum of a rotating body : It is defined as "The sum of the moments of linear momentum of all the particles of a rotating rigid body about the axis of rotation and is called angular momentum about that axis."

Consider a rigid body which is moving around an axis of rotation. Let the be a particle of mass m at distance r from the axis of rotation. If ω is the angular velocity then the linear velocity of the particle of mass m is $r\omega$

$\therefore$ Linear momentum = mass × velocity

$$P = m \times r\,\omega$$

The moment of this momentum is

$$P \times r = m \times r\,\omega \times r = mr^2\omega$$

i.e., angular momentum of a particle of mass $m = mr^2\omega$

Therefore angular momentum of the body

$$L = \Sigma\, mr^2\omega$$

$$L = \omega \Sigma\, mr^2$$

$$\therefore \quad \boxed{L = I\,\omega} \qquad [\because\ I = \Sigma\, mr^2]$$

Thus "angular momentum of the body is equal to the product of the moment of inertia and the angular velocity of the body about the axis."

Now we have $\quad K = \frac{1}{2} I\omega^2$ and $L = I\omega$

$$\therefore \quad K = \frac{1}{2}\frac{l^2\omega^2}{l} = \frac{L^2}{2l}$$

$$K = \frac{L^2}{2l}$$

This is relationship between angular momentum and kinetic energy of the body.

(c) Power and Work done by a Torque : Let us consider a rigid body rotating about a fixed axis on which a torque acts.

This torque produces the angular acceleration and increases the kinetic energy of the body. We know that the rate of change of kinetic energy (work done) is equal to the power delivered by the torque *i.e.*,

Power = rate of change of kinetic energy or rate of change of work done

$$P = \frac{d\omega}{dt} = \frac{d}{dt}\left[\frac{1}{2} I\,\omega^2\right]$$

$$= I\omega\frac{d\omega}{dt} = I\,\alpha\,\omega \qquad [\text{where } \alpha = \text{angular acceleration}]$$

$$\frac{d\omega}{dt} = P = \tau\omega$$

where τ is a torque acting on the body.

The work done due to small angular displacement $d\theta$ is

$$d\omega = \tau\,\omega\,dt$$

$$d\omega = \tau\, d\theta \qquad \left[\omega = \frac{d\theta}{dt}\right]$$

Total work done

$$\boxed{W = \int_{\theta_1}^{\theta_2} \tau\, d\theta}$$

This is the work done by the torque.

(d) Relation between torque and angular acceleration : Let us consider a r body rotating about a given axis with a uniform angular acceleration α, and let a tor τ act on the body.

Consider that m_1, m_2, m_3 ... are the masses of the particles of the body perpendicular distances $r_1, r_2\, r_3$ ... respectively from the axis of rotation.

Since body is rigid so the angular acceleration of all particles of the body rem same while their linear acceleration is different due to different distances of the par from the axis.

Let a_1, a_2, a_3... be the linear accelerations of the particles then

$$a_1 = r_1\,\alpha,\ a_2 = r_2\,\alpha,\ a_3 = r_3\,\alpha, \ldots$$

Force on particle of mass m_1

$$f_1 = m_1 a_1 = m_1 r_1 \alpha$$

Moment of this force about the axis of rotation

$$= f_1 \times r_1 = (m_1 r_1 \alpha) \times r_1 = m_1 r_1^2 \alpha$$

Similarly, moment of forces on other particles about the axis of rotation $m_2 r_2^2\,\alpha,\ m_3 r_3^2\,\alpha$ ...

$\therefore$ Torque acting on the body

$$\tau = m_1 r_1^2 \alpha + m_2 r_2^2\,\alpha + m_2 r_3^2\,\alpha + \ldots$$

$$= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2)\,\alpha$$

$$= (\Sigma\, m r^3)\,\alpha$$

$$\boxed{\tau = I\alpha} \qquad [\because\ I = \Sigma m r^2$$

If $\alpha = 1$ then $\tau = I$

Hence "moment of inertia of a body about a given axis is numerically equal to tor acting on the body rotating with unit angular acceleration about it."

The above relation in vector from may be written as

$$\boxed{\vec{\tau} = I\vec{\alpha}}$$

This equation is called **fundamental equation of rotation or law of rotatio**

• 2.5. THEOREMS OF MOMENT OF INERTIA

There are two important theorems to determine the moment of inertia. They help determining the moment of inertia about any axis if moment of inertia about one axi known. They are :

(a) Theorem of parallel axes : According to this theorem, **"Moment of inertia a body about any axis is equal to its moment of inertia about a parallel a**

through its centre of mass plus Mh^2, where M is the mass of the body and h the perpendicular distance between the two axes." *i.e.*,

$$I = I_{cm} + Mh^2$$

This is the "theorem of parallel axes."

Proof : Let us consider a particles of the body of mass m at a distance r from the line AB. Here we have to calculate the moment of inertia of the body about the line GH which is parallel to the centre of mass axis AB. Let h be the perpendicular distance between AB and GH as shown in the fig. 4.

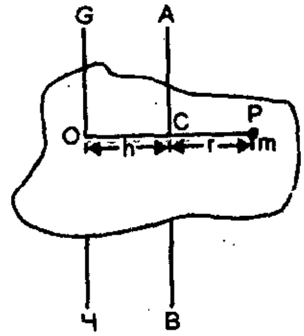


Fig. 4.

Therefore the moment of inertia of the body about centre of mass axis AB is

$$I_{cm} = \Sigma mr^2$$

Now, moment of inertia of the body about the line GH is

$$I = \Sigma m\,(r+h)^2$$

$$= \Sigma m\,[r^2 + h^2 + 2rh]$$

$$= \Sigma mr^2 + \Sigma mh^2 + \Sigma\, 2\,mrh$$

$$= I_{cm} + h^2 \Sigma m + 2h\,\Sigma\, mr$$

$$= I_{cm} + Mh^2 + 0$$

where M = Total mass of the body and Σmr = sum of the moments of the masses of particles constituting the body about an axis through its mass must be zero.

$\therefore$ $$\boxed{I = I_{cm} + Mh^2}$$ **Hence proved.**

(b) Theorem of perpendicular axes : According to this theorem, **"The moment of inertia of a plane lamina (a two-dimensional body) about an axis perpendicular to its plane (OZ) is equal to sum of the moments of inertia about any two mutually perpendicular axes OX and OY in its plane intersecting on the first axis."**

i.e., $$I_z = I_x + I_y$$

where x, y, z axes are mutually perpendicular to each other.

Proof : According to this theorem, the sum of the moments of inertia of a plane lamina about any two mutually perpendicular axes in its plane is equal to its moment of inertia about an axis perpendicular to the plane of the lamina and passing through the point of intersection of the first two axes.

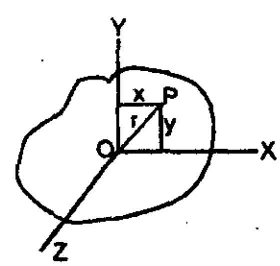


Fig. 5.

i.e., $$I_z = I_x + I_y$$

Now we have to prove it.

Let us consider a particle P of mass m at distance r from O and at distances x and y from OY and OX, respectively. Then the moment of inertia about OX is my^2 and hence I_x, the moment of inertia of the lamina about OX is

$$I_x = \Sigma m.y^2 \qquad \text{... (ii)}$$

and the moment inertia of the lamina about OZ is

$$I_z = \Sigma mr^2$$

$$I_z = \Sigma m\,(x^2 + y^2) \qquad [\because\ r^2 = x^2 + y^2]$$

$$I_z = \Sigma mx^2 + \Sigma my^2$$

$$I_z = I_y + I_x$$

Hence $\boxed{I_z = I_x + I_y}$ [from eqs. (i) and (ii)]

This proves the theorem of perpendicular axis.

• 2.6. MOMENT OF INERTIA OF A CIRCULAR DISC

(i) Moment of inertia of a circular disc about an axis through its centre and perpendicular to its plane : Let us consider a circular disc of mass M and radius R with centre 'O'. We have to calculate the moment of inertia of the disc about the line yy' *i.e.*, the axis which passes through the centre O and is perpendicular to the plane of disc.

$\because$ Mass of the disc $= M$

and Area of the disc $= \pi R^2$

$\therefore$ Mass per unit area $= \dfrac{M}{\pi R^3}$

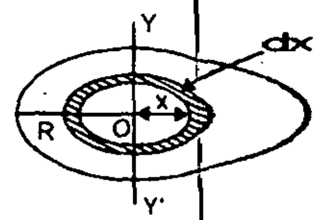


Fig. 6.

Consider a small element of the disc which is also circular in shape of radius x and width dx.

$$\text{Area of the element} = 2\pi x\, dx$$

$$\therefore \quad \text{Mass of the element} = \frac{M}{\pi R^2}(2\pi x\, dx)$$

$$= \frac{2M x\, dx}{R^2}$$

$$\text{M.I. of this element about } yy' = \frac{2M x\, dx}{R^2}.(x^2)$$

$$= \frac{2M x^3 dx}{R^2}$$

$\therefore$ M.I. of the circular disc about yy' is

$$I = \int_0^R \frac{2Mx^3}{R^2}\, dx = \frac{2M}{R^2}\int_0^R x^3\, dx$$

$$\boxed{I = \frac{1}{2} M R^2}$$

(ii) About diameter : M.I. of the circular disc about any diameter can be obtained by using theorem of perpendicular axis, *i.e.*,

$$I = I_{AB} + I_{CD} = 2 I_{AB} \qquad [\because AB = CD]$$

$$\boxed{I_{AB} = \frac{1}{4} MR^2}$$

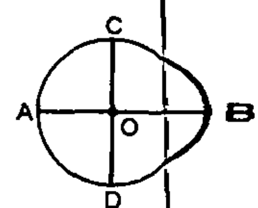


Fig. 7 (a)

(ii) About tangent : M.I. of the disc about tangent EF can be obtained by using the theorem of parallel axis, *i.e.*,

$$I_{EF} = I_{AB} + MR^2$$

$$= \frac{1}{4} MR^2 + MR^2$$

$$= \frac{5}{4} MR^2$$

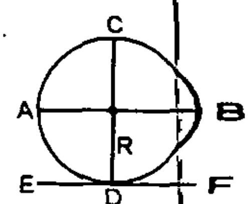


Fig. 7 (b)

• 2.7. MOMENT OF INERTIA OF AN ANNULAR DISC

(i) Moment of inertia of an annular disc about an axis through its centre and perpendicular to its plane : Let us consider an annual disc of mass M and inner radius R_1 and outer radius be R_2 with centre 'O'. Consider a small strip of this disc, it will be a ring. Let x be the radius of this ring.

Mass of the disc $= M$

$\therefore$ Mass per unit area $= \dfrac{M}{\pi(R_2^2 - R_1^2)}$

$\because$ Area of small ring $= 2\pi x\, dx$

$\therefore$ Mass of the ring $= \dfrac{M}{\pi(R_2^2 - R_1^2)} 2\pi x^2\ dx$

$$= \frac{2M}{(R_2^2 - R_1^2)} x\, dx$$

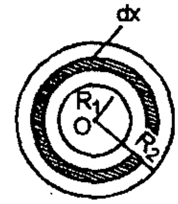


Fig. 8

M.I. of this ring about the axis passing through its centre and perpendicular to its plane

$$= \text{mass} \times (\text{distance})^2$$

$$= \frac{2M}{\pi\,(R_2^2 - R_1^2)} x\, dx.\, x^2 = \frac{2M}{(R_2^2 - R_1^2)} x^3\ dx$$

M.I. of the disc about an axis which passes through the centre and is perpendicular to the plane is

$$I = \int_{R_1}^{R_2} \frac{2M}{(R_2^2 - R_1^{2)}} x^3\ dx = \frac{2M}{(R_2^2 - R_1^2)} \left[\frac{x^4}{4}\right]_{R_1}^{R_2}$$

$$= \frac{M}{2(R_2^2 - R_1^2)} [R_2^4 - R_1^4] = \frac{M}{2}(R_2^2 + R_1^2)$$

$$\boxed{I = \frac{M}{2}(R_2^2 + R_1^2)}$$

(ii) About the diameter : Moment of inertia of the annular disc about any diameter can be obtained by using the theorem of perpendicular axis.

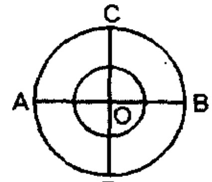


Fig. 9

$$I = I_{AB} + I_{CD}$$

$$I = 2I_{AB} \qquad [\because\ AB = CD]$$

$$I_{AB} = \frac{1}{2} I$$

$$I_{AB} = \frac{1}{2} \times \frac{M}{2}(R_2^2 + R_1^2)$$

$$\boxed{I_{AB} = \frac{M}{4}(R_2^2 + R_1^2)}$$

This is the desired result.

(iii) About the tangent : Moment of inertia of the annular disc can be obtained by using theorem of parallel axis, *i.e.*,

$$I_{EF} = I_{cm} + M\, R_2^2$$

$$= I_{AB} + M\, R_2^2$$

$$= \frac{M}{4}(R_2^2 + R_1^2) + M\, R_2^2$$

$$I_{EF} = \frac{M}{4}(R_1^2 + 5R_2^2)$$

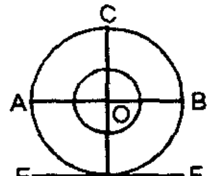


Fig. 10

• 2.8. MOMENT OF INERTIA OF A SOLID CYLINDER

Moment of inertia of a solid cylinder about an axis perpendicular to geometrical axis and passing through its centre.

Let us consider a solid cylinder of mass M and radius R with centre of mass 'O'. Let l be the length of the cylinder as shown in the fig. 11. Let XX' be the axis of the cylinder and YY' be the axis about which M.I. is to be determined.

Consider a small part of this cylinder. It will be a disc of the same radius R and width dx. Let x be the distance of this disc from YY'.

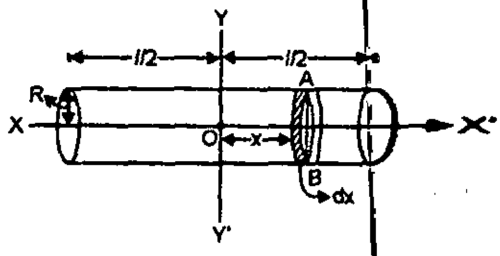


Fig. 11

$\because$ Mass of the cylinder $= M$

and volume of the cylinder $= \pi R^2 l$

$\therefore$ Mass per unit volume $= \dfrac{M}{\pi R^2 l}$

Volume of the disc = surface area × width $= \pi R^2 dx$

$\therefore$ Mass of disc $= \dfrac{M}{\pi R^2 l} . \pi R^2 dx = \dfrac{M}{l} dx$

M.I. of this disc about its diameter, *i.e.*, about AB

$$= \frac{1}{4} \text{mass} \times \text{radius}^2$$

$$= \frac{1}{4}\frac{M}{l} dx . R^2 = \frac{1}{4}\frac{MR^2}{l} dx$$

M.I. of the disc about YY' can be obtained by using theorem of parallel axes

$$= \frac{1}{4}\frac{M}{l} dx . R^2 + \frac{M}{l} dx . x^2$$

$$= \frac{1}{4}\frac{M}{l}(R^2 + 4x^2)\, dx$$

M.I. of the solid cylinder about YY'

$$I = \int_{-l/2}^{l/2} \frac{1}{4}\frac{M}{l}(R^2 + 4x^2)\, dx$$

$$= \frac{1}{4}\frac{M}{l}\left[R^2 \int_{-l/2}^{l/2} dx + 4\int_{-l/2}^{l/2} x^2 dx\right]$$

$$= \frac{1}{4}\frac{M}{l}\left[R^2 . \frac{2l}{2} + \frac{4}{24} . 2l^3\right]$$

$$= \frac{1}{4}\frac{M}{l}\left[R^2 l + \frac{8}{24} . l^3\right]$$

$$\boxed{I = M\left[\frac{R^2}{4} + \frac{l^2}{12}\right]} \quad \ldots (1)$$

(b) If ρ is the density of its material then

$$M = \pi R^4 / \rho$$

$$\therefore \quad R^2 = \frac{M}{\pi l \rho}$$

so by (1) we get

$$I = M\left(\frac{l^2}{12} + \frac{M}{4\pi l \rho}\right)$$

Now, I will be minimum if $\dfrac{dI}{dl} = 0$ *i.e.*,

$$\frac{d}{dt}\left[M\left(\frac{l^2}{12}+\frac{M}{4\pi l\rho}\right)\right]=0$$

$$M\left[\frac{2l}{12}-\frac{M}{4\pi l^2\rho}\right]=0$$

$$\frac{l}{6}=\frac{M}{4\pi l^2\rho}=\frac{\pi R^3 l\rho}{4}$$

$$\therefore \qquad \boxed{\frac{l}{R}=\sqrt{\frac{3}{2}}}$$

This is the required relation between l and R.

(c) If the rod is very thin so that its radius is negligible ($R = 0$) then from equation (1) M.I. of the rod about the axis in part (a) is

$$I=\frac{Ml^2}{12}$$

If K is the radius of gyration of the rod about this axis then

$$I=\frac{Ml^2}{12}=MK^2$$

$$K=\frac{l}{\sqrt{12}}$$

• 2.9. MOMENT OF INERTIA OF A CYLINDER ABOUT ITS OWN AXIS

(a) Moment of inertia of cylinder about its own axis : Let us consider a cylinder of mass M and radius R. Let l be the length of the cylinder as shown in the fig. 12. Let XX' be its geometrical axis about which its moment of inertia is to be determined. Consider a small part of this cylinder. It will be a disc. Let the mass of this disc be m.

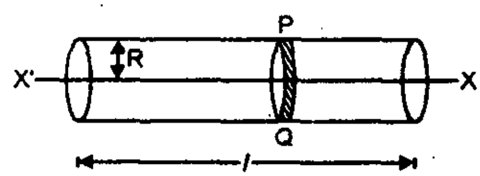


Fig. 12

M.I. of one disc PQ about

$$XX'=\frac{1}{2}\times(\text{mass})\times(\text{radius})^2$$

$$=\frac{1}{2}mR^2$$

M.I. of the whole cylinder about XX' will be equal to the sum of the moment of inertia of these discs, *i.e.*,

$$I=\frac{1}{2}MR^2+\frac{1}{2}mR^2+\frac{1}{2}mR^2+\ldots$$

$$I=\Sigma\frac{1}{2}mR^2=\frac{1}{2}R^2\,\Sigma m$$

$$I=\frac{1}{2}MR^2$$

where $\Sigma m = M$ = mass of the cylinder.

Now the M.I. of the cylinder about the line parallel to its axis and touching its surface can be obtained by using theorem of parallel axes, *i.e.*,

$$\frac{1}{2}MR^2+MR^2=\frac{3}{2}MR^2$$

(b) The M.I. of the cylinder about its own axis

$$=\frac{1}{2}MR^2$$

M.I. of the cylinder about the equatorial axis is

$$= M\left[\frac{l^2}{12}+\frac{R^2}{4}\right]$$

when these are equal, then

$$M\left[\frac{l^2}{12}+\frac{R^2}{4}\right] = \frac{1}{2}MR^2$$

$$\frac{l^2}{12}+\frac{R^2}{4} = \frac{R^3}{2}$$

$$\frac{l^2}{12} = \frac{R^2}{4}$$

$$l^2 = 3R^2$$

$$\boxed{l = \sqrt{3}\,R}$$

This is the required relation between l and R.

• 2.10. MOMENT OF INERTIA OF A THIN SPHERICAL SHELL

(1) About a diameter : Let us consider a thin spherical shell of mass M and radius R with centre O.

Consider a small part of this shell. This lies between two parallel planes AB and CD and is perpendicular to XX'. This small part will be a ring. Let its thickness be dx at a distance x from the centre O as shown in fig. 13.

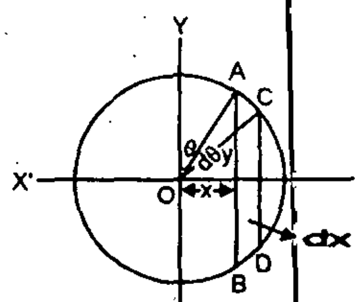


Fig. 13

From the fig.

Radius of ring = $R\cos\theta$

$$y = \sqrt{R^2 - x^2}$$

and $\quad x = R\sin\theta$

$\therefore \quad dx = R\cos\theta\, d\theta$

$\because$ Mass of the spherical shell = M

Surface area of the shell = $4\pi R^3$

$\therefore$ Mass per unit area $= \dfrac{M}{4\pi R^3}$

area of the ring = circumference × width

$$= 2\pi y\, AC$$

$$= 2\pi . R\cos\theta\, .\, R\, d\theta$$

$$= 2\pi R\, dx$$

$\therefore \quad$ Mass of ring $= \dfrac{M}{4\pi R^2}\, .\, 2\pi R\, dx = \dfrac{M}{2R}\, dx$

M.I. of ring about XX' = mass × (radius)^2

$$= \frac{M}{2R}\times y^2 = \frac{M}{2R}\times (R^2 - x^2)\, dx$$

M.I. of the spherical shell is

$$I = \int_{-R}^{R}\left[\frac{MR^2}{2R} - \frac{Mx^2}{2R}\right] dx = \left[\frac{MR}{2}x - \frac{Mx^3}{6R}\right]_{-R}^{R}$$

$$\boxed{I = \frac{2}{3}MR^2}$$

(ii) About the tangent : In case when spherical shell is parallel to any diameter at a distance R from the centre. Now from theorem of parallel axis M.I. of the shell about tangent is

$$I_t = \frac{2}{3}MR^2 + MR^2 = \frac{5}{3}MR^2$$

• 2.11. MOMENT OF INERTIA OF SOLID SPHERE ABOUT DIAMETER

Let us consider a solid sphere of mass M and radius R with centre O. We have to calculate the moment of inertia of this sphere about its diameter. *i.e.*, about XX' as shown in the fig. 14.

Volume of the solid sphere $= \frac{4}{3}\pi R^3$

Mass per unit volume of the solid sphere

$$= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

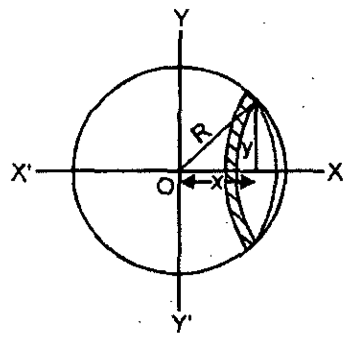


Fig. 14

Consider a small part of this sphere, it will be a disc. Let x be the radius of this disc and dx be the thickness of this disc with centre O.

From the figure the radius of this thin disc is

$$y = \sqrt{R^2 - x^2}$$

Volume of this disc $= \pi(R^2 - x^2)\,dx$

$\therefore$ Mass of disc $= \frac{3M}{4\pi R^3} \cdot \pi(R^2 - x^2)\,dx$

$$= \left(\frac{3M}{4\pi R^2} \cdot \pi R^2 - \frac{3M}{4\pi R^3} \cdot \pi x^2\right) dx = \left(\frac{3M}{4R} - \frac{3M}{4R^3}x^2\right) dx$$

$\therefore$ M.I. of this disc about $XX' = \frac{1}{2}$ mass $\times$ (radius)^2

$$= \frac{1}{2}\frac{3M}{4R^3}(R^2 - x^2)\,dx(R^2 - x^2)$$

$$= \frac{1}{2}\frac{3M}{3R^3}(R^2 - x^2)^2\,dx$$

$$= \frac{1}{2} \cdot \frac{3M}{4R^3}(R^4 + x^4 - 2R^2x^2)\,dx$$

$\therefore$ M.I. of the solid sphere about XX' is

$$I = \int_{-R}^{R} \frac{1}{2}\frac{3M}{4R^3}(R^4 + x^4 - 2R^2x^2)\,dx$$

$$= \frac{1}{2}\frac{3M}{4R^3}\int_{-R}^{R}(R^4 + x^4 - 2R^2x^2)\,dx$$

$$= \frac{1}{2}\frac{3M}{4R^3}\left[R^4x - 2R^2 \cdot \frac{x^3}{3} + \frac{x^5}{5}\right]_{-R}^{R}$$

$$\boxed{I = \frac{2}{5}MR^2}$$

(ii) About a tangent : Any tangent to the sphere at any point is parallel to one of its diameters so by the theorem of parallel axes

$$I_t = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$

• 2.12. BODY ROLLING DOWN AN INCLINED PLANE

(a) Body rolling down an inclined plane : Let us consider a body of mass M and radius R rolling down an inclined plane, which makes an angle θ with horizontal. When a body rolls without slipping then it rotates about a horizontal axis through its centre of mass and also its centre of mass moves. So, the rolling may be assumed as a rotational motion about an axis through its centre of mass plus a translational motion of the centre of mass.

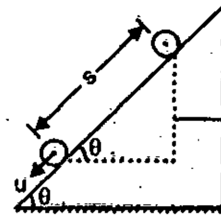


Fig. 15

In fig. 15, a body starts to rool down at certain height. When it rolls then it suffers loss in gravitational potential energy, but gains kinetic energy that of rotation. This loss in kinetic energy must be equal to the total gain in kinetic energy, provided no energy is lost due to friction between the body and the plane.

Let υ be the linear velocity of its centre of mass and ω be the angular velocity about the centre of mass after rolling down the plane a distances.

The loss in gravitational potential energy = weight × loss in vertical height $= Mgs \sin\theta$

Now, translational E.K. gained by the body $= \frac{1}{2} M\upsilon^2$

and the rotational K.E. $= \frac{1}{2} I\omega^2$

where I is the moment of inertia about rotational axis.

$$\text{Total energy gained by the body} = \frac{1}{2} M\upsilon^2 + \frac{1}{2} I_{cm}\omega^2 \qquad \text{... (1)}$$

If K be the radius of gyration of the body about the axis of rotation, then

$$I = MK^2 \quad \text{also} \quad \omega = \frac{\upsilon}{R}$$

∴ by eqn. (1)

Total energy gained by the body

$$= \frac{1}{2} M\upsilon^2 + \frac{1}{2}(MK^2)\frac{\upsilon^2}{r^2}$$

$$= \frac{1}{2} M\upsilon^2 \left(1 + \frac{K^2}{R^2}\right)$$

Assuming that no energy is used up against friction, the loss in potential energy is equal to the total gain in kinetic *i.e.*,

$$Mgs \sin\theta = \frac{1}{2} M\upsilon^2 \left(1 + \frac{K^2}{R^2}\right)$$

$$\upsilon^2 = 2s = \frac{g \sin\theta}{1 + \frac{K^2}{R^2}}$$

$$\upsilon^2 = \frac{2gs \sin\theta}{1 + \frac{K^2}{R^2}}$$

$$\therefore \quad \upsilon = \sqrt{\left(\frac{2gs \sin\theta}{1 + \frac{K^2}{R^2}}\right)} \qquad \text{... (2)}$$

This is the required expression for the final velocity of the body rolling on an incline plane.

$$\because \quad v^2 = 2as$$

$$a = \frac{v^2}{2s}$$

$$\boxed{a = \frac{g \sin \theta}{1 + \frac{K^2}{R^2}}} \qquad \text{... (3)}$$

This is the expression for the final acceleration.

(b) For different rolling bodies, acceleration is obtained as follows :

(i) Solid sphere : The moment of inertia of a solid sphere about its diameter is given by

$$I = MK^2 = \frac{2}{5}MR^2$$

$$\therefore \quad \frac{K^2}{R^2} = \frac{2}{5}$$

so by eqn. (3)

$$a = \frac{g \sin \theta}{1 + \frac{2}{5}} = \frac{5}{7} g \sin \theta \qquad \text{... (4)}$$

(ii) Disc : The M.I. of a disc about the axis passing through its centre and perpendicular to its plane is given by

$$I = MK^2 = \frac{1}{2}MR^2$$

$$\frac{K^2}{R^2} = \frac{1}{2}$$

so by eqn. (3)

$$a = \frac{g \sin \theta}{1 + \frac{1}{2}} = \frac{2}{3} g \sin \theta \qquad \text{... (5)}$$

This is also the acceleration for cylinder.

(iii) Spherical shell : Its M.I. about the diameter is given by

$$I = MK^2 = \frac{2}{3}MR^2$$

$$\frac{K^2}{R^2} = \frac{2}{3}$$

so by eqn. (3)

$$a = \frac{g \sin \theta}{1 + \frac{2}{3}} = \frac{3}{5} g \sin \theta \qquad \text{... (6)}$$

(iv) Ring : Its M.I. about the axis passing through its centre and perpendicular to its plane is given by

$$I = MR^2 = MR^2$$

$$\frac{K^2}{R^2} = 1$$

so by eqn. (3)

$$a = \frac{g \sin \theta}{1 + 1} = g \sin \theta \qquad \text{... (7)}$$

From eqns. (4), (5), (6) and (7) we get the ratio of acceleration of solid sphere, d: spherical shell and ring as

$$= \frac{5}{7} : \frac{2}{3} : \frac{3}{5} : \frac{1}{2} = 150 : 140 : 126 : 105$$

Hence the acceleration of solid sphere, disc, shell and ring are in a decreasing ord Therefore, if all of them start rolling at the same instant then, the sphere will rea down the plane first, then the disc, then the shell and then the ring.

Student Activity

3. Find the relation between length and the radius of the cylinder so that M.I. minimum.

4. If earth were to shrink suddenly, what would happen to the length of the day ?

• 2.13. PRECESSION

When a torque is exerted perpendicular to the axis of rotation of a rotating body the the rate of rotation of the body remains constant but the direction of the axis of rotatio changes, *i.e.*, the axis of rotation itself rotates. The motion of the axis of rotation about fixed axis due to an external torque is called **precession.** The axis about which th direction of rotation of the body processes is called the axis of precession.

In other words we can say that **the turning of the axis of rotation is calle precession.**

Gyroscope : In majority cases, the body, subjected to precessional motion, i supported at a point, away from the vertical line through its centre of gravity, where th axis of rotation is free to turn about the centre of gravity of the body.

Such a body, with its axis of spin supported at a point away from its centre c gravity and with the precessional rate of its spin axis maintained by the gravitationa torque (due to its weight) about that point is called **a gyroscope or a top.**

Procession of a top spinning in earth's gravitational field : Top is symmetrical body rotating about an axis, one point of which is fixed. In the fig. 16 (a) to is spinning with angular velocity ω about its own axis of symmetry, O is the fixed poin at the origin of an inertial reference frame. Its angular momentum is $\vec{L}$, pointing alon the axis of rotation. This axis makes an angle θ with the vertical.

Let the position of centre of mass be $\vec{r}$ with respect to O.

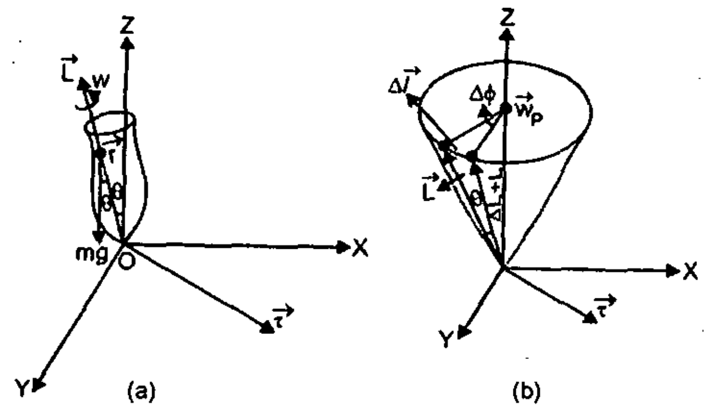


Fig. 16

The weight of the top is mg which exerts a torque about the fixed point O.

We have $\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times m\vec{g}$

Its magnitude is $\tau = rmg\,(180° - \theta) = rmg \sin\theta$ ... (1)

According to right-hand-rule the direction of torque is perpendicular to the plane containing $\vec{r}$ and mg. This means that the torque $\vec{\tau}$ is perpendicular to $\vec{L}$ or perpendicular to the axis of rotation of the top.

The torque $\vec{\tau}$ change the angular momentum $\vec{L}$ of the top. The change ΔL is also in the direction of the torque, *i.e.*, perpendicular to $\vec{L}$.

If this change takes place in a time Δt, the torque is given by

$$\vec{\tau} = \frac{\Delta \vec{L}}{\Delta L} \quad \text{... (2)}$$

The angular momentum $\vec{L} + \Delta\vec{L}$, after a time Δt is the vector sum of $\vec{L}$ and $\Delta\vec{L}$. When $\Delta\vec{L}$ the perpendicular to $\vec{L}$ and is very small so the new angular momentum vector $\vec{L} + \Delta\vec{L}$ has the same magnitude as the initial angular momentum $\vec{L}$, but a different direction. *i.e.*, the angular momentum remains constant in magnitude but varies in direction. The top of the angular momentum vector L describes a circle around z-axis. In time Δt the radius of this circle moves through an angle $\Delta\phi$. This angular velocity of precession ω_p is defined as the rate at which the axis of rotation itself rotates about a fixed axis OL in the laboratory.

Now,
$$\Delta\phi = \frac{\Delta L}{L \sin\theta} = \tau \frac{\Delta t}{L \sin\theta}$$

$$\omega p = \frac{\Delta\phi}{\Delta t} = \frac{\tau \dfrac{\Delta t}{L \sin\theta}}{\Delta t}$$

$$\omega p = \frac{\tau}{L \sin\theta} \quad \text{... (3)}$$

From eq. (1) putting the value of τ in eqn. (3) we get

$$\omega p = \frac{rmg \sin\theta}{L \sin\theta}$$

$$\omega p = \frac{rmg}{L}$$

Thus the angular velocity of procession is independent of θ and is inversely proportional to the magnitude of angular momentum. Larger the angular momentum smaller will be the precessional velocity. As the spinning to P slows down, its angular momentum L (= $I\omega$) decreases and the angular velocity of precession increases, $\vec{\omega}_p$ is a vector pointing vertically upwards as shown in fig. 18(b).

From
$$\omega p = \frac{\tau}{L \sin\theta}$$

$$\tau = \omega p L \sin\theta$$

From the fig. 16 (b) it is clear that θ is the angle between $\vec{\omega}_p$ and $\vec{L}$ and $\vec{\tau}$ is a vector perpendicular to the plane formed by $\vec{\omega}_p$ and $\vec{L}$. So $\omega_p L \sin\theta$ is a vector product of $\vec{\omega}_p$ and $\vec{L}$, *i.e.*,

$$\vec{\tau} = \vec{\omega}_p \times \vec{L}$$ **Hence Proved.**

2.14. RELATIONS BETWEEN THE ELASTIC CONSTANTS

The four elastic constants Y, K, η and σ of a body are not independent but related with each other and when two of them are known others can be calculated.

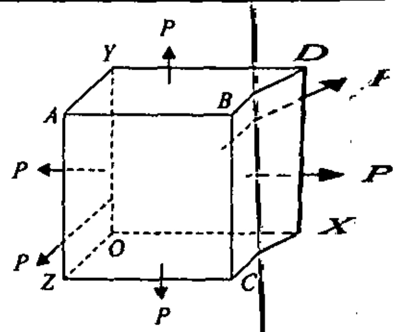


Fig. 17.

Consider a cube *ABCZ YDXO* of unit side and let a force P act normally outwards on each of its six faces (fig. 17). Each force produces extension in its own direction and compressions in perpendicular directions. Thus the forces acting parallel to the X-axis on the planes *BDXC* and *AYOZ* produce extensions along the X-axis and compressions along Y and Z-axes.

Now Young's modulus, $Y = \dfrac{\text{longitudinal stress}}{\text{longitudinal strain}}$

$$\therefore \quad \text{Longitudinal strain} = \frac{\text{longitudinal stress}}{Y}$$

$$= \frac{P/1}{Y} = \frac{P}{Y} \qquad [\because \text{ face area is unity}]$$

$$\because \quad \text{Extension along the } X\text{-axis} = \frac{P}{Y}.$$

Again,
$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

$$\Rightarrow \quad \text{Lateral strain} = \sigma \times \text{longitudinal strain} = \sigma \times \frac{P}{Y}.$$

$$\therefore \quad \text{Contraction along the } Y \text{ and } Z\text{-axes} = \sigma \times \frac{P}{Y}$$

or
$$\text{Extension along the } Y \text{ and } Z\text{-axes} = -\sigma \frac{P}{Y}.$$

Thus the effect of the forces acting parallel to the X-axis is to produce extension $P/Y, -\sigma P/Y$ and $-\sigma P/Y$ along X, Z and Z-axes respectively. In a similar way, considering the pairs of forces acting along Y and Z-axes, we have the total extensions e_x, e_y and e_z along X, Y and Z-axes respectively as follows :

$$e_x = \frac{P}{Y} - \frac{\sigma P}{Y} - \frac{\sigma P}{Y} = \frac{P}{Y}(1 - 2\sigma),$$

$$e_y = -\frac{\sigma P}{Y} + \frac{P}{Y} - \frac{\sigma P}{Y} = \frac{P}{Y}(1 - 2\sigma),$$

$$e_z = -\frac{\sigma P}{Y} - \frac{\sigma P}{Y} + \frac{P}{Y} = \frac{P}{Y}(1 - 2\sigma).$$

Thus each side of the cube becomes $1 + \dfrac{P}{Y}(1 - 2\sigma)$. Now considering all the six faces together, they constitute a normal stress of magnitude P. The cube, therefore, suffers a change in volume. The new volume is

$$\left\{1 + \frac{P}{Y}(1 - 2\sigma)\right\}^2 = 1 + \frac{3P}{Y}(1 - 2\sigma) \text{ approx.}$$

Since the original volume was unity, change in volume

$$= 1 + \frac{3P}{Y}(1 - 2\sigma) - 1$$

$$= \frac{3P}{Y}(1 - 2\sigma)$$

Hence volume strain $= \dfrac{\text{change in volume}}{\text{original volume}} = \dfrac{3P}{Y}(1 - 2\sigma).$

The bulk modulus K is, therefore,

$$K = \frac{\text{Volume stress}}{\text{Volume strain}}$$

$$= \frac{P}{\frac{3P}{Y}(1-2\sigma)} = \frac{Y}{3(1-2\sigma)}.$$

or $$\boxed{Y = 3K(1-2\sigma).} \quad \ldots (1)$$

Now instead of extensional stresses on each of the six faces of the cube, consider compressional stresses on the faces $YDBA$ and $OXCZ$ parallel to Y-axis and an equal extensional stress on faces $AYOZ$ and $BDXC$ parallel to X-axis (Fig. 18).

Then extensional stress P, parallel to X-axis, will produce extension P/Y along X-axis, and compression $\sigma P/Y$ along each of the Y and Z-axes. Similarly compressional stress P parallel to the axis of Y will produce compression P/Y along the Y-axis and extensions $\sigma P/Y$ along each of the X and Z-axes. The net extensions e_x, e_y and e_z along the three axes of X, Y and Z are, therefore,

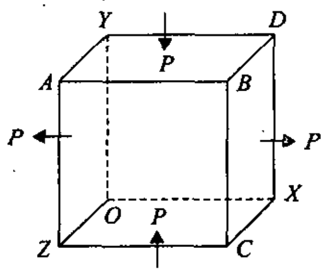


Fig. 18.

$$e_x = \frac{P}{Y} + \frac{\sigma P}{Y} = \frac{P}{Y}(1+\sigma),$$

$$e_y = -\frac{\sigma P}{Y} - \frac{P}{Y} = -\frac{P}{Y}(1+\sigma),$$

$$e_z = -\frac{\sigma P}{Y} + \frac{\sigma P}{Y} = 0.$$

Thus we have equal extension and compression along X and Y-axes. But we know that the sum of simultaneous equal compression and extension at right angles to each other are equivalent to a shear θ and hence

$$\frac{P}{Y}(1+\sigma) + \frac{P}{Y}(1+\sigma) = \theta$$

or $$\frac{2P}{Y}(1+\sigma) = \theta$$

or $$\frac{P}{\theta} = \frac{Y}{2(1+\sigma)}.$$

The extensional stress P and the compressional stress P at right angles to each other are equivalent to a shearing stress P.

Therefore, the modulus of rigidity

$$\eta = \frac{\text{shearing stress}}{\text{shear}}$$

$$= \frac{P}{\theta} = \frac{Y}{2(1+\sigma)}.$$

$\therefore$ $$Y = 2\eta(1+\sigma). \quad \ldots (2)$$

Now from (1), we have

$$\frac{Y}{3K} = 1 - 2\sigma$$

and from (2) $$\frac{Y}{\eta} = 2 + 2\sigma.$$

Adding these two,

$$\frac{Y}{3K} + \frac{Y}{\eta} = 3$$

or $$\boxed{Y = \frac{9\eta K}{3K+\eta}.} \quad \ldots (3)$$

Similarly eliminating Y from (1) and (2), we have

$$3K(1-2\sigma) = 2\eta(1+\sigma)$$

or
$$3K - 2\eta = \sigma(2\eta + 6K)$$

or
$$\boxed{\sigma = \frac{3K - 2\eta}{6K + 2\eta}} \qquad \ldots (4$$

Equations (1), (2), (3) and (4) are the relations connecting the four elastic constants.

Limiting values of σ : Equating the right hand sides of equations (1) and (2) we get

$$3K(1-2\sigma) = 2\eta(1+\sigma). \qquad \ldots (5$$

Since both K and η are positive quantities, σ may either be a positive or negative quantity. If σ is positive, right hand side of equation (5) is positive and for the left hand side to be positive,

$$1 - 2\sigma > 0$$

or
$$1 > 2\sigma$$

or
$$\sigma < \frac{1}{2}, \quad \text{or} \quad \sigma < 0.5$$

If σ is a negative quantity, left hand side of (5) is positive and for the right hand side to be positive,

$$1 + \sigma > 0$$

or
$$\sigma > -1.$$

Thus for a homogeneous isotropic material the value of σ *must lie between* 0.5 *and* — 1. It cannot be greater than $+\frac{1}{2}$ and cannot be less than – 1. In actual practice, however, σ is not negative. A negative value of σ would mean that on being extended, a body should also expand laterally. Since no substance behaves in this way, σ in practice lies between 0 and 0.5.

• 2.15. BENDING OF BEAMS

[A] Some important Definitions.

Beam : A beam is a bar of uniform rectangular (or circular) cross section whose length is much greater as compared to its thickness.

Longitudinal filament : A rectangular beam can be regarded as made up of a larger number of horizontal layers placed one above the other (Fig. 19 a). Further each layer may be considered to be made up of a large number of fibres lying parallel to the length of the beam. These fibres are called longitudinal filaments (Fig. 19 b).

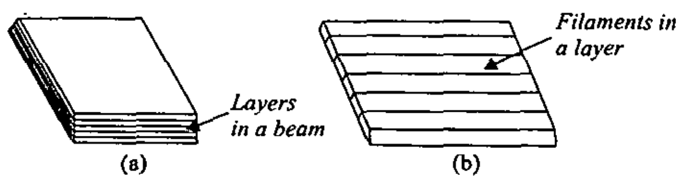


Fig. 19.

Neutral surface : When equal and opposite couples are applied at the ends of a beam in a plane parallel to its length, the beam bends into a circular arc [Fig. 20 (a)]. Due to bending, the filaments on the convex side are lengthened while those on the concave side are shortened. There is, however, a layer in between in which the filaments are neither lengthened nor shortened but remain constant in length. This layer is called ***neutral layer*** or ***neutral surface*** [Fig. 20 (b)].

Plane of Bending : The plane in which the beam bends is called the plane of bending. It is the same as the plane of applied couple. If the beam is placed horizontally, it is obviously the vertical plane [Fig. 20 (b)].

Neutral axis : The line of intersection of neutral layer with the plane of bending is called the neutral axis [Fig. 20 (b)]. In a horizontal beam the neutral axis coincides with the geometrical axis.

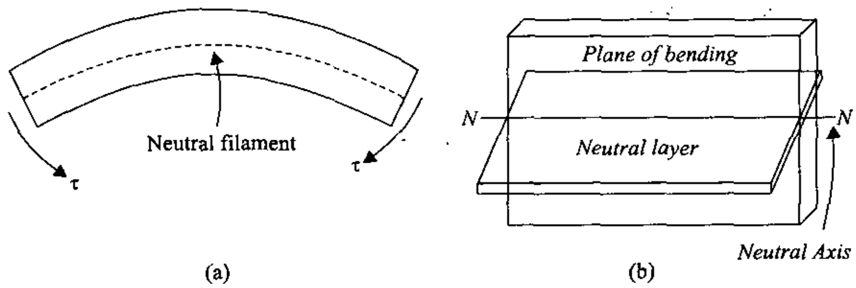


Fig. 20.

Bending Moment : Let $ABCD$ be the longitudinal section of a beam bent under the action of equal and opposite couples τ, τ at its ends. Let NN' be its neutral axis. Consider the beam to be divided into two parts by a transverse plane through PS. Due to bending, the filaments of the beam above the neutral surface are lenghened and are, therefore, in tension. Their portions to the left of PS exert a ***pulling force*** on their portions to the right of PS. The filaments of the beam below the neutral axis are shortened and hence their portions to the left of PS exert a pushing force on their portions to the right. The resultant of these pulling and ***pushing force*** is a restoring couple which balances the external couple responsible for the bending of the beam. ***The magnitude of this restoring couple is called the bending moment.*** Thus ***bending moment*** may be defined as ***the total moment of all the forces arising in a bent beam and trying to resist its deformation caused by an external couple.***

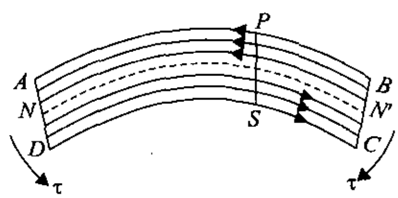


Fig. 21.

[B] Expression for Bending Moment

Let us now consider a small part of the beam bounded by two transverse sections PS and QR. Let R be the radius of curvature of the neutral axis NN' and let it subtend an angle ϕ at its centre of curvature O. Consider then a filament KL at a distance $KN = z$ from the neutral axis, so that $KO = (R + z)$. It follows from fig. (22) that

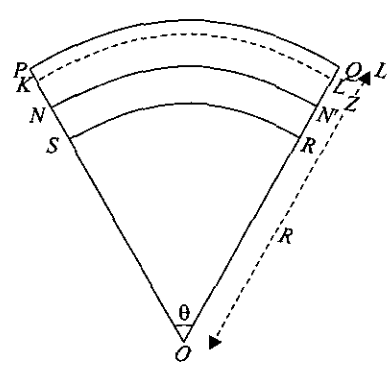


Fig. 22.

$$KL = (R + z)\,\phi$$

and $$NN' = R\phi.$$

Before bending, the length of the filament KL was equal to $NN' = R\phi$. Hence the extension in the filament KL is

$$(R + z)\,\phi - R\phi = z\phi.$$

$\therefore$ longitudinal or extensional strain for this filament is

$$\frac{\text{increase in length}}{\text{original length}} = \frac{z\phi}{R\phi} = \frac{z}{R}$$

Thus the strain in a filament is directly proportional to its distance from the neutral axis.

If Y is the Young's modulus of elasticity of the beam, then

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

or longitudinal stress = $Y \times$ longitudinal strain = $Y \times \dfrac{z}{R}$.

Now, if 'a' be the area of cross-section of the filament depicted in fig. 23, the fc
acting on this area = stress × area

$$= Y \frac{z}{R} \times a.$$

The moment of this force about a line through the neutral axis and perpendicular to the plane of bending *i.e.* through NN' is given by

$$Y \cdot \frac{z}{R} \times a \times z = \frac{Ya}{R} \cdot z^2.$$

The sum of the moments of all these forces of extension and compression acting over the whole cross-section of the beam *i.e.* the ***internal bending moment*** is given by

$$G = \Sigma \frac{Ya}{R} \cdot z^2 = \frac{Y}{R} \Sigma a z^2 = \frac{Y}{R} I.$$

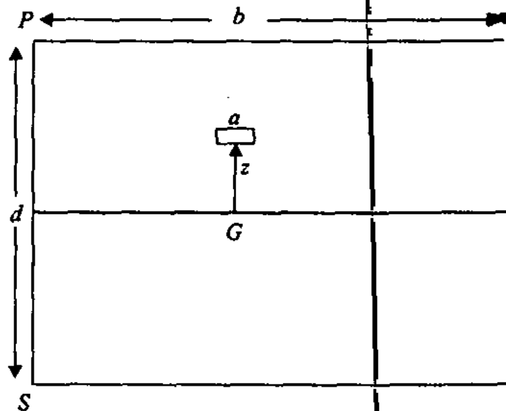


Fig. 23.

where $I = \Sigma a \cdot z^2$ is a quantity analogous to moment of inertia about the axis $z = 0$, t
only difference here being that the mass of the element is replaced by its area. Tl
quantity is, therefore, called ***Geometrical moment of inertia*** of the cross-secti
about an axis through its centroid and perpendicular to the plane of bending. T
quantity YI is known as ***flexural rigidity.***

Now if A is the area of cross-section of the beam and k the radius of gyration of t
cross-section about the axis passing through the centroid and perpendicular to t
plane of bending, then $I = Ak^2$

And, therefore, the internal bending moment $G = \dfrac{YAk^2}{R}$.

For a beam of rectangular cross-section with breadth 'b' and thickness 'd' the area
cross-section is bd and (radius of gyration)^2 is equal to $\dfrac{d^2}{12}$ and hence

$$I = bd \times \frac{d^2}{12} = \frac{bd^3}{12}.$$

Similarly for a beam of circular cross-section of radius r the geometrical moment
inertia of the second area moment is

$$= \pi r^2 \times \frac{r^2}{4} = \frac{\pi r^4}{4}.$$

• 2.16. THE CANTILEVER

A beam fixed horizontally at one end and loaded at the other is called a cantileve
Let AB represent the neutral axis of a cantilever of length l, fixed at the end A ar
loaded at B with a mass M (fig. 24). The end B is thus depressed downward compared
A and the neutral axis takes up the new position AB'. It is assumed that the weight
the beam is negligible and produces no bending.

Let us take the axis of X horizontally in the direction of unbent beam and axis of Y vertically downward with the fixed end A of the beam as the origin of co-ordinates. Consider then the equilibrium of a transverse section of the beam at P with co-ordinates (x, y). The distance of this section at P from the free end is $(l - x)$ and hence the moment of external couple at this section due to the load Mg is $Mg\,(l - x)$. This moment of external couple is maximum for the smaller value of x, *i.e.* at the fixed end. ***Hence a cantilever of uniform section is more likely to break near its fixed end.*** For equilibrium of this section

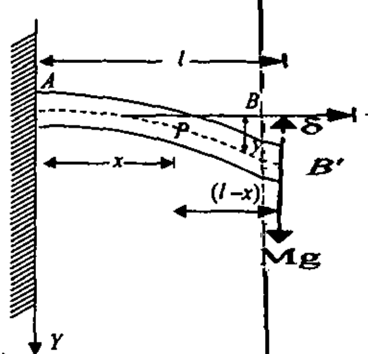


Fig. 24.

at P, the moment of external couple must be equal to the internal bending moment YI/R, where R is the radius of curvature of the neutral axis at the section P. Hence

$$\frac{YI}{R} = Mg(l-x) \quad \text{or} \quad \frac{1}{R} = \frac{Mg}{YI}(l-x).$$

If now y be the depression of the beam at the section at P, the radius of curvature R at this section is given by

$$\frac{1}{R} = \frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}$$

where $\frac{dy}{dx}$ is the slope of the tangent at the point (x, y). But here the depression is small, the slope will be small and consequently $\left(\frac{dy}{dx}\right)^2$ is negligible compared to unity. Thus

$$\frac{1}{R} = \frac{d^2y}{dx^2} \qquad \text{... (2)}$$

Comparing equations (1) and (2), we get

$$\frac{d^2y}{dx^2} = \frac{Mg}{YI}(l-x).$$

Integrating this equation, we get

$$\frac{dy}{dx} = \frac{Mg}{YI}\left(lx - \frac{x^2}{2}\right) + C_1, \qquad \text{... (3)}$$

where C_1 is an integration constant.

Since the curvature at the fixed end A is zero, the tangent is horizontal and hence at $x = 0$, $\frac{dy}{dx} = 0$. Substituting this in (3), we get $C_1 = 0$ and thus

$$\frac{dy}{dx} = \frac{Mg}{YI}\left(lx - \frac{x^2}{2}\right)$$

Integrating again, we get

$$y = \frac{Mg}{YI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

where C_2 is another constant of integration. But since at $x = 0$, $y = 0$, so that $C_2 = 0$ hence

$$y = \frac{Mg}{YI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) \qquad \text{... (4)}$$

Now at the free end of the cantilever, $x = l$ and the depression y is maximum, say δ. Substituting $x = l$ and $y = \delta$ in equation (4), we have the depression at the free end

$$\delta = \frac{Mg}{YI}\left(l \cdot \frac{l^2}{2} - \frac{l^3}{6}\right)$$

$$= \frac{Mgl^3}{3YI}. \qquad \text{... (5)}$$

If the beam is of rectangular cross-section of breadth b and thickness d, its geometrical moment of inertia, $I = \frac{bd^3}{12}$ and hence from (5).

$$\delta = \frac{4Mgl^3}{Ybd^3}. \qquad \text{... (6)}$$

If the beam is of circular cross-section of radius r, $I = \frac{\pi r^4}{4}$ and hence

$$\delta = \frac{4Mgl^3}{3Y\pi r^4}. \quad \ldots (\text{'}$$

• 2.17. POTENTIAL ENERGY AND OSCILLATIONS OF A LOADED CANTILEVE

Consider a cantilever of length l clamped at one end and loaded at its free end wit mass M. When the loaded end is depressed slightly and then released, the cantileve executes transverse vibrations. If at any instant, the loaded end has a depression then the force causing this depression is

$$F = Mg = \frac{3YI}{l^3} \cdot y \qquad \left[\because \quad y = \frac{Mgl^3}{3YI} \text{ at the free end}\right.$$

The amount of work done in a further displacement dy is given by

$$dW = F \cdot dy = \frac{3YI}{l^3} \cdot y\, dy.$$

Hence the ***total work done*** in producing a depression y of the loaded end i.e., th ***potential energy of the cantilever*** when its end has a displacement y relative to th undisplaced position is

$$W = \int_0^y \frac{3YI}{l^3} \cdot y\, dy = \frac{3YI}{l^3} \cdot \frac{y^2}{2} \quad \ldots (1$$

As the weight of the cantilever is assumed negligible, the whole kinetic energy i confined in mass M attached at the free end and hence given by

$$E = \frac{1}{2} Mv^2 = \frac{1}{2} M \left(\frac{dy}{dt}\right)^2 \quad \ldots (2$$

According to the principle of conservation of energy

$$\frac{1}{2} M \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} \frac{3YI}{l^3} y^2 = \text{constant.}$$

Differentiating, we get

$$\frac{1}{2} M \times 2 \frac{dy}{dt} \cdot \frac{d^2y}{dt^2} + \frac{1}{2} \frac{3YI}{l^3} \cdot 2y \cdot \frac{dy}{dt} = 0$$

or
$$\frac{d^2y}{dt^2} + \frac{3YI}{Ml^3} \cdot y = 0 \qquad \text{or} \qquad \frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \ldots (3)$$

where $\omega = \sqrt{\left(\frac{3YI}{Ml^3}\right)}$ is a constant. This equation represents a simple harmonic motion whose time period is given by

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\left(\frac{Ml^3}{3YI}\right)} \quad \ldots (4)$$

• 2.18. BEAM SUPPORTED AT ITS ENDS AND LOADED IN THE MIDDLE

Let AB be a beam supported symmetrically on two knife edges K_1 and K_2 at distance l apart in the same horizontal line, with equal but small lengths of the beam projecting beyond the knife edges (fig. 25). Let the beam be loaded at the middle point O with a mass M. The reaction at each knife edge is $Mg/2$ acting vertically upwards. The beam is bent as shown, the maximum depression being at the loaded point O. Moreover, the portion of the beam in the neighbourhood of O will be horizontal. The beam may, therefore, be looked as made up of two inverted

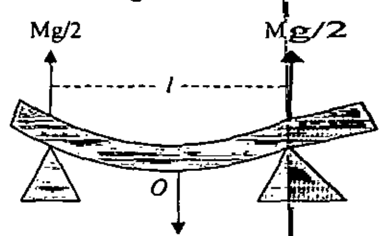


Fig. 25.

cantilevers, each of the length $l/2$ fixed at O, and carrying a load $\frac{Mg}{2}$ at K_1 and K_2. The depression of the loaded beam at O is the same as produced in cantilevers of length $l/2$ $l/2$ loaded by a weight $\frac{Mg}{2}$ at the free end.

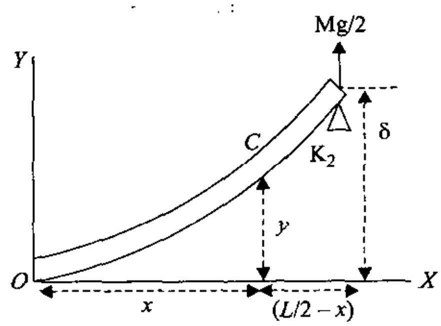


Fig. 26.

Let us consider a section of inverted cantilever at C, at a distance x from O (fig. 26). Consider the equilibrium of the part CK_2. Since the beam is fixed at O, the load $Mg/2$ at K_2 exerts a torque on CK_2 which tends to rotate it anticlockwise. Its magnitude is $\frac{Mg}{2}\left(\frac{l}{2}-x\right)$. In equilibrium position, this torque is balanced by the internal restoring torque which arises due to elastic reaction against the extension of filaments on one side of the neutral surface and compression on the other. The magnitude of this restoring torque is $\frac{YI}{R}$ where Y is the Young's modulus of the material of the beam, I the geometrical moment of inertia of the section C about the neutral surface and R is the radius of curvature at C. Hence in equilibrium

$$\frac{YI}{R}=\frac{Mg}{2}\left(\frac{l}{2}-x\right) \qquad \text{... (1)}$$

Choosing O as the origin of the co-ordinate system, the co-ordinates of C are (x, y) where y is the elevation of C. The radius of curvature at C is given by

$$\frac{1}{R}=\frac{d^2y/dx^2}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}=\frac{d^2y}{dx^2}$$

because $\left(\frac{dy}{dx}\right)^2$ can be neglected in comparison to one when y is small.

Substituting this value of $\frac{1}{R}$ in eqn. (1), we get

$$\frac{Mg}{2}\left(\frac{l}{2}-x\right)=YI\,\frac{d^2y}{dx^2}$$

or

$$\frac{d^2y}{dx^2}=\frac{Mg}{2YI}\left(\frac{l}{2}-x\right)$$

Integrating this eqn., we get

$$\frac{dy}{dx}=\frac{Mg}{2YI}\left(\frac{l}{2}x-\frac{x^2}{2}\right)+C_1$$

where C_1 is constant of integration.

At the fixed point O, the tangent is horizontal *i.e.*, at $x=0$, $\frac{dy}{dx}=0$. Therefore, $C_1=0$.

Hence

$$\frac{dy}{dx}=\frac{Mg}{2YI}\left(\frac{l}{2}x-\frac{x^2}{2}\right)$$

Integrating again, we get

$$y=\frac{Mg}{2YI}\left(\frac{l}{2}\cdot\frac{x^2}{2}-\frac{x^3}{6}\right)+C_2$$

where C_2 is another constant of integration. But since at $x=0$ (at point O). $y=0$, so that $C_2=0$ and we have

$$y = \frac{Mg}{2YI}\left(\frac{lx^2}{4} - \frac{x^3}{6}\right)$$

Now at the end K_2, we have $x = \frac{l}{2}$ and the elevation y is maximum. Putting $x = \frac{l}{2}$ and $y = \delta$ (say) in eqn. (2), we get

$$\delta = \frac{Mg}{2YI}\left(\frac{l}{4}\cdot\frac{l^2}{4} - \frac{l^3}{48}\right) = \frac{Mgl^3}{48YI}$$

For a beam of rectangular cross section of breadth b and thickness d.

$$I = \frac{bd^3}{12}$$

and hence
$$\delta = \frac{Mgl^3}{48Y\cdot\frac{bd^3}{12}} = \frac{Mgl^3}{4Y\,bd^3}$$

If the beam is of circular cross section of radius r, $I = \frac{\pi r^4}{4}$ and hence

$$\delta = \frac{Mgl^3}{12Y\pi r^4}$$

Equations (3) and (4) give the depression at the middle point O for a rectangular and a circular beam respectively.

Determination of Young's Modulus Y.

Equation (3) may be used for the determination of the value of Y for the material of the beam. The beam is placed symmetrically on two strong knife edges K_1 and K_2 as shown in Fig. 27 A hanger with hook is hung from it at a point O, exactly midway between t-he two knife edges. The load may be applied by placing weights on the hanger. The central screw of the spherometer is made to touch the hanger for each load and the reading taken. The position is indicated by the deflection in the galvanometer. Readings are taken first with the load increasing and then load decreasing in equal steps, and their mean taken. This gives the mean depression δ. A graph is then plotted between δ and the corresponding load Mg (Fig. 28). The slope of the straight line obtained gives Mg/δ. The distance l between the knife edges is measured by a metre scale. Thickness d and breadth b of the beam are measured by screw gauge and vernier callipers respectively. Substituting the value of Mg/δ, l, b and d in equation (3) Y can be calculated.

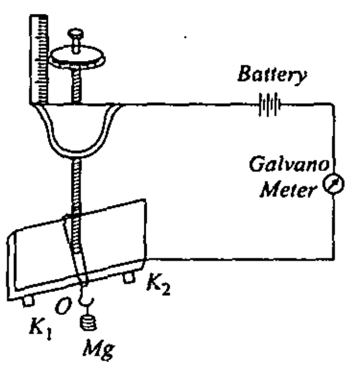


Fig. 27.

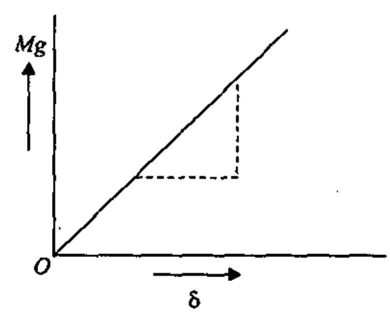


Fig. 28.

• 2.19. APPLICATIONS OF BENDING OF BEAMS

(i) In girders of rectangular cross section the longer side is used as depth. The function of girders is to support heavy loads which produce depression in the girder. For a given load, the depression

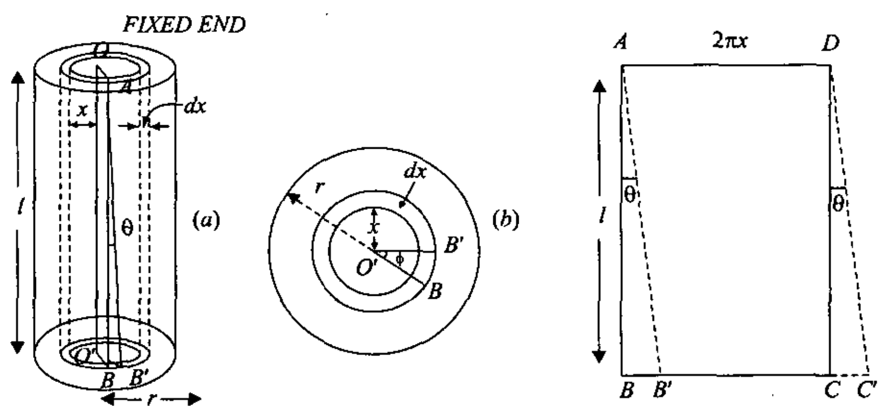
$$\delta \propto \frac{l^3}{bd^3} \qquad \left(\because \quad \delta = \frac{Mgl^3}{4Ybd^3}\right)$$

i.e., the depression in the middle of a beam for a given load is directly proportional to the cube of its length and inversely proportional to its breadth and cube of depth or thickness. Obviously, for the depression to be small, the length or span of the girder should be small, while its breadth and depth should be large. Moreover, the depth in the denominator occurs in third power and hence an increase in depth reduces the depression much more as compared to the same increase in breadth. Thus the depth d is more effective than breadth b in reducing the depression of the beam. Therefore, ***in the girder is of rectangular cross-section, the larger side is used as its depth***, so that the girder may not bend appreciably.

(ii) Steel girders and rails are generally made I shaped : Girders standing on pillars at their ends support load. Consequently the girder suffers bending and its middle part is depressed. In this process the filaments in the upper half are compressed while those in the lower half are extended. These extensions and compressions are greatest near the surface, the stresses produced there are also maximum and decreases towards the neutral surface from either side. Therefore, the upper and lower faces of the girder should be much stronger than its inner portions. The inner parts may, therefore, be made of smaller breadth than the upper and lower parts. It is why the ***steel girders are usually manufactured with their sections in the form of letter I***. Thus a good deal of materials is saved without sacrificing the strength of the girder.

• 2.20. TORSION OF A CYLINDER

Consider a cylindrical rod of length l and radius r of a material of coefficient of rigidity η. Let the upper end of the rod be clamped rigidly and at the lower free end a twisting couple be applied in an anticlockwise direction in a plane perpendicular to the length of the rod. Then each cross-section of the rod rotates about the axis of the rod by an amount proportional to the distance of the cross-section from the fixed upper end. ***The angle through which any cross-section rotates is called the angle of twist.*** Its value is zero at the fixed end and greatest at the free end. In fig. 29 (a) each radius like $O'B$ of the free lower end rotates through an angle ϕ to a new position $O'B'$, also shown separately in fig. 29 (b). Thus ϕ is the angle of twist at the free end.



(a) (b) (c)

Fig. 29.

As the rod is twisted, a restoring couple is set up in it due to the elasticity of the material. In equilibrium position, the restoring couple is equal and opposite to the twisting couple. In order to find the value of this couple, imagine the cylindrical rod to be divided into a large number of thin coaxial cylindrical shells and consider one such shell of radius x and thickness dx. Before twisting the rod, let AB be a line parallel to the axis of the rod. When the rod is twisted, the point B shifts to B', [Fig. 29 (a)] and the line AB is displaced into the position AB', such that if before twisting, the cylindrical shell could be cut along the line AB and flattened out, it would form a rectangle $ABCD$, but after twisting if it is cut along AB' and flattened out, a parallelogram $AB'C'D$ is obtained [Fig. 29 (c)]. Thus the twisting couple shears the cylindrical shell through an angle $BAB' = \theta$. This angle is called the ***angle of shear***, being maximum for outermost layer of the rod and decreasing to zero for the innermost layer.

Referring to fig. 29 (a), we have from $\Delta BO'B'$,

$$\text{Arc } BB' = \text{angle} \times \text{radius} = \phi \times O'B = x\phi$$

Also from $\Delta BAB'$, we have

$$\text{Arc } BB' = \theta \times AB = l\theta.$$

$\therefore$ $$l\theta = x\phi$$

or $$\theta = x\phi / l \qquad \ldots$$

Now let F be the tangential force acting on the base of elementary cylindrical sh considered. Since face area of this shell is $2\pi x\ dx$, the shearing stress acting

$$= \frac{\text{force}}{\text{area}} = \frac{F}{2\pi x\ dx}$$

And hence $$\eta = \frac{\text{shearing stress}}{\text{shearing strain}}$$

$$= \frac{F/2\pi x\,dx}{\theta} = \frac{F}{2\pi x\,dx} \cdot \frac{l}{x\phi}$$

[substituting the value of θ from (

$\therefore$ $$F = \frac{2\pi\eta\phi}{l} x^2 dx. \qquad \ldots$$

The moment of this force about the axis OO' of the rod

$$= \text{force} \times \text{distance}$$

$$= F \times x$$

$$= \frac{2\pi\eta\phi}{l} x^2\ dx \times x$$ [Substituting the value of F from (2

$$= \frac{2\pi\eta\phi}{l} x^3\ dx. \qquad \ldots$$

This is the couple required to twist the elementary shell of radius x through an an ϕ. Therefore the twisting couple τ, required to twist the whole rod (solid cylinder) radius r, is obtained by integration of expression (3) between the limits $x = 0$ and $x =$ Hence

$$\tau_s = \int_0^r \frac{2\pi\eta\phi}{l} x^3\ dx$$

$$= \frac{2\pi\eta\phi}{l}\left[\frac{x^4}{4}\right]_0^r$$

$$= \frac{\pi\eta\phi}{2l} . r^4 \text{ radian.} \qquad \ldots$$

The couple required is thus proportional to the twist ϕ (in radians). Hence *the cou required to produce a twist of one radian is*

$$C_s = \frac{\tau_s}{\phi} = \frac{\pi\eta r^4}{2l}. \qquad \ldots$$

This quantity is called the *torsional rigidity or restoring couple per unit radian tw* of the **solid cylinder** (or wire).

If, however, the cylinder is hollow with inner and outer radii r_1 and r_2 respectively then the ***couple required*** is given by integration of (3) between $x = r_1$ and $x =$ Hence.

$$\tau_h = \int_{r_1}^{r_2} \frac{2\pi\eta\phi}{l} x^3\ dx$$

$$= \frac{\pi\eta\phi}{2l}(r_2^4 - r_1^4). \qquad \ldots$$

This is the **torsional couple for a hollow cylinder.**

The torsional rigidity of the hollow cylinder

$$C_h = \frac{\tau_h}{\phi} = \frac{\pi\eta\,(r_2^4 - r_1^4)}{2l}. \quad \ldots (7)$$

• 2.21. WORK DONE IN TWISTING A WIRE OR CYLINDER

The restoring torsional couple on the end of a cylinder of length l and radius r when it is twisted by ϕ radians keeping the other end clamped is given by

$$\tau = \frac{\pi\eta r^4}{2l}\,\phi = C\,.\,\phi$$

The work required to be done in twisting the cylinder by a further angle $d\phi$ against this torsional couple

$$dW = \tau\,.\,d\phi = C\,.\,\phi\,.\,d\phi$$

Hence the total work done in twisting the cylinder through an angle ϕ is

$$W = \int dW = \int_0^{\phi} C\,.\,\phi\,.\,d\phi = \frac{1}{2}\,C\phi^2$$

This work is stored in the cylinder as the elastic potential energy. Hence this *expression gives the potential energy of a twisted cylinder when it has a twist* ϕ

• SUMMARY

- Angular displacement is the angle described by the position vector $\vec{r}$ about the axis of rotation.
- The rate of change of angular displacement is known as angular velocity.
- The rate of change of angular velocity of a body about the axis of rotation is known as angular acceleration.
- Torque is defined as the external force acting on the body which rotates the body about fixed axis.
- The rate of change of angular momentum of a particle is equal to the torque acting on the particle.
- The moment of inertia of a rigid body about a given axis of rotation is the sum of the products of the masses of the various particles and squares of their perpendicular distance from the axis of rotation. It is given by $I = \Sigma\, mr^2$.
- Theorem of parallel axes states that Moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass plus Mh^2, where M is the mass of the body and h the perpendicular distance between the two axes.
- The equation $\vec{\tau} = I\vec{\alpha}$ is called fundamental equation of relation of law of rotations, where τ = Torque, I = Moment of Inertia, α = Angular acceleration.
- *Theorem of perpendicular axes* states that "The moment of inertia of a plane lamina (a two-dimensional body) about an axis perpendicular to its plane (OZ) is equal to sum of the moments of inertia about any two mutually perpendicular axes OX and OY is its plane intersecting on the first axis."
- The motion of the axis of rotation about a fixed axis due to an external torque is called precession.
- The axis about which the direction of rotation of the body precesses is called the axis of precession.

Student Activity

1. Define elasticity, perfectly elastic body and perfectly plastic body.

2. Define stress, strain and shear.

3. Define modulus of elasticity.

4. What is Hooke's law ?

5. Which of the two-glass and rubber is more elastic and why ?

6. What is meant by "plane of bending" ?

7. What is neutral axis ?

8. What is bending moment ?

• TEST YOURSELF

1. Define torque $\vec{\tau}$ acting on a particle about an axis. Obtain its angular momentum. Find out the relationship between torque and angular momentum.
2. Define moment of inertia of a body. Give its physical significance.
3. Derive an expression for the kinetic energy of a body rotating about an axis, hence define the M.I. of the body.
4. Prove that for a rigid body the angular momentum about an axis of rotation is equal to the product of M.I. and the angular velocity about that axis. Hence show that the K.E. of rotation is $L^2 / 2I$.
5. Obtain a relation between the torque applied and the angular acceleration produced in the body.
6. State and prove that theorem of parallel axes.
7. State and prove the theorem of perpendicular axis.
8. Obtain an expression for M.I. of a thin circular disc (i) about an axis passing through its centre and perpendicular to its plane (ii) about a diameter and (iii) about a tangent in its plane.

9. Find M.I. of annular disc of mass M, inner radius R_1 and outer radius R_2 : (i) about an axis passing through its center and perpendicular to its plane (ii) about a diameter and (iii) about the tangent in its plane.
10. *A solid sphere rolls on a table. What fraction of its total* K.E. *is rotational ?*
11. Deduce expressions for the gravitational potential and attraction due to thin uniform spherical shell at a point (a) outside and (b) inside the shell.
12. If the rotational moment of a body about an axis is to be changed then we must apply about the axis :
    (a) Torque (b) Torque and force
    (c) Force (b) None of these
13. The moment of inertia of a thin uniform circular disc of mass M and radius R about any tangent is :
    (a) $\frac{MR^2}{4}$ (b) $\frac{MR^2}{2}$ (c) $\frac{5MR^2}{2}$ (b) None of these
14. The *moment of inertia of uniform solid sphere of mass* M and radius R about diameter is :
    (a) $\frac{2}{5}MR^2$ (b) $\frac{2}{3}MR^2$ (c) $\frac{2}{7}MR^2$ (b) None of these
15. The moment of inertia of uniform solid cylinder of mass M, radius R and length l about long axis of symmetry is :
    (a) $\frac{1}{2}MR^2$ (b) $\frac{M\rho^2}{12}+\frac{MR^2}{4}$ (c) MR^2 (b) None of these
16. The rate of change of angular momentum is equal to :
    (a) Force (b) Angular Acceleration
    (c) Torque (b) Moment of Inertia
17. A gymanst is *sitting on a rotating stool with her arms outstretched.* Suddenly she folds her arms near the body. Which of the following in correct :
    (a) Angular speed decreases (b) Moment of inertia decreases
    (c) Angular momentum decreases (d) Angular speed remains constant
18. Moment of momentum is called :
    (a) Torque (b) Weight
    (c) Moment of inertia (d) Angular momentum
19. When the torque acting on a system is zero, which of the following will be constant ?
    (a) Force (b) Linear momentum
    (c) Angular momentum (d) Linear impulse
20. A solid sphere of mass M rolls down an inclined plane without slipping from rest at the top of the inclined plane. The linear speed of the sphere *at the bottom of the* inclined plane is V. The K.E. of the sphere is :
    (a) $\frac{1}{2}MV^2$ (b) $\frac{5}{3}MV^2$ (c) $\frac{2}{5}MV^2$ (d) $\frac{7}{10}MV^2$
21. Two circular discs A and B have equal mass and thickness but are made of metals with densities d_A and d_B $(d_A > d_B)$. If their moments of inertia about an axis passing through the centre and normal to circular faces by I_A and I_B then :
    (a) $I_A = I_B$ (b) $I_A > I_B$ (c) $I_A < I_B$ (d) $I_A \geq I_B$
22. What must be the relation between I and R if the moment of inertia of the cylinder about its axis is be the same as the moment of inertia about the equatorial axis :
    (a) $l = \sqrt{3}R$ (b) $R = \sqrt{3}\,l$ (c) $l = \frac{\sqrt{3}}{2}R$ (d) $R = \frac{3}{\sqrt{2}}l$
23. The total kinetic energy of a rolling uniform disc is equal to
    (a) $\frac{2}{3}$ *translational kinetic energy* (b) $\frac{3}{2}$ translational kinetic energy
    (c) 2 translational kinetic energy (d) None of these

24. The moment of inertia of the spherical shell of mass M and radius R about a tange
is :
(a) $\frac{5}{3}MR^2$ (b) $\frac{2}{3}MR^2$ (c) $\frac{7}{5}MR^2$ (d) $\frac{2}{5}MR^2$

25. A body of mass M and radius R rolls down a plane inclined at an angle θ to t
horizontal without slipping, the acceleration of a body will depend upon :
(a) Mass (b) Angle θ (c) Height (d) None of these

26. A sphere, a spherical shell, a ring and a cylinder are allowed to roll dov
simultaneously an inclined plane from the same height without slipping which w
reach earlier :
(a) Shell (b) Cylinder (c) Sphere (d) Ring

27. The relation between angular momentum and angular velocity is :
(a) $\vec{J} = \vec{r} \times \vec{\omega}$ (b) $\vec{J} = \vec{\omega} \times \vec{r}$ (c) $\vec{J} = \frac{1}{\vec{\omega}}$ (d) $\vec{J} = I\vec{\omega}$

28. Dimensional formula for modulus of elasticity is :
(a) $[MLT^{-2}]$ (b) $[ML^{-1}T^{-2}]$ (c) $[ML^{-2}T^{-1}]$ (d) $[ML^{-2}T^{-2}]$

29. Energy per unit volume in a stretched wire is :
(a) $\frac{1}{2}$ × load × strain (b) load × strain
(c) stress × strain (d) $\frac{1}{2}$ × stress × strain

30. The Young's modulus of a wire is numerically equal to the stress which will :
(a) not change the length of wire
(b) double the length of wire
(c) increase the length by 50%
(d) change the area of cross-section of wire to half

31. The Poisson ratio can not have the value :
(a) 0.7 (b) 0.2
(c) 0.1 (d) 0.5

32. What is the relation between Y, B and η for isotopic material ?
(a) $\eta = \frac{3BY}{9B+Y}$ (b) $\eta = \frac{9BY}{4B+Y}$
(c) $\eta = \frac{9BY}{9B-Y}$ (d) $Y = \frac{9B\eta}{3B+\eta}$

33. Poisson's ratio is equal to :
(a) $\frac{\text{Lateral Strain}}{\text{Longitudinal Strain}}$ (b) $\frac{\text{Longitudinal Strain}}{\text{Lateral Strain}}$
(c) Longitudinal Strain × Lateral Strain
(d) None of these

34. On increasing temperature, the value of Young's modulus :
(a) decreases (b) increases
(c) remains constant (d) has no effect

ANSWERS

12. (a) 13. (c) 14. (a) 15. (d) 16. (c) 17. (b) 18. (c) 19. (c) 20. (d) 21. (c)
22. (a) 23. (b) 24. (a) 25. (b) 26. (a) 27. (d) 28. (b) 29. (d) 30. (b) 31. (a)
32. (d) 33. (a) 34. (a)

UNIT 3

MOTION UNDER CENTRAL FORCES : PLANETS AND SATELLITES

STRUCTURE

LEARNING OBJECTIVES

After learning this chapter, you will be able to know ....

- *Main features of central force. Angular momentum, Relative and centre of mass.coardinates.*
- Universal law of gravitation, Kepler's law.
- Newton's law of gravitation, motion of satellities. Escape velocity and orbital velocity.

• 3.1. CENTRAL FORCE

A central force is defined as a force whose line of action always passes through a given centre or fixed point and whose magnitude depends only upon the distance (*r*) from the fixed point. These forces are always directed towards or away from this fixed point. If we consider two particles which exert forces on each other along the line which connects them, then the force on each particle is directed towards the centre of mass of the system because centre of mass lies on the line joining the two particles. It may, therefore, be said to be an example of central force. The force of attraction between the sun and the planets is also a central force.

A central force is a function of r only. Hence the central force on a body, in general, is expressed as

$$\vec{F} = f(r)\,\hat{r}$$

where $f(r)$ is a function of distance r and $\hat{r}$ represents the unit vector along the joining the two bodies.

Accordingly, a central force may be represented as shown in Fig. 1. It represents central force acting on a particle P whose polar coordinates are r and θ. Here O is centre of force, which is taken as the origin of coordinate system. $\hat{\mathbf{r}}$ is the unit ve along the radius vector r of the particle with respect to the fixed origin.

Examples of Central Forces

(1) The gravitational force exerted on a particle of mass m_1 by another statio particle of mass m_2 is a central force and can be written as

$$\mathbf{F}_{12} = -G\,\frac{m_1 m_2}{r^2}\,\hat{r}$$

where r is the distance between the two particles. Here negative sign indicates tha force is attractive.

But $\qquad \mathbf{F} = f(r)\,\hat{r}$

$$\therefore \qquad f(r) = -G\,\frac{m_1 m_2}{r^2} = -\frac{C}{r^2}$$

where $C = -G\,m_1 m_2$ is a constant.

Thus $\qquad f(r) \propto \dfrac{1}{r^2}$

which is the famous **inverse square law.**

Thus the earth moves around the sun under a central force which is always dire towards the sun.

(2) The electrostatic force exerted on a charged particle q_1 by another statio charged particle q_2 is a central force and is given by

$$\mathbf{F}_{12} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1 q_2}{r^2}\,\hat{r}$$

But $\qquad \mathbf{F} = f(r)\,\hat{r}$

$$\therefore \qquad f(r) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1 q_2}{r^2} = \frac{C}{r^2}$$

or $\qquad f(r) \propto \dfrac{1}{r^2}$

Thus, electron in a hydrogen atom moves under a central force which is alw directed towards the nucleus.

(3) A mass attached to one end of a spring whose other end is fixed is also an exa of central force. The spring always pulls towards the fixed end or pushes away from an elastic force

$$F = -kx$$

where x is the distance of the mass from the unstretched position of the spring and spring constant.

3.2. MAIN FEATURES OF CENTRAL FORCE

(i) The general form of the central force is represented by what may be called inverse **nth power law** viz

$$\mathbf{F} = \frac{C}{r^n}\,\hat{r}$$

where C is a constant.

(ii) The central force is attractive if $F(r) < 0$ *i.e.*, negative and repulsive if $F(r) > 0$ positive.

(iii) Central force is a conservative force *i.e.*, work done by the force in movi particle from one point to another is independent of the path followed.

(iv) In a motion under central force, the torque acting on a particle is always zero.

(v) When a particle moves under a central force, its angular momentum remains conserved.

(vi) In case of a central force, motion of the particle is confined to a plane.

(vii) The areal velocity of a particle moving under a central force remains constant.

We shall discuss these features in the articles to follow.

3.3. CONSERVATIVE NATURE OF CENTRAL FORCE

A force is said to be conservative when the work done by the force in moving a particle from a point A to a point B is independent of the path followed between A and B.

Let us consider two arbitrary paths between two points A and B represented by two curves APB and AQB as shown in fig. 1. Let the central force be directed away from the point O. With O as centre, we draw arcs of circles with radii r and $r + dr$ respectively. Then $\mathbf{F}_1$ and $\mathbf{F}_2$ represent the central forces acting on particles at points P and Q taken on paths APB (path 1) and AQB (path 2) respectively away from the centre.

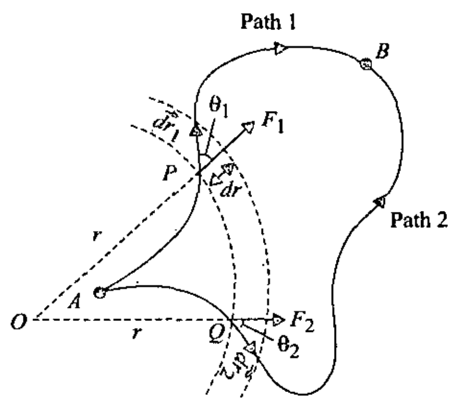


Fig. 1.

Let $d\mathbf{r}_1$ and $d\mathbf{r}_2$ be the displacements of the particles between the arcs along paths 1 and 2 respectively. If θ_1 and θ_2 are the angles between $\mathbf{F}_1$ and $d\mathbf{r}_1$, and between $\mathbf{F}_2$ and $d\mathbf{r}_2$, then

$$\mathbf{F}_1 \cdot d\mathbf{r}_1 = F_1 \, dr_1 \cos\theta_1$$

and

$$\mathbf{F}_2 \cdot d\mathbf{r}_2 = F_2 \, dr_2 \cos\theta_2$$

But magnitudes of $\mathbf{F}_1$ & $\mathbf{F}_2$ are equal because they are acting at the same distance from the centre O. Also the projections of vectors $d\mathbf{r}_1$ and $d\mathbf{r}_2$ on $\mathbf{F}_1$ & $\mathbf{F}_2$ are equal *i.e.*, $dr_1 \cos\theta_1 = dr_2 \cos\theta_2$. Therefore

$$\mathbf{F}_1 \cdot d\mathbf{r}_1 = \mathbf{F}_2 \cdot d\mathbf{r}_2$$

taken over the path segments. In the same way, we can obtain the same results for every pair of segments along paths APB (Path 1) and AQB (path 2). Hence

$$\underset{\text{(Path 1)}}{\int_A^B \mathbf{F} \cdot d\mathbf{r}} = \underset{\text{(Path 2)}}{\int_A^B \mathbf{F} \cdot d\mathbf{r}}$$

Thus the work done by the forces along the two paths is equal *i.e.*,

$$W_{APB} = W_{AQB}$$

It means that the **work done** by a central force acting on a **particle in moving from a point A to point B is independent of the path followed between two points. Hence the central force is conservative.**

3.4. FOR CENTRAL FORCES (CONSERVATIVE FORCES), THE WORK DONE AROUND A CLOSED PATH IS ZERO

The work done in moving the particle from point A to B via path 1 is

$$W_{A \to B} = \underset{\text{(Path 1)}}{\int_A^B \mathbf{F} \cdot d\mathbf{r}_1} = \underset{\text{(Path 1)}}{\int_A^B \mathbf{F} \cdot d\mathbf{r}}$$

The work done in moving the particle back from B to A via path 2 is

$$W_{B \to A} = \underset{\text{(Path 2)}}{\int_B^A \mathbf{F} \cdot d\mathbf{r}} = -\underset{\text{(Path 2)}}{\int_A^B \mathbf{F} \cdot d\mathbf{r}}$$

But for conservative force

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r}$$
(Path 1) (Path 2)

$\therefore$ $$W_{A \to B} = -W_{B \to A}$$

or $$W_{A \to B} + W_{B \to A} = 0$$

Thus the total work done in moving the particle around the closed path $A \to B$ — is zero.

• 3.5. ANGULAR MOMENTUM UNDER A CENTRAL FORCE IS CONSERV

According to the principle of conservation of angular momentum, **if no exter torque acts on a body rotating about a fixed point, its angular moment remains conserved** *i.e.*, there is no change in its direction and magnitude.

Proof. Let us consider that a body is subjected to a central force which is expres as

$$\mathbf{F} = f(r)\,\hat{\mathbf{r}}$$

The torque acting on the body is defined by

$$\vec{\tau} = \frac{d\mathbf{J}}{dt} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

where $\mathbf{J}$ is the angular momentum of the body.

Substituting the value of $\mathbf{F}$ from (i) in (ii), we have

$$\vec{\tau} = \frac{d\mathbf{J}}{dt} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \mathbf{r} \times f(r)\,\hat{\mathbf{r}}$$

$$= \vec{\mathbf{r}} \times f(r)\frac{\vec{\mathbf{r}}}{r} = 0$$

because $\mathbf{r} \times \mathbf{r}$ *i.e.*, the vector product of two parallel vectors is zero. Thus

$$\vec{\tau} = \frac{d\mathbf{J}}{dt} = 0$$

or $$\mathbf{J} = \text{constant}$$

Thus when a body moves under the action of central force.

(i) Torque acting on the body is always zero.

(ii) Angular momentum of the body remains constant.

Examples : (i) Earth moves around the sun under a central force which is direc towards the centre of the sun. Hence the angular momentum of the earth rema constant.

(ii) In hydrogen atom, the electron revolves around the nucleus under a central fo Hence angular momentum of the electron relative to nucleus is constant.

• 3.6. MOTION UNDER A CENTRAL FORCE TAKES PLACE IN A FIXED PLA

To prove it, consider that at any instant, the position vector r and velocity vector a particle lie in the $X - Y$ plane. The its angular momentum is given by

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$

which by the definition of cross product will be directed along the Z-axis. Since un the influence of a central force, $\mathbf{J}$ remains constant both in magnitude and direction means that $\mathbf{J}$ is always along the Z-axis. To satisfy this condition, $\mathbf{r}$ and $\mathbf{v}$ that is orbit of the particle must always lie in the $X - Y$ plane. Thus the **motion of particle is confined to a plane.**

Example. Earth revolves round the sun under the influence of a central force wh is the gravitational force exerted by the sun on the earth. Consequently, its angu momentum $\mathbf{J} = \mathbf{r} \times m\mathbf{v}$ with respect to sun is constant in magnitude and direct

Since vectors **r** and **v** are always perpendicular to **J**, it means that the **orbit of the planet always lies in a plane perpendicular to J**.

• 3.7. AREAL VELOCITY UNDER CENTRAL FORCE REMAINS CONSTANT

To prove that the areal velocity of a particle moving under a central force remains constant, let us consider the motion of the earth around sun (Fig. 2). Let **r** be the radius vector of the earth with respect to sun at any instant. Let in a small time interval Δt, the earth moves from E to E' where the radius vector is $\mathbf{r} + d\mathbf{r}$. Then the vector area swept out by the radius vector in time interval Δt is

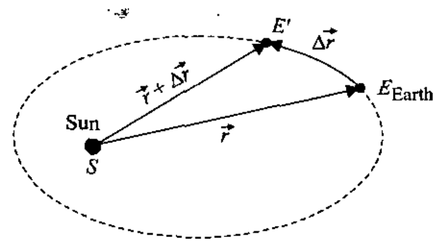


Fig. 2.

$$\Delta \mathbf{A} = \text{Area of triangle SEE}'$$
$$= \frac{1}{2} \text{ base} \times \text{height} = \frac{1}{2} \mathbf{r} \times \Delta \mathbf{r}$$

Dividing both sides by Δt, we have

$$\frac{\Delta \mathbf{A}}{\Delta t} = \frac{1}{2} \mathbf{r} \times \frac{\Delta \mathbf{r}}{\Delta t}$$

When $\Delta t \to 0$, it gives

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2} \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \frac{1}{2} \mathbf{r} \times \mathbf{v}$$
$$= \frac{1}{2m} \mathbf{r} \times m\mathbf{v}$$

Since $\mathbf{r} \times m\mathbf{v} = \mathbf{J}$, the angular momentum of earth with respect to sun, it gives

$$\frac{d\mathbf{A}}{dt} = \frac{\mathbf{J}}{2m} = \text{Constant}$$

because under central force, the angular momentum **J** remains constant.

Thus the **radius vector of the planet sweeps out equal areas in equal times** *i.e.*, the areal velocity under central force remains constant.

Student Activity

(1) Show that angular momentum under a central force is conserved.

----

(2) Show that the Areal Velocity under central force remains constant.

----

(3) Show that Motion under central force take place in a fixed plane.

• 3.8. REDUCTION OF TWO BODY PROBLEM TO ONE BODY PROBLEM : (MOTION OF A SYSTEM OF TWO PARTICLES UNDER CENTRAL FORCE)

A two particle system can be reduced effectively to a one particle system by introducing the concept of central forces. Suppose a system is composed of two masses, then for an inertial observer, the relative motion of these masses can be represented by a fictitious particle. The mass of such a fictitious particle is known *as reduced mass*.

Let us consider two particles of masses m_1 and m_2 whose instantaneous position vectors

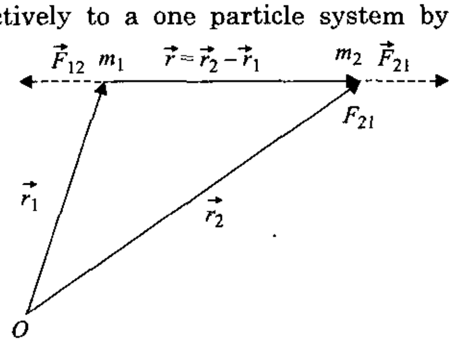


Fig. 3.

in any inertial frame with origin O are $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively (Fig. 3). Let us conside that no external force acts on this system *i.e.*, it is acted only by interaction force (gravitational or electrostatic). Let $\mathbf{F}_{12}$ be the force experienced by m_1 due to m_2 and $\mathbf{F}_{21}$ be the force on m_2 due to m_1 on account of mutual gravitational attraction. Then th equations of motion of the particles are given by

$$\mathbf{F}_{12} = m_1 \frac{d^2\mathbf{r}_1}{dt^2} \quad \text{and} \quad \mathbf{F}_{21} = m_2 \frac{d^2\mathbf{r}_2}{dt^2};$$

$$\therefore \quad \frac{d^2\mathbf{r}_1}{dt^2} = \frac{\mathbf{F}_{12}}{m_1} \quad \text{and} \quad \frac{d^2\mathbf{r}_2}{dt^2} = \frac{\mathbf{F}_{21}}{m_2}$$

From these equations, we get

or $$\frac{d^2\mathbf{r}_2}{dr^2} - \frac{d^2\mathbf{r}_1}{dt^2} = \frac{\mathbf{F}_{21}}{m_2} - \frac{\mathbf{F}_{12}}{m_1}$$

But from fig., $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$;

$$\therefore \quad \frac{d^2\mathbf{r}}{dt^2} = \frac{\mathbf{F}_{21}}{m_2} - \frac{\mathbf{F}_{12}}{m_1}.$$

But according to Newton's third law,

$$\mathbf{F}_{21} = -\mathbf{F}_{12} = \mathbf{F}\ (\text{say});$$

$$\therefore \quad \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}\left(\frac{1}{m_2} + \frac{1}{m_1}\right) = \mathbf{F}\left(\frac{m_1 + m_2}{m_1 m_2}\right)$$

or $$\mathbf{F} = \frac{m_1 m_2}{m_2 + m_2} \cdot \frac{d^2\mathbf{r}}{dt^2} = \mu \frac{d^2\mathbf{r}}{dt^2}$$

where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is called the **reduced mass** of the two particles having mass m_1 and m_2. This equation represents a one body problem because it is similar to t equation of motion of a single particle of mass μ placed at m_2 at a distance $\mathbf{r}$ from ma m_1 considered as the fixed origin of the inertial frame. Thus the two body problem reduced to one body problem and the relative motion is represented by the motion o fictitious particle of mass μ acted on by the internal force F.

Thus the two body problem may be reduced to one body problem if

(i) A fictitious particle of mass μ is placed at the location of lighter particle m_2

(ii) The motion of fictitious particle is considered relative to heavier particle m_1

(iii) The force on particle μ is the same as that on m_2.

Examples of hydrogen atom and positronium.

(a) The hydrogen atom consists of a proton and an electron. If m_1 is the mass of t proton and m_2 that of electron, the reduced mass of hydrogen atom, μ, is given by

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

or $$\mu = \frac{m_1 m_2}{m_1 + m_2} = m_2 \frac{1}{1 + \dfrac{m_2}{m_1}}$$

$$= m_2\left(1 - \frac{m_2}{m_1}\right) \qquad \text{by binomial theore}$$

But $$\frac{m_2}{m_1} = \frac{1}{1836}$$

$\therefore$ Reduced mass, μ, of hydrogen atom $\equiv m_2$ = mass of electron.

In other words ***the lighter of the two masses tends to dominate the value reduced mass.***

(b) Positronium is made up of a positron and electron. A positron is a particle having mass equal to that of electron and is positively charged, its charge in magnitude being equal to the charge of electron.

Therefore in the case of positronium, $m_1 = m_2 = m$.

The reduced mass μ is given

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{m} + \frac{1}{m} = \frac{2}{m}.$$

$$\therefore \quad \mu = \frac{m}{2}.$$

Thus the **reduced mass of positronium is about one half that of the hydrogen atom.**

Student Activity

(1) What is the idea of reduced mass ?

----

(2) Explain the example of Hydrogen and Positronium.

• 3.9. RELATIVE AND CENTRE OF MASS COORDINATES

Suppose m_1 and m_2 are two mass particles having position vectors $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ with respect to an origin O in laboratory reference frame (Fig. 4). The position vector of the centre of mass C of the particles is given by

$$\vec{\mathbf{r}}_{cm} = \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2}{m_1 + m_2} \qquad \text{... (i)}$$

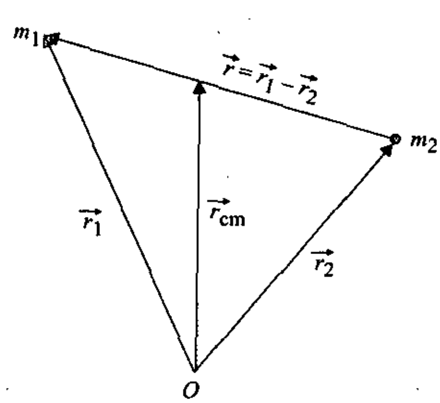


Fig. 4.

Now, if $\vec{\mathbf{r}}$ be the position vector of m_1 relative to m_2, then

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2. \qquad \text{... (ii)}$$

Now putting the value of $\vec{\mathbf{r}}_2$ $(= \vec{\mathbf{r}}_1 - \vec{\mathbf{r}})$ in eq. (i), we get

$$\vec{\mathbf{r}}_{cm} = \frac{m_1 \vec{\mathbf{r}}_1 + m_2 (\vec{\mathbf{r}}_1 - \vec{\mathbf{r}})}{m_1 + m_2}$$

$$= \frac{(m_1 + m_2) \vec{\mathbf{r}}_1 - m_2 \vec{\mathbf{r}}}{m_1 + m_2} = \vec{\mathbf{r}}_1 - \frac{m_2}{m_1 + m_2} \vec{\mathbf{r}}$$

$$\therefore \quad \vec{\mathbf{r}}_1 = \vec{\mathbf{r}}_{cm} + \frac{m_2}{m_1 + m_2} \vec{\mathbf{r}}.$$

If μ is the reduced mass of the system, then

$$\frac{m_2}{m_1 + m_2} = \frac{\mu}{m_1}$$

$$\therefore \quad \vec{\mathbf{r}}_1 = \vec{\mathbf{r}}_{cm} + \frac{\mu}{m_1} \vec{\mathbf{r}}.$$

Similarly, $$\vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_{cm} - \frac{\mu}{m_2} \vec{\mathbf{r}}.$$

In the centre-of-mass coordinate system (*i.e.*, when origin is taken at *C*) $\vec{r}_{cm} = 0$

Therefore, if $\vec{r}'_1$ and $\vec{r}'_2$ be the centre of mass coordinates of m_1 and m_2, then

$$\vec{r}_{1'} = \frac{\mu}{m_1}\vec{r}$$

and

$$\vec{r}_{2'} = -\frac{\mu}{m_2}\vec{r}.$$

These equation give relation between relative and centre-of-mass coordinats.

• 3.10. THE LAW OF UNIVERSAL GRAVITATION

In 1687, **Newton** formulated the law of gravitation in his monumental **work Principial** which is hailed as **the greatest production of the human mind.**

The law states that *every particle of matter in the universe attracts every other particle with a force that acts along the line joining the particles and has a magnitude that is directly proportional to the product of their masses and inversely proportional to the square of distance between them.*

Thus if two particles of masses m_1 and m_2 be placed at a distance *r* from each other (Fig. 5), the force of gravitational attraction between them is given by

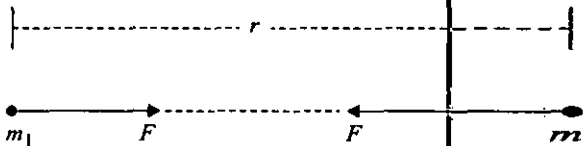


Fig. 5.

$$F \propto \frac{m_1 m_2}{r^2}$$

or

$$F = -G\frac{m_1 m_2}{r^2} \quad \text{... (i}$$

where *G* is a universal constant known as **gravitational constant**. The negative sign indicates that the force is always attractive.

Characteristics of Gravitational Forces

The characteristics of gravitational forces between two particles are :

(i) They are **always attractive and very weak** as compared with other type of forces (electrical or magnetic) between material particles.

(ii) They are **central** *i.e.*, act along the line joining the particles.

(iii) They always **occur in action reaction pair.** The first particle exerts a force on the second particle and the second particle exerts an equal and opposite force on the first.

(iv) They are completely *independent* of the presence of *other bodies* and the nature of the *medium* between the particles.

Gravitational Constant G.

If we put in eq. (1) $m_1 = m_2 = 1$ and $r = 1$, then

$$G = F$$

Thus *gravitational constant G is numerically equal to the force with which two particles, each of unit mass, attract each other when placed at a unit distance apart.*

Writing the dimensions of various quantities in equation (i), we get

$$G = \frac{Fr^2}{m_1 m_2} = \frac{[MLT^{-2}]\,[L^2]}{[M]\,[M]}$$
$$= [M^{-1}L^3T^{-2}]$$

Accordingly, **units of G** are kg^{-1} m^3 sec^{-2} and its **accepted value i** 6.673×10^{-11} kg^{-1} m^3 sec^{-2}.

G is a **scalar quantity.** It is **universal** *i.e.*, does not change with the change of place. It does not depend upon the nature of intervening medium or masses and is not affected by external conditions such as pressure or temperature. The question of variability of G with time, however, is still undecided. In order to explain the theory of expanding universe, Dirac has suggested that G is a function of time and is decreasing as the age of universe is increasing.

Universal Nature of the Law.

The law of gravitation is universal in nature. The apple (or any object) falling under gravity on the earths surface, the moon revolving around earth, Jupiter or saturn orbiting the sun, all obey the same law of gravitation.

Newton verified his inverse square law by calculating the period of revolution of moon about the earth. The moon revolves round the earth in a nearly circular orbit of radius $r = 3.8 \times 10^8$ *meter which is nearly 60 times the radius of the earth*. Its' centripetal acceleration, directed towards the centre of the earth, is given by

$$\alpha = \frac{v^2}{r} = \frac{r^2\omega^2}{r} = r\omega^2 \qquad \text{... (ii)}$$

where ω is the angular velocity.

Now the value of g (acceleration due to gravity) on earth is 9.8 m/s^2. If inverse square law is correct, the gravitational attraction on moon must be

$$g_m = \frac{9.8 \text{ m/s}^2}{(60)^2} \qquad \text{... (iii)}$$

Equating (ii) and (iii),

$$\frac{9.8}{3600} = r\omega^2 = (3.8 \times 10^8)\,\omega^2$$

or $$\omega = 2.65 \times 10^{-6} \text{ radian/sec.}$$

Therefore, time of one revolution

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.65 \times 10^{-6}} \text{ sec}$$

$$= \frac{2\pi}{2.65 \times 10^{-6} \times 3600 \times 24} \text{ days}$$

$$= 27.4 \text{ days}$$

which is very near to the observed value of 27.3 days. It is a very strong evidence in favour of Newton's inverse square law.

• 3.11. MOTION UNDER INVERSE SQUARE LAW—KEPLER'S LAW OF PLANETARY MOTION

Inverse Square Law Force

If the force acting on the moving particle towards the fixed particle or the origin is inversely proportional to the square of the distance between two particles, then the force is said to obey **inverse square law.** For example, if a planet of mass m rotates round the sun of mass M in an orbit of radius r, then the **gravitational force of attraction** between them obeys inverse square law and is given by

$$F = -G\frac{mM}{r^2} = -\frac{C}{r^2}$$

This force is always directed towards the centre of the sun and hence it is an example of central force.

Kepler's Laws of Planetary Motion

German astronomer John Kepler studied the motion of planets (celestial bodies revolving round the sun) and gave three important laws which are known as Kepler's **laws of planetary motion.** These laws are :

First Law (The law of orbits)

Each planet revolves around the sun in an elliptical orbit having the sun as one of the foci of the orbit.

Second Law (The Law of axis)

The radius vector drawn from the sun to a planet sweeps out equal areas in equal times *i.e.*, the areal velocity (area swept out per unit time) of the planet is constant.

Third law (Harmonic law or law of time period)

The square of the period of revolution of a planet around the sun is proportional to the cube of the semi-major axis of the elliptical orbit.

Deduction of Kepler's Laws

I Law : Let S be the sun (mass M) and a planet of mass m, revolving around the sun in elliptical orbit (Fig. 6). The force acting between planet and sun is gravitational and given by

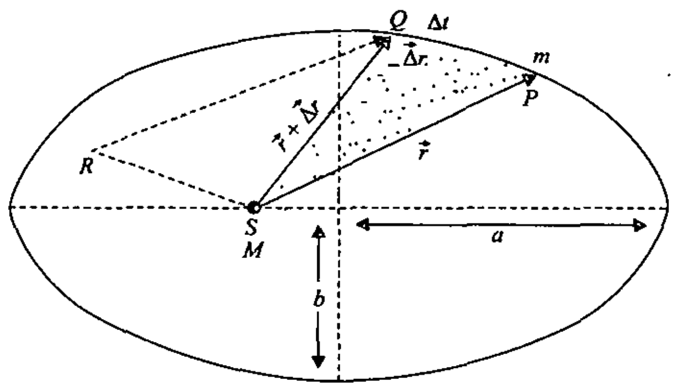


Fig. 6.

$$\mathbf{F} = -\frac{GMm}{r^2}\,\hat{\mathbf{r}},$$ r being position vector of planet relative to sun

$$= -\frac{GMm}{r^3}\,\mathbf{r} = f(r)\,\mathbf{r}$$

with $$f(r) = -\frac{GMm}{r^3} \text{ (a scalar function).}$$

Torque acting on planet,

$$\vec{\tau} = \mathbf{r}\times\mathbf{F} = \mathbf{r}\times f(r)\,\mathbf{r} = f(r)\,\mathbf{r}\times\mathbf{r} = 0.$$

As torque acting on planet is zero, the angular momentum of the planet relative to sun is conserved.

The angular momentum of planet relative to sun is

$$\mathbf{J} = \mathbf{r}\times\mathbf{p} = \mathbf{r}\times m\mathbf{v} = \text{constant}.$$

It follows that $\mathbf{J}$, being constant, must be perpendicular to the plane containing position vector $\mathbf{r}$ and velocity $\mathbf{v}$ that is ***the orbit of the planet must lie in a plane.***

II Law. Let P be the instantaneous position of planet relative to sun. The position vector of P relative to sun is $\mathbf{r}$. After time Δt, the planet is at Q, having position vector $\mathbf{r} + \Delta\mathbf{r}$ such let $\Delta\mathbf{r} = \vec{PQ}$.

The area swept out by radius vector $\vec{SP}$ in time Δt.

$$\Delta \mathbf{A} = \text{Area of triangle } PSQ$$
$$= \frac{1}{2} \text{ Area of parallelogram } SRQP$$
$$= \frac{1}{2} \mathbf{r} \times \Delta \mathbf{r}$$

Area swept by radius vector per second

or Areal velocity, $$\frac{d\mathbf{A}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{A}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{1}{2}\mathbf{r} \times \Delta \mathbf{r}}{\Delta t}$$
$$= \frac{1}{2} \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \frac{1}{2} \mathbf{r} \times \mathbf{v}$$

From (i) $$\mathbf{r} \times \mathbf{v} = \frac{\mathbf{J}}{m} = \text{constant}$$

Areal velocity, $$\frac{d\mathbf{A}}{dt} = \frac{1}{2} \cdot \frac{\mathbf{J}}{m} = \text{constant.}$$

It follows that the ***areal velocity of the radius vector of planet relative to sun remains constant. This is Kepler's II Law.***

III Law. Let T be period of revolution of the planet in elliptical orbit,

Area of ellipse $= \pi ab$, a and b being semi-major and semi-minor axes of the ellipse.

Area of ellipse $$\frac{d\mathbf{A}}{dt} = \frac{\mathbf{J}}{2m}$$

$\therefore$ Period of revolution

$$T = \frac{\text{Area of ellipse}}{\text{Areal velocity}} = \frac{\pi ab}{J/2m}$$

$$\therefore \quad T^2 = \frac{4\pi^2 m^2 a^2 b^2}{J^2}$$

If l is the semi-latus rectum of ellipse, then

$$l = \frac{b^2}{a}$$

$$\therefore \quad T^2 = \frac{4\pi^2 m^2 l a^3}{J^2}$$

i.e., $T^2 \propto a^3$ (Since all other quantities are constant)

This is Kepler's III Law.

3.12. CONCLUSIONS OF NEWTON FROM KEPLER'S LAWS

Newton derived some important conclusions from Kepler's laws. For it, he assumed that the orbit of a planet around the sun is almost circular which is true for all planets except mercury and pluto.

(i) According to Kepler's second law, the areal velocity of a planet around the sun remains constant. Then the linear speed of a planet will remain constant in circular orbit.

(ii) If T is period of revolution and r is radius of orbit, than from Kepler's III law

$$T^2 \propto r^3 \quad \text{or} \quad T^2 = Kr^3 \qquad \dots (1)$$

where K is a constant for all planets.

Now in a circular orbit, a centripetal force F acts on a planet which is given by

$$F = \frac{mv^2}{r} \qquad \dots (2)$$

where linear speed v of planet is given by

$$v = \frac{\text{Distance travelled in one revolution}}{\text{period of revolution}} = \frac{2\pi r}{T} \quad \ldots (3$$

Substituting this value in eq. (2), we have

$$F = \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2 = \frac{4\pi^2 mr}{T^2}$$

$$= \frac{4\pi^2 mr}{kr^3} = \frac{4\pi^2 m}{Kr^2} \text{ using (i)}$$

$$\therefore \quad F \propto \frac{m}{r^2} \quad \ldots (4)$$

Thus Newton derived following conclusions from Kepler's laws :

(i) A planet revolving round the sun is acted upon by a force whose direction is towards the centre (sun).

(ii) Force acting on the planet is directly proportional to the mass m of the planet, *i.e.*,

$$F \propto m$$

(iii) Force acting on the planet is inversely proportional to the square of the distance of planet from the sun (inverse square law). Thus

$$F \propto \frac{1}{r^2}$$

• 3.13. KEPLER'S LAWS FROM NEWTON'S LAW OF GRAVITATION

Consider a *planet of mass m* moving in the gravitational field of the sun of mass M. According to Newton's law of gravitation, the attractive force acting on the planet due to sun is given by,

$$F = -\frac{GMm}{r^2} \quad \ldots (i)$$

where r is the distance of the planet from the sun.

The gravitational force is a *central* force. Hence the angular momentum J is conserved in magnitude and direction. Consequently, **the motion of the planet must take place in a fixed plane and the areal velocity of its radius vector should be constant.** This is Kepler's second law.

Now areal velocity is given by

$$\frac{dA}{dt} = \frac{J}{2m}.$$

If we put $\frac{dA}{dt} = \frac{h}{2}$, where h is a constant and put $J = I\omega = mr^2 \frac{d\theta}{dt}$. Then,

$$\frac{h}{2} = \frac{1}{2m}\left(mr^2 \frac{d\theta}{dt}\right)$$

or

$$h = r^2 \frac{d\theta}{dt}. \quad \ldots (ii)$$

Now, the radial force on the planet is (mass × radial acceleration)

$$F = m\left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]$$

But $F = -\frac{GMm}{r^2}$ from eq. (i).

$$\therefore \quad \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2}$$

Using eq. (ii), we get

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{GM}{r^2}.$$

Now introducing a new variable u such that

$$r = \frac{1}{u} \qquad \text{... (iii)}$$

The last expression becomes

$$\frac{d^2r}{dt^2} - h^2u^3 = -GMu^2 \qquad \text{...(iv)}$$

Now, differentiating eq. (iii), we get

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

Putting $\frac{d\theta}{dt} = \frac{h}{r^2}$ from eq. (ii) and then $\frac{1}{r^2} = u^2$ from eq. (iii), we get

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}(hu^2)$$

$$= -h\frac{du}{d\theta}.$$

Differentiating again,

$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}$$

Again putting $\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2$ from eq. (ii) and (iii),

$$\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Putting this value of $\frac{d^2r}{dt^2}$ in eq. (iv), we have

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -GMu^2$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

or

$$\frac{d^2u}{d\theta^2} + \left(u - \frac{GM}{h^2}\right) = 0.$$

Since $\frac{GM}{h^2}$ is a constant quantity, we may write this equation as

$$\frac{d^2}{d\theta^2}\left(u - \frac{GM}{h^2}\right) + \left(u - \frac{GM}{h^2}\right) = 0.$$

The solution of this differential equation is of the form:

$$u - \frac{GM}{h^2} = -A\cos\theta$$

or

$$\frac{1}{r} = \frac{GM}{h^2} + A\cos\theta$$

Dividing this equation by $\frac{GM}{h^2}$

$$\frac{1/r}{GM/h^2} = 1 + \frac{A\cos\theta}{GM/h^2} \qquad \text{...(v)}$$

This equation is similar to the polar equation of ellipse

$$\frac{h}{r} = 1 + e\cos\theta \qquad \ldots$$

Comparing the two equation, semi latus rectum of ellipse

$$l = \frac{h^2}{GM}$$

and eccentricity $e = \dfrac{Ah^2}{GM}$.

It indicates that the path of the planet is an elliptical orbit with the sun at one of foci which is the law of elliptical orbits.

Thus, **the orbit of the planet around the sun is an ellipse. This is Kepler's first law.**

To establish Kepler's third law, we consider the semi-latus rectum l of the elliptical orbit. If a and b are the semi-major and semiminor axes of the ellipse, then

$$l = \frac{b^2}{a} = \frac{h^2}{GM} \qquad \ldots(v$$

If T be the period of the planet around the sun, then

$$T = \frac{\text{area of elliptical path}}{\text{areal velocity}} = \frac{\pi ab}{h/2}$$

or

$$T^2 = \frac{4\pi^2 a^2 b^2}{h^2}$$

$$= \frac{4\pi^2 a^2 b^2}{GMb^2/a} = \frac{4\pi^2 a^3}{GM} \qquad \text{[Using eq. (vii)]}$$

Since $\dfrac{4\pi^2}{GM}$ is a constant quantity,

$$\therefore \qquad T^2 \propto a^3.$$

Thus, **the square of the period revolution of the planet around the sun is proportional to the cube of the semimajor axis of the ellipse.**

This is the third law of Kepler.

Student Activity

(1) Discuss Kepler's Law

----

(2) How Newton's Law of gravitation can be derived from Kepler's Law.

3.14. MOTION OF SATELLITES

Among celestial bodies, a statellite is what may be called a minor or junior member of the solar system. Satellite is a secondary body which revolves round a planet in its own prescribed orbit, planet being a celestial body revolving round the sun. Our earth is thus a planet and moon is a natural satellite of earth, while Aryabhatta, Rohini etc. are artificial satellites of earth.

Orbital velocity

Let us consider a satellite in a circular orbit and let it be concentric and coplanar with the equator of the earth. The revolving satellite is kept in its circular orbit by the force of attraction between the satellite and the earth, which provides the necessary centripetal force. If M_e be the mass of the earth, M_s the mass of the satellite and r be the radius of revolution (Fig. 7), then the force of attraction between the earth and satellite, in accordance of the law of gravitation, in vector from is

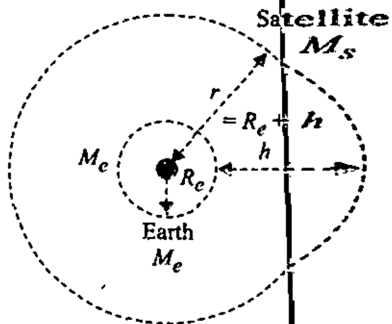


Fig. 7.

$$F = -\frac{GM_e\, M_s}{r^2}\ \hat{r} = -\frac{GM_e\, M_s}{r^3}\, r \qquad \text{... (1)}$$

Now if v and ω be respectively the linear and angular velocities of the satellite in orbit, then the centripetal force acting on it is given by

$$\mathbf{F} = -M_s\ \omega^2 \mathbf{r} \qquad \text{... (2)}$$

Therefore for a satellite in circular motion, we have [equating (1) and (2)]

$$-M_s \omega^2 \mathbf{r} = -\frac{GM_e\, M_s}{r^3}\, \mathbf{r}$$

i.e.,
$$\omega^2 = -\frac{GM_e}{r^3} \qquad \text{... (3)}$$

or
$$\frac{v^2}{r^2} = \frac{GM_e}{r^3} \quad i.e., \quad v_0 = \sqrt{\frac{GM_e}{r}} = \sqrt{\frac{GM}{R_e + h}} \qquad \text{... (4)}$$

This velocity is known as orbital velocity and hence is denoted by v_0. Here R_e is the radius of the earth and h is the height above the earths surface at which satellite (moon) is revolving.

In terms of acceleration due to gravity g at earths surface, we have

$$GM_e = gR_e^2$$

$$\therefore \quad v_0 = \sqrt{\frac{gR_e^2}{r}} = R_e\sqrt{\frac{g}{r}} = R_e\sqrt{\frac{r}{R_e + h}}$$

When the satellite is very close to the earth

$$v_0 = R_e\sqrt{\frac{g}{R_e}} = \sqrt{gR_e}$$

$$= \sqrt{9.8 \times 6.4 \times 10^3 \times 1000}$$

$$= \mathbf{8000\ m / s = 8\ km / sec}$$

Period of Revolution

If we express the angular velocity in terms of the period of revolution T, then $\omega = \frac{2\pi}{T}$ and we obtain from (3)

$$\frac{4\pi^2}{T^2} = \frac{GM_e}{r^3} \qquad \text{... (5)}$$

Eqn. (5) may be written as

$$T^2 = \frac{4\pi^2}{GM_e}\, r^3 \qquad \text{... (6)}$$

This equation can be considered as the basic of satellite motion. From (6), we have

$$T^2 \propto r^3 \qquad \left(\text{since } \frac{4\pi^2}{GM_e} = \text{constant}\right)$$

***i.e.* the square of the period of revolution of a satellite moving in a circular orbit round the earth is proportional to the cube of its distance from the centre of the earth.**

From eq. (6), period of revolution can be expressed as

$$T = \frac{2\pi(R_e + h)^{3/2}}{\sqrt{gR_e^2}} = \frac{2\pi(R_e + h)^{3/2}}{R_e\sqrt{g}}$$

In terms of the density of earth material ρ,

$$M_e = \frac{4}{3}\pi R_e^3\, \rho$$

Therefore, eq. (6) becomes

$$T = \frac{2\pi(R_e + h)^{3/2}}{\left(G \cdot \frac{4}{3}\pi R_e^3 \rho\right)^{1/2}} \approx \frac{3\pi}{G\rho}$$

• 3.15. GEOSTATIONARY SATELLITE

A satellite which appears stationary when viewed by observers fixed on the eart[illegible] called geostationary satellite. For such a satellite,

(i) the sense of rotation of the earth and satellite will be the same

(ii) period of the revolution of the satellite should be the same as that of the ea[illegible] about its axis of rotation.

In other words angular velocity ω for the satellite orbit must be equal to the angu[illegible] velocity ω_e of the earth about its axis, *i.e.* $\omega = \omega_e$

Now the angular velocity of the earth is

$$\omega_e = \frac{2\pi}{T} = \frac{2\pi}{24 \times 60 \times 60} = \frac{2\pi}{8.64 \times 10^4} \quad \ldots$$

$$= 7.3 \times 10^{-5}\ s^{-1}$$

Substituting the value of ω_e from (1), in place of ω in (3) and also substituting t[illegible] value of G and M_e, the mass of the earth, we have

$$r^3 = \frac{(6.67 \times 10^{11}\ \mathrm{N \cdot m^2/kg^2})\ (5.98 \times 10^{24}\ \mathrm{kg})}{(7.3 \times 10^{-5}\ \mathrm{s^{-1}})^2}$$

$$= 75 \times 10^{21}$$

Therefore, the desired radius

$$r = 4.2 \times 10^7 \text{ metre.} \quad \ldots ($$

It indicates that if the radius of satellite orbit be 4.2×10^7 metre and rotation of t[illegible] earth and satellite be in the same sense, then the satellite will appear stationary to t[illegible] observer fixed on the earth.

Therefore, height of geostationary satellite above earth surface

$$(\because\ r = R_e + $$

$$h = r - R_e = 4.2 \times 10^7 \text{ metre} - 6.4 \times 10^6 \text{ metre}$$

$$= (4.2 - 0.64) \times 10^7 \text{ metre}$$

$$= \mathbf{35600\ Km}$$

• 3.16. ESCAPE VELOCITY AND ORBITAL VELOCITY

The escape velocity for a body projected from the surface of earth

$$v_e = \sqrt{2gR} \quad \ldots (1)$$

where R is radius of earth.

The orbiting velocity of satellite around the earth

$$v_0 = R\sqrt{\left(\frac{g}{R+h}\right)}$$

If satellite is very near the earth, then $h << R$. Then

$$v_0 = R\sqrt{\frac{g}{R}} = \sqrt{gR} \quad \ldots (2)$$

Comparing (1) and (2).

$$\frac{v_e}{v_0} = \frac{\sqrt{2gR}}{\sqrt{gR}} = \sqrt{2}$$

or $$v_e = \sqrt{2}\ v_0 \quad \ldots (3)$$

SUMMARY

- Magnituole of central force depends only upon distance from fixed point.
- When a particle move under central force its angular momentum remains constant.
- Curl of conservative force is always zero.
- Gravitational forces are always attractive and weak.
- G is scalar and universal quantity.
- Kepler gave three laws which describe the motion of the planets around the sun. These are :

  1. Law of Orbit : The path of each planet about the sun is an ellipse with the sun at one focus.

  2. The Law of Areas : Each planet moves in such a way that an imaginary line drawn from the sun of the planet sweeps out equal areas in equal time, *i.e.*, the areal velocity of the radius vector in constant.

  3. Law of Period : The square of the period of revolution of any planet around the sun in proportional to the cube of the semi-major axis of its elliptical orbit.
- $T = 2\pi\sqrt{\dfrac{r^3}{GM}}$ is the expression for the period of revolution and is derived using Kepler's laws.
- Escape velocity if $v_e = \sqrt{2gR}$ and orbital velocity is $v_0 = \sqrt{gR}$.

TEST YOURSELF

1. What is a central force ? State Kepler's laws of planetary motion.
2. What are central forces ? Give some examples of such forces.
3. What do you understand by reduced mass ? How will you reduce two body problem to one body problem under influence of a central force ?
4. What do you mean by geostationary satellites ? Estimate the height of such a satellite from the surface of the earth.
5. Prove that for a particle moving under central force, the areal velocity of the radius vector remains constant.
6. Show that a particle under a central force moves in a fixed plane.
7. If an earth satellite moves to lower orbit and there is some dissipation of energy, yet the satellite speed increases. Explain.
8. A geo-stationary satellite is that which has a time period of revolution equal to 24 hours is it True or False? Give reasons to support your answer.
9. Write Newton's law of gravitation and hence define universal gravitational constant G.
10. Determine the reduced mass of positronium if mass of an electron is m_e.
11. What do you mean by binding energy of a satellite? Why is there weightlessness inside a satellite?
12. The angular momentum of a satellite of mass m, energy E and path radius R is :

    (a) $\sqrt{2mR^2E}$ (b) $\dfrac{2mR^2}{E}$

    (c) $\dfrac{2mE}{R^2}$ (d) $\dfrac{8mR^2}{E^2}$
13. The earth is revolving about the sun under gravitational force. What is conserved for the system?

    (a) linear momentum (b) angular momentum

    (c) both of the above (d) neither (a) mor (b)
14. If the mass of the earth becomes double, the period of rotation of earth around sun as compared to initial period becomes

    (a) double (b) half

    (c) one fourth (d) approximately same.

15. The total energy of a satellite round the earth is :
(a) zero
(b) infinite
(c) positive
(d) negative

16. The distance of two satellites from the surface of the earth are R and $7R$. The time period of rotation will be in the ratio.
(a) 1 : 8
(b) 1 : 64
(c) 1 : 7
(d) None of these

17. A satellite in a circular orbit about the earth has a kinetic energy E_k. What is minimum amount of energy to be added so that it escapes from the earth?
(a) $2E_K$
(b) E_K
(c) $E_{K/2}$
(d) $E_{K/4}$

18. The shape of the orbit of a planet depends on :
(a) angular momentum
(b) total energy
(c) both (a) and (b)
(d) none of these

19. Kepler's third law is (T = time period, a = semimajor axis, b = semi minor axi. elliptical orbit)
(a) $T^2 \propto a^3$
(b) $T^2 \propto \left(\frac{a+b}{2}\right)^3$
(c) $T^2 \propto b^3$
(d) $T^2 \propto (ab)^3$

20. If mass of hydrogen atom is m_H, the reduced mass of hydrogen molecule will be
(a) $2m_H$
(b) m_H
(c) $\frac{1}{2} m_H$
(d) $3m_H$

21. The expression for reduced mass is :
(a) $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$
(b) $\mu = m_1 + m_2$
(c) $\mu = m_1 - m_2$
(d) $\mu = \sqrt{m_1 m_2}$

22. If mass of electron is m_e, the reduced mass of positronium will be :
(a) $2m_e$
(b) m_e
(c) $\frac{1}{2} m_e$
(d) $3m_e$

23. If mass of oxygen atom is m, the reduced mass of O_2 will be :
(a) m
(b) $2m$
(c) $\frac{m}{2}$
(d) $4m$

24. What remains constant in the field of central force?
(a) potential energy
(b) kinetic energy
(c) angular momentum
(d) none

25. Central force is
(a) Conservative
(b) Non-conservative
(c) Conservative or non-conservative
(d) None of these

26. The orbital velocity of a satellite revolving in an orbit near the earth's surface do not depend upon
(a) mass of the satellite
(b) radius of the orbit
(c) man of the earth
(d) radius of the earth

ANSWERS

12. (a) 13. (b) 14. (d) 15. (d) 16. (a) 17. (b) 18. (b) 19. (a) 20. (a) 21. (c)
22. (c) 23. (c) 24. (c) 25. (a) 26. (a)

UNIT

4

SIMPLE HARMONIC MOTION FREE, DAMPED AND FORCED VIBRATIONS

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :
- Oscillations and Harmonic motions of a body.
- Motion of different types of pendulum *i.e.*, simple, compound, torsion.
- Energy stored in body executing Harmonic motion.
- The curve traced by the particle executing Harmonic motion or the Lissajous figures.

• 4.1. SIMPLE HARMONIC OSCILLATIONS

(a) Periodic Motion : The motion of a body is said to be periodic motion if its motion is repeated identically after a fixed interval of time and this fixed interval of time is known as **period of motion.**

Examples : (i) The revolution of earth around the sun is an example of periodic motion. Its period of revolution is one year.

(ii) The rotation of earth about its polar axis is a periodic motion whose period of rotation is **one day**.

(iii) The revolution of moon around the earth is also an example of periodic motion whose period of motion is 27.3 days.

(b) Oscillatory motion : When a body moves to and fro repeatedly about its mean position in a definite interval of time then this motion is known as oscillatory or vibratory motion.

Thus we can say that a periodic and bounded motion of a body about a fixed po called an oscillatory motion. The oscillatory motion can be expressed in terms c and cosine functions of their combinations. Due to this, the **oscillatory motion is called harmonic motion.**

Examples : (i) The motion of the pendulum of a wall clock is an examp oscillatory motion.

(ii) When the bob of simple pendulum is displaced from its mean position and itseolf then the motion of bob is known as oscillatory motion.

(c) Time period : In periodic motion the time taken by the body in one per known as time period. It is denoted by T and S.I. unit of T is second.

(d) Frequency : The number of periodic motion made by the body in one seco known as frequency. Its S.I. unit if hertz. Thus the frequency is the reciprocal c periodic time.

$$\boxed{\text{Frequency} = \frac{1}{\text{Time period}}}$$

Phase : Phase of an oscillatory particle at any instant is a physical quantity w completely expresses the position and direction of motion of the particle at that ins with respect to its mean position.

• 4.2. SIMPLE HARMONIC MOTION

When a particle moves to and fro repeatedly about its mean position und restoring force, which is always directed towards the mean position and w magnitude at any instant is directly proportional to the displacement of the par from the mean position at that instant, this motion of the particle is known as sim **harmonic motion.** *i.e.*,

$$\text{retoring force } (F) \propto -\text{ (displacement)}$$

$$F = -kx$$

where k is constant and this constant is known as force constant.

Here negative sign shows that the restoring force is always directed towards mean position.

Characteristics of Simple Harmonic Motion

(i) Displacement : The distance of the particle from the mean position any instant, is known as displacement of the particle at that instant.

(ii) Amplitude : The maximum displacement of the particle from me position is known as amplitude of motion.

(iii) Velocity : In simple harmonic motion the velocity of the particle at a instant is equal to the rate of change of displacement at that instant.

The displacement of the particle at time t is given by

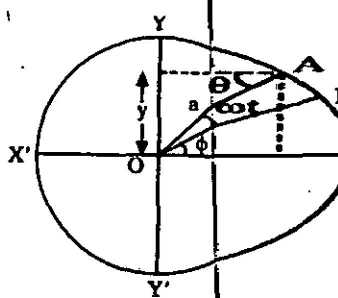


Fig. 1.

$$y = a \sin \omega t$$

$$\therefore \quad \text{velocity} \qquad v = \frac{dy}{dt} = \frac{d}{dt}(a \sin \omega t)$$

$$v = a\omega \cos \omega t$$

$$= a\omega\sqrt{1 - \sin^2 \omega t}$$

$$= a\omega\sqrt{1 - \frac{y^2}{a^2}}$$

$$\boxed{v = \omega\sqrt{a^2 - y^2}}$$

This is the expression for the velocity of the particle at any instant.

(iv) Acceleration : This acceleration of a particle in simple harmon motion at any instant is equal to the rate of change of velocity at that insta *i.e.*,

$$\text{acceleration } (\alpha) = \frac{dv}{dt}$$

$$\alpha = \frac{d}{dt}(a\omega \cos \omega t)$$

$$= -\omega^2 a \sin \omega t$$

$$\boxed{\alpha = -\omega^2 y}$$

This is the expression for acceleration. This is the condition of S.H.M.

At mean position $y = 0$ $[\because\ \theta = 0]$

$\therefore$ $\boxed{\alpha = 0}$ *i.e.*, $\alpha = \min$

At extreme position $y = a$ $[\because\ \theta = 90°]$

$\therefore$ $\boxed{\alpha = -\omega^2 a}$ *i.e.*, max.

Thus, for above it is clear that acceleration and velocity are not uniform in the whole motion. The maximum value of velocity is known as **velocity amplitude** in S.H.M. and the maximum value of acceleration is called **acceleration amplitude.**

(v) Time Period : The time taken by a particle in simple harmonic motion to complete one period is known as time period.

We know acceleration $\alpha = \omega^2 y$ (neglecting negative sign)

$$\therefore \quad \omega = \sqrt{\frac{\alpha}{y}}$$

$$\therefore \quad \text{time period,} \quad T = \frac{2\pi}{\omega}$$

$$\text{or} \quad T = 2\pi\sqrt{\frac{y}{\alpha}}$$

$$\text{or} \quad T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

and frequency is given by

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\text{acceleration}}{\text{displacement}}}$$

• 4.3 EQUATION OF MOTION OF A SIMPLE HARMONIC OSCILLATOR

A particle executing simple harmonic motion is known as harmonic oscillator. Consider a particle of mass m executing simple harmonic motion along a straight line (fig. 2).

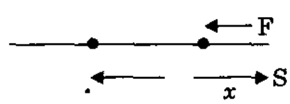


Fig. 2.

Let x be the displacement of the particle from mean position O at any time t, then from the basic condition of simple harmonic motion, the restoring force F is proportional to the displacement x with negative sign *i.e.*,

$$F \propto -x$$

$$F = -kx \qquad \dots (1)$$

where k is constant and this constant is known as force constant. The equation of motion can be obtained by Newton's second law. *i.e.*,

$$F = m\frac{d^2x}{dt^2}$$

From eq. (1),

$$F = -kx$$

Therefore

$$-kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Putting $\frac{k}{m} = \omega^2$, then we get

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \quad \text{... (}$$

This is the differential equation of simple harmonic oscillator.

Let the solution of equaion (2) be

$$x = Ce^{\alpha t} \quad \text{... (3}$$

where C and α are constants

From equation (3),

$$\frac{dx}{dt} = C\alpha\, e^{\alpha t}$$

and $$\frac{d^2x}{dt^2} = C\alpha^2 e^{\alpha t}$$

Putting these values in equation (2), we get

$$C\alpha^2 e^{\alpha t} + \omega^2 Ce^{\alpha t} = 0$$

$$Ce^{\alpha t}(\alpha^2 + \omega^2) = 0$$

as $$Ce^{\alpha t} \neq 0$$

$\therefore$ $$\alpha^2 + \omega^2 = 0$$

$\therefore$ $$\alpha = \pm\sqrt{(-\omega^2)} = \pm i\omega$$

where i is imaginary number, $i = \sqrt{(-1)}$. Thus two solutions of equation (2) are possibl
Therefore

$$x = Ce^{i\omega t} \text{ and } x = Ce^{-i\omega t}$$

The general solution will be

$$x = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \quad \text{... (4)}$$

where C_1 and C_2 are constants.

From equation (4),

$$x = C_1[\cos\omega t + i\sin\omega t] + C_2[\cos\omega t - i\sin\omega t]$$

$$= (C_1 + C_2)\cos\omega t + (iC_1 - iC_2)\sin\omega t$$

Let $C_1 + C_2 = a\sin\phi$

and $i(C_1 - C_2) = a\cos\phi$

where $a \sin\phi$ are constants.

Therefore

$$x = a\sin\phi\cos\omega t + \cos\phi\sin\omega t$$

$$\boxed{x = a\sin(\omega t + \phi)}$$

This is the required solution of equation (2). This equation gives displacement o
particle executing simple harmonic motion at any time t.

(a) Velocity : We have

$$x = a\sin(\omega t + \phi)$$

Differentiating with respect to time t, we get

$$v = \frac{dx}{dt} = a\cos(\omega t + \phi) \,.\, \omega$$

$$= a\omega\sqrt{1 - \sin^2(wt + \phi)}$$

$$= \omega\sqrt{(a^2 - a^2\sin^2(\omega t + \phi))}$$

$$\boxed{v = \omega\sqrt{a^2 - x^2}}$$

This is the expression for the velocity at any displacement x

(b) Period : We have

$$\omega^2 = \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\therefore \quad \boxed{T = \frac{2T}{\omega} - 2\pi\sqrt{\frac{m}{k}}}$$

This is the expression for the time period.

(c) Frequency :

$$f = \frac{1}{T}$$

$$f = \frac{\omega}{2\pi}$$

$$\therefore \quad \boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

This is the expression for the frequency.

(d) Importance of S.H.M. The importance of S.H.M. in physics is due to the following reasons :

(i) The physical problems in mechanics, optics, electricity and in atomic and molecular physics in which the force is directly proportional to its displacement from some equilibrium position the resulting motion is represented by the simple harmonic model.

(ii) The complicated periodic motions occuring in physical problems can also be represented by the combination of a number of simple harmonic motions having frequencies which are multiples of the complicated motion. The vibrations of atoms in solids, the electrical and acoustical oscillations in a cavity can be analysed in this manner.

• 4.4. ENERGY OF A PARTICLE EXECUTING SIMPLE HARMONIC MOTION

When a particle oscillates about its mean position then it has potential energy as well as kinetic energy. The potential energy is due to displacement from the mean position and kinetic energy is due to its velocity. These energies vary during oscillation while the total energy of the particle remains conserved.

(a) Potential Energy : Let us consider a particle of mass m executing S.H.M. Let x be its displacement from the means position at any time 't'. In this position force F acting on the particle is given by

$$F = -kx \qquad \text{... (1)}$$

where k is the force constant. In terms of potential energy, the force is given by

$$F = -\frac{dU}{dx}$$

so from eq. (1)

$$\frac{dU}{dx} = kx \qquad \text{.... (2)}$$

On integrating eq. (2), we get

$$U = \frac{1}{2}kx^2 + C$$

where C is constant of integration.

Now at $x = 0$, $U = 0$, then $C = 0$

so by eq. (3)

$$U = \frac{1}{2}kx^2$$

But the simple harmonic motion, we have

$$x = a \sin(\omega t + \phi)$$

where $\omega^2 = \frac{k}{m}$ From eqs. (4) and (5), we get

$$\boxed{U = \frac{1}{2}ka^2 \sin^2(\omega t + \phi)}$$

This is the expression for the potential energy of the particle at any time 't'. From this expression it is clear that when $\sin^2(\omega t + \phi) = 1$, then V is maximum.

(b) Kinetic Energy : Kinetic energy of the particle at any time 't' is

$$K.E. = \frac{1}{2}mv^2 \quad \ldots ($$

We have

$$x = a \sin(\omega t + \phi)$$

so by eq. (6)

$$K.E. = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\,[\omega a \cos(\omega t + \phi)]^2$$

$$= \frac{1}{2}m\omega^2 \alpha^2 \cos^2(\omega t + \phi)$$

$$\boxed{K.E. = \frac{1}{2}ka^2 \cos^2(\omega t + \phi)} \quad \ldots (7$$

This is the expression for kinetic energy. From this expression it is clear that $\cos^2(wt + \phi)$ is 1 then kinetic energy will be maximum which is $\frac{1}{2}ka^2$.

Total energy of the particle is given by

$$E = P.E. + K.E.$$

$$E = \frac{1}{2}ka^2 \sin^2(\omega t + \phi) + \frac{1}{2}ka^2 \cos^2(\omega t + \phi)$$

$$E = \frac{1}{2}ka^2 [\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

$$E = \frac{1}{2}ka^2$$

But $k = \omega^2 m$ and $\omega = 2\pi x$

where n is the frequency.

$\therefore$

$$E = \frac{1}{2}ka^2$$

$$k = \frac{1}{2}\omega^2 ma^2$$

$$E = \frac{1}{2}(2\pi n)^2\ ma^2$$

$$\boxed{E = 2\pi^2 n^2 ma^2}$$

This is the expression for the total energy of the oscillating particle.

From this expression we see that total energy is proportional to the square of the amplitude (a^2) and also inversely proportional to the quare of the time period (T^2)

$$\left(\because\ n \propto \frac{1}{T}\right)$$

(c) Since $x = a \sin(\omega t + \phi)$

$$\therefore \quad v = \frac{dx}{dt} = \omega a \cos(\omega t + \phi)$$

$$= \pm \sqrt{\left(\frac{k}{m}\right)} \sqrt{[a^2 - a^2 \sin^2(\omega t + \phi)]}$$

$$= \pm \sqrt{\frac{k}{m}} \sqrt{(a^2 - x^2)}$$

Total energy of the particle is

$$E = K.E. + P.E.$$

$$= \frac{1}{2} mv^2 + \frac{1}{2} kx^2$$

$$= \frac{1}{2} k(a^2 - x^2) + \frac{1}{2} kx^2$$

$$= \frac{1}{2} ka^2$$

When the displacement is one half of the amplitude *i.e.*, $x = \frac{a}{2}$, then we get

$$(U)_{a/2} = \frac{1}{2} kx^2 + \frac{1}{2} k\left(\frac{1}{2} a\right)^2$$

$$= \frac{1}{8} ka^2$$

$$= \frac{1}{4} E$$

and

$$(K)_{a/2} = \frac{1}{2} k(a^2 - x^2)$$

$$= \frac{1}{2} k\left(a^2 - \frac{a^2}{4}\right)$$

$$= \frac{3}{8} ka^2$$

$$= \frac{3}{4} E.$$

Thus, the potential energy is one fourth and the kinetic energy is three fourth of the total energy.

Now let x be the displacement at which the energy is half potential and half kinetic energy *i.e.*,

$$U = k = \frac{1}{2} E$$

$$U = \frac{1}{2} E = \frac{1}{2} kx^2$$

$$\frac{1}{2}\left(\frac{1}{2} ka^2\right) = \frac{1}{2} kx^2$$

$$\therefore \quad \boxed{x = \frac{a}{\sqrt{2}}}$$

• 4.5. TIME-AVERAGE OF KINETIC ENERGY AND POTENTIAL ENERGY

The kinetic energy of a particle of mass m executing S.H.M. under a force constant is given by

$$K = \frac{1}{2} mv^2 = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2$$

$$= \frac{1}{2} m\omega^2 a^2 \cos^2(\omega t + \phi), \text{ where } \omega^2 = \frac{k}{m}$$

The time-average of the kinetic energy over a period T of the motion is

$$\overline{K} = \frac{\int_0^T K\, dt}{T} = \frac{1}{2} m\omega^2 a^2 \frac{\int_0^{2\pi/\omega} \cos^2 (\omega t + \phi)\, dt}{2\pi/\omega}$$

$$= \frac{m\omega^3 a^2}{4\pi} \int_0^{2\pi/\omega} \frac{1 - \sin^2 (\omega t + \phi)}{2} dt$$

$$= \frac{m\omega^3 a^2}{8\pi} \left[\int_0^{2\pi/\omega} dt + \int_0^{2\pi/\omega} \cos 2 (\omega t + \phi)\, dt\right]$$

$$= \frac{m\omega^3 a^2}{8\pi} \left[\frac{2\pi}{\omega} - 0\right]$$

$$= \frac{1}{4} m\omega^2 a^2 = \frac{1}{4} ka^2$$

The potential energy of the particle is

$$U - \frac{1}{2} kx^2 = \frac{1}{2} ka^2 \sin^2(\omega t + \phi)$$

The time average of the potential energy over a period T of the motion is

$$\overline{U} = \frac{\int_0^t U dt}{T} = \frac{\frac{1}{2} ka^2 \int_0^{2\pi/\omega} \sin^2(\omega t + \phi)}{2\pi/\omega}$$

$$= \frac{k\omega a^2}{4\pi} . \int_0^{2\pi/\omega} \frac{1 - \cos 2 (\omega t + \phi)}{2} dt$$

$$= \frac{k\omega a^2}{8\pi} \left[\int_0^{2\pi/\omega} dt - \int_0^{2\pi/\omega} \cos 2 (\omega t + \phi)\, dt\right]$$

$$= \frac{k\omega a^2}{8\pi} \left[\frac{2\pi}{\omega} + 0\right]$$

$$= \frac{1}{4} ka^2.$$

Hence, the time average kinetic energy $\overline{K}$ is equal to the time average potential energy $\overline{U}$ and each is equal to $\frac{1}{4} ka^2$.

Now the position average of the kinetic and potential energy is obtained as follows. The kinetic energy is

$$K = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \frac{1}{2} k (a^2 - x^2)$$

The position average over the displacement from $x = 0$ to $x = a$ is

$$\overline{K} = \frac{\int_0^a K dx}{a} = \frac{\frac{1}{2} k \int_0^a (a^2 - x^2)\, dx}{a}$$

$$= \frac{1}{2}\frac{k}{a}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{1}{3}ka^2$$

the potential energy is

$$U = \frac{1}{2}kx^2$$

The position average over the displacement from $x = 0$ to $x = a$ is

$$\overline{U} = \frac{\int_0^a Udx}{a}$$

$$= \frac{\frac{1}{2}k\int_0^a x^2dx}{a}$$

$$= \frac{1}{2}\frac{k}{a}\left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{1}{6}ka^2$$

Hence, the position average kinetic energy $\left(\frac{1}{2}ka^2\right)$ is not equal to the position average potential energy $\left(\frac{1}{6}ka^2\right)$.

• 4.6. PERIOD OF OSCILLATION OF A MASS SUSPENDED BY A SPRING

Let us consider a weightless spring of length l, hanging vertically as shown in fig. 3 when a weight of mass m is attached to its lower end then its length increases by x'. In this position the spring exerts a restoring vertical force F on the mass m.

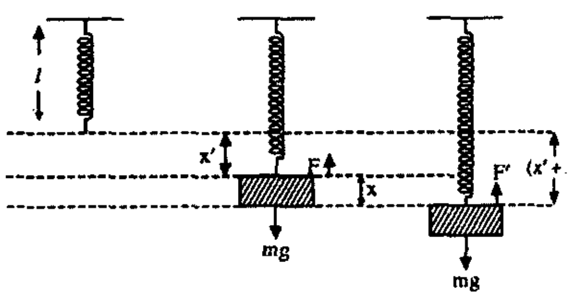


Fig. 3.

Now from Hooke's law

$$F = -kx'$$

where k is constant and this constant is known as **force constant of the spring**. In equation (1) the negative sign shows that the direction is opposite to the expension in length of the spring. The force due to mass m is mg acting downward.

Since there is no acceleration in the body so net force should be zero *i.e.*,

$$F + mg = 0$$

$$-kx' + mg = 0 \qquad \text{[from eq. (1)]}$$

$$x' = \frac{mg}{k} \qquad \dots (2)$$

when the body is displaced by small distance and left, then the body starts to vibrate about its mean position. Let x be the displacement from its equilibrium position. In this position the tolat increment in the length of the spring is $(x'+x)$.

Again, from Hooke's law

$$F' = -k(x'+x)$$

$$F' = -k\left(\frac{mg}{k} + x\right) \qquad \text{[from eq. (2)]}$$

$$F' = -mg - kx \quad \ldots (3$$

Therefore, total force is given by

$$F'' = F' + mg$$

$$F'' = (-mg - kx) + mg \quad \text{[from eq. (3)}$$

$$F'' = -kx \quad \ldots (4$$

From Newton's second law force F'' will be equal to the product of mass m and acceleration $\frac{d^2x}{dt^2}$ *i.e.*,

$$F'' = m\frac{d^2x}{dt^2} = -kx$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where $\omega^2 = \frac{k}{m}$

This equation shows that the motion of spring is simple harmonic.

In this position the time period is given by

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}$$

• 4.7 FREQUENCY OF MASS CONNECTED WITH TWO SPRINGS IN HORIZONTAL POSITION

When the mass m oscillates, then at any instant one spring is stretched and the order is compressed and vice-versa.

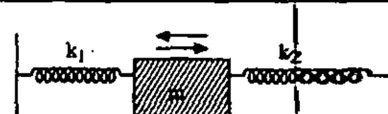


Fig. 4.

Let x be the displacement of mass m from its mean position then

$$m\frac{d^2x}{dt^2} = -k_1x - k_2x$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{k_1 + k_2}{m}x$$

or $$\frac{d^2x}{dt^2} = -\omega^2 x$$

where $$\omega^2 = \frac{k_1 + k_2}{m}$$

This shows that, the motion is simple harmonic.

The time period is given by

$$T = \frac{2\pi}{\omega}$$

$$\boxed{T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}}$$

and frequency is given by

$$\boxed{n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}}}$$

• 4.8 SIMPLE PENDULUM

When a heavy particle is suspended by an inextensible, weightless and flexible string from a rigid support, then this system is known as simple pendulum.

Let us consider a particle of mass m. Let which is suspended by an inextensible, weightless and flexible string of length l.

Let a pendulum and displaced from its mean position and allowed to oscillate as shown in fig. 5.

Let at time 't' the particle be at point P. In this position the force acting on the particle vertically downward in mg.

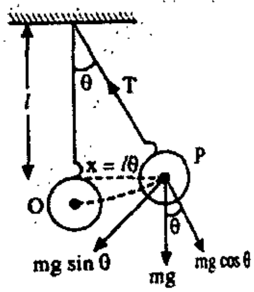


Fig. 5.

Now resolving mg into two components :

(1) Force along the string = $mg\cos\theta$

(2) Force perpendicular to the string = $mg\sin\theta$

Let the tension in the string be T which is balanced by the component $mg\cos\theta$.

i.e., $$T = mg\cos\theta$$

Hence $-mg\sin\theta$ is the only force which acts on the oscillating particle

$$F = -mg\sin\theta \quad \text{... (1)}$$

Here the negative sign shows that the acceleration is directed towards the mean position.

$\because$ $$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \ldots\ldots$$

If θ is very small then $\sin\theta = \theta$.

so by eq. (1)

$$F = -mg\theta \quad \text{... (2)}$$

The displacement is

$$x = l\theta$$

acceleration $\dfrac{d^2x}{dt^2} = l\,\dfrac{d^2\theta}{dt^2}$

$\therefore$ $$\text{Force} = ml\frac{d^2\theta}{dt^2} \qquad [\because F = ma] \quad \text{... (3)}$$

From eqns. (2) and (3), we get

$$ml\frac{d^2\theta}{dt^2} = -mg\theta$$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0} \quad \text{... (4)}$$

This is the expression for the equation of motion of simple pendulum which is similar to the eq. of simple harmonic motion *i.e.*,

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \text{... (5)}$$

From eqns. (4) and (5)

$$\omega^2 = \frac{g}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

$$\text{Time period } T = \frac{2\pi}{\omega}$$

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

This is the expression for the time period of simple pendulum.

• 4.9 COMPOUND PENDULUM

When a rigid body which is capable of oscillating freely in a vertical pla about a fixed axis, passing through the body but not through its centre gravity then the system is called compound pendulum and this fixed point is known as point of suspension.

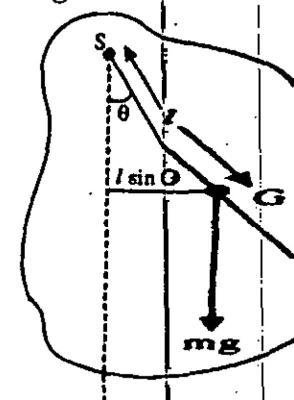


Fig. 6

Let us consider a rigid body of mass m. Let G be the centre of gravity of the body and S be the point of suspension. When this rigid body is displaced from its mean position, then SG makes an angle θ with the vertical.

The force acting vertically downwards = mg

and restoring moment of this force

$$= -mg\, l \sin\theta$$

This restoring moment produces angular acceleration in the pendulum.

Let I be the moment of ienrtia of the pendulum about an axis passing through S and perpendicular to its length. In this position the angular acceleration is $\frac{d^2\theta}{dt^2}$.

so
$$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

where $I\frac{d^2\theta}{dt^2}$ is the torque.

Here negative sign shows that the force is directed towards the mean position.

If θ is very small than $\sin\theta = \theta$

$$\therefore \quad I\frac{d^2\theta}{dt^2} = -mgl\,\theta$$

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\theta}$$

$$\frac{d^2\theta}{dt^2} + \frac{mgl}{I}\theta = 0$$

This is the equation of motion of compound pendulum. Thus, motion of compou pendulum is simple harmonic.]

Here
$$\omega^2 = \frac{mgl}{I} \Rightarrow \omega = \sqrt{\frac{mgl}{I}}$$

$\therefore$ Time period
$$T = \frac{2\pi}{\omega}$$

$$\boxed{T = 2\pi\sqrt{\frac{I}{mgl}}}$$

From theorem of parallel axes the moment of inertia of the pendulum about an ax passing through S and perpendicular to its plane

$$= mk^2 + ml^2$$
$$= m(k^2 + l^2)$$

where k is the radius of gyration about an axis passing through the centre of gravity G pendulum.

-so $$\frac{d^2\theta}{dt^2} + \left(\frac{lg}{k^2 + l^2}\right)\theta = 0$$

$\therefore$ $$\omega^2 = \frac{lg}{k^2 + l^2}$$

$$\omega = \sqrt{lg / k^2 + t^2}$$

$\therefore$ $$\text{Time period } T = \frac{2\pi}{\omega}$$

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{lg}}$$

$$\boxed{T = 2\pi\sqrt{\frac{\frac{k^2}{l} + 1}{g}}}$$

Here $\left(\frac{k^2}{l} + l\right)$ is called the equivalent length of simple pendulum.

• 4.10 TORSION PENDULUM

When one end of a very thing and long wire is clamped to a rigid support and the other end is attached to the centre of a heavy disc or sphere, then this arrangement is known as torsion pendulum.

If the disc is turned through an angle θ then the wire is also twisted through the same angle θ. In this position the restoring torsional couple $(-C\theta)$ beings to act which tends to bring the pendulum to its initial position, where c is the torsional constant or **couple per unit twist**.

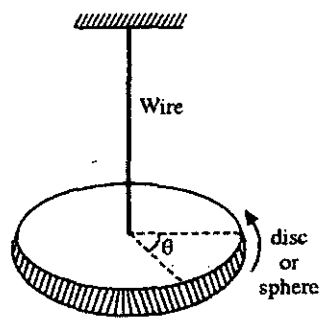


Fig. 7.

Thus torsional couple $\tau = -C\theta$ ... (1)

Let I be the moment of inertia of the disc about the wire as the axis. Here $\frac{d^2\theta}{dt^2}$ is the angular acceleration and the couple due to the acceleration is given by $I\frac{d^2\theta}{dt^2}$. This couple is balanced by the restaring torsional couple. Thus,

$$I\frac{d^2\theta}{dt^2} = -C\theta$$

$$I\frac{d^2\theta}{dt^2} + C\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$$

or $$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

where $\omega^2 = \frac{C}{I}$

This equation presents the simple harmonic motion where period is given by

$$T = \frac{2\pi}{\omega}$$

$$\boxed{T = 2\pi\sqrt{\frac{I}{C}}}$$

From mechanics of materials, 'C in case of wire is given by

$$C = \frac{\pi \eta r^4}{2l}$$

where r = radius of the wire

l = length of the wire

η = modulus of rigidity

Student Activity

1. What is the importance of S.H.M. ?

2. What do you mean by restoring force ?

• 4.11. DAMPED HARMONIC OSCILLATOR

So far we have considered entirely free motion of the harmonic oscillator. In suc ideal case the energy of the oscillator remains constant throughout the motion. Bu actual practice when a body oscillates in a medium (like air), then medium offe resistance to its motion. As a result the amplitude of oscillations goes on decreasing ultimately it becomes zero. Why is it so? Actually various frictional forces which a from air resistance or internal forces, act on the oscillating body tending to reduce amplitude of the motion. The motion of the oscillator is said to be damped by fric and is called damped harmonic motion and the oscillating system is called dam harmonic oscillator. The magnitude of the friction depends upon the velocity of oscillator. In most cases, the frictional or damping force is proportional to the velocit the oscillating body; but it is oppositely directed.

The frictional force proportional to velocity is given by

$$F_D = -b\frac{dx}{dt}$$

Therefore, total force acting on the oscillator $= -Kx - b\frac{dx}{dt}$.

According to Newton's second law this force is equal to $m\frac{d^2x}{dt^2}$.

Therefore, equation of motion is given by $m\frac{d^2x}{dt^2} = -Kx - b\frac{dx}{dt}$.

or
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + Kx = 0 \quad \ldots$$

or
$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{K}{m}x = 0. \quad \ldots$$

Putting $\frac{b}{m} = \frac{1}{\tau}$ and $\frac{K}{m} = \omega_0^2$. (τ being known as relaxation tir

we have for the equation of motion

$$\frac{d^2x}{dt^2} + \frac{1}{\tau}\frac{dx}{dt} + \omega_0^2 x = 0. \quad \ldots$$

Let the trial solution of equation (3) be

$$x = e^{\alpha t}.$$

$$\therefore \quad \frac{dx}{dt} = \alpha e^{\alpha t}, \quad \frac{d^2x}{dt^2} = \alpha^2 e^{\alpha t}$$

$$\therefore \quad \alpha^2 e^{\alpha t} + \left(\frac{1}{\tau}\right) . \alpha e^{\alpha t} + \omega_0^2 e^{\alpha t} = 0$$

or $$\alpha^2 + \frac{1}{\tau}\alpha + \omega_0^2 = 0$$

or $$\alpha = \frac{\left[-\frac{1}{\tau} \pm \sqrt{\left(\frac{1}{\tau^2} - 4\omega_0^2\right)}\right]}{2}$$

or $$\boxed{\alpha = \left[-\frac{1}{2\tau} \pm \sqrt{\left(\frac{1}{2\tau}\right)^2 - \omega_0^2}\right]}$$

$$\therefore \quad \alpha_1 = -\frac{1}{2\tau} + \sqrt{\left[\left(\frac{1}{2\tau}\right)^2 - \omega_0^2\right]}$$

$$= -\frac{1}{2\tau} + \beta, \text{ where } \beta = \sqrt{\left[\left(\frac{1}{2\tau}\right)^2 - \omega_0^2\right]}$$

$$\alpha_2 = -\frac{1}{2\tau} - \sqrt{\left[\left(\frac{1}{2\tau}\right)^2 - \omega_0^2\right]} = -\frac{1}{2\tau} - \beta$$

Therefore, the general solution of eq. (3) is given by

$$x = e^{-1/2\tau}\{A_1 e^{\beta t} + A_2 e^{-\beta t}\} \qquad \ldots (4)$$

where $$\beta = \sqrt{\left\{\left(\frac{1}{2\tau}\right)^2 - \omega_0^2\right\}}$$

Now there arise three cases :

Case (i) Overdamped motion. When β is real *i.e.* $\frac{1}{4\tau^2} > \omega_0^2$ the solution is given by eq. (4), where A_1 and A_2 are constants to be determined from initial conditions.

Case (ii) Critically damped motion : When $\beta = 0$, *i.e.*, $\frac{1}{4\tau^2} = \omega_0^2$.

In this case equation (3) has the general solution

$$x = e^{-1/2\tau}(A_1 + A_2 t) \qquad \ldots (5)$$

where A_1 and A_2 are constants to be found from initial conditions.

Case (iii) Under Damped motion : In this case β is imaginary, *i.e.*

$$\frac{1}{4\tau^2} < \omega_0^2$$

$$\therefore \quad \sqrt{\left(\frac{1}{4\tau^2} - \omega_0^2\right)} = i\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}$$

Substituting these values in eq. (4), we have

$$x = e^{-t/2\tau}\left\{A_1 e^{i\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}t} + A_2 e^{-i\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}t}\right\}$$

$$= e^{-t/2\tau}\left[A_1\left\{\cos\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}t + i\sin\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}t\right\}\right.$$

$$\left. + A_2\left\{\cos\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)} - i\sin\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}t\right\}\right]$$

$$= e^{-t/2\tau}\left[C\cos\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}\,t + D\sin\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}\,t\right]$$

where $C = A_1 + A_2$, $D = i(A_1 -$

$$= e^{-t/2\tau}\sqrt{(C^2+D^2)}\left[\frac{C}{(C^2+D^2)}\cos\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}\,t + \frac{D}{\sqrt{(C^2+D^2)}}\sin\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}\right.$$

Putting $\dfrac{C}{\sqrt{(C^2+D^2)}} = \sin\phi$, $\qquad \dfrac{D}{\sqrt{(C^2+D^2)}} = \cos\phi$

$$= e^{-t/2\tau}\sqrt{(C^2+D^2)}\left[\sin\phi\,\cos\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}\,t + \cos\phi\sin\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}\,\tau\right]$$

$$= \sqrt{(C^2+D^2)}\,e^{-t/2\tau}\sin\left\{\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}\,t + \phi\right\}$$

$$= ae^{-t/2\tau}\sin\left\{\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}\,t + \phi\right\}$$

where $\sqrt{(C^2+D^2)} = a$ is called the initial amplitude and ϕ is the phase angle and a determined from initial conditions.

In the first case the frictional force or damping is so large that oscillations do n take place and the particle of mass m returns to its equilibrium position gradually.

In the second case, the particle of mass m returns to its equilibrium position fast than that in the first case.

In the third case, oscillations take place about the equilibrium position. Th amplitude of these oscillations goes on decreasing and finally it becomes zero.

All the three cases discussed above are shown in figure 5.14. with initial condition at $x = a$, $\dfrac{dx}{dt} = 0$.

The time interval between two successive maxima or minima in the damped oscillator motion is known as the "period of the motion" and is give

$$T = \frac{2\pi}{\sqrt{\left(\omega_0^2 - \frac{1}{4\tau^2}\right)}} \qquad \ldots (\text{?}$$

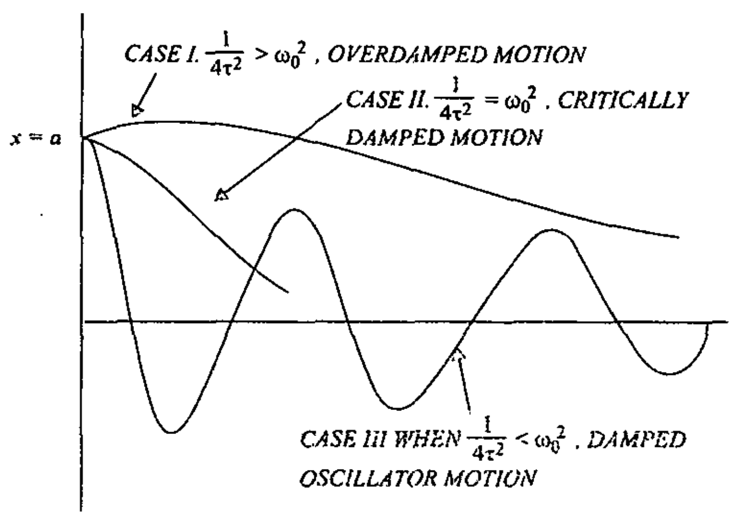


Fig. 8.

$$= \frac{2\pi}{\sqrt{\left(\frac{K}{m} - \frac{b^2}{4m^2}\right)}} \qquad \left(\text{since } \omega_0^2 = \frac{K}{m} \text{ and } \frac{1}{\tau} = \frac{b}{m}\right)$$

Frequency of the motion is given by

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\left(\frac{K}{m} - \frac{b^2}{4m^2}\right)} \qquad \ldots (8)$$

From eqn. (6) the *amplitude of damped motion at any instant t is* $ae^{-t/2\tau}$.

If $b = 0$, or damping is negligibly small the motion of the oscillator is undamped and the frequency of the motion is called **"natural frequency"** and is given by

$$n_0 = \frac{1}{2\pi}\sqrt{\left(\frac{K}{m}\right)} \qquad \ldots (9)$$

also **natural period of motion**

$$T = 2\pi\sqrt{\left(\frac{m}{K}\right)}. \qquad \ldots (10)$$

4.12. POWER DISSIPATION IN THE WEAK DAMPING LIMIT

When the damping is weak, $\left(\frac{1}{2\tau}\right)^2 << \omega_0^2$. *i.e.,* $\frac{b^2}{4m^2} << \frac{K}{m}$, for convenience let us consider that the initial phase difference is equal to zero. Then equation (5) may be written as

$$x = ae^{-t/2\tau} \sin \omega_0 t. \qquad \ldots (11)$$

$$\therefore \quad \frac{dx}{dt} = -a \cdot \left(\frac{1}{2\tau}\right) e^{-t/2\tau} \sin \omega_0 t + ae^{-t/2\tau} \omega_0 \cos \omega_0 t. \qquad \ldots (12)$$

There the kinetic energy is given by

$$T_E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \frac{1}{2} ma^2 \left[\omega_0^2 e^{-t/2\tau} \cos \omega_0 t - \left(\frac{1}{2\tau}\right)^2 e^{-t/2\tau} \sin^2 \omega_0 t\right]^2$$

$$= \frac{1}{2} ma^2 e^{-t/\tau}\left[\omega_0^2 \cos^2 \omega_0 t + \left(\frac{1}{2\tau}\right)^2 \sin^2 \omega_0 t - \frac{\omega_0}{\tau} \sin \omega_0 t \cos \omega_0 t\right] \qquad \ldots (13)$$

Average kinetic energy is obtained by taking the time average over one vibration of equation (13). It is represented by $\overline{T}_E$ and is given by

$$\overline{T}_E = \frac{\int_0^T \frac{1}{2} ma^2 e^{-t/\tau}\left[\omega_0^2 \cos^2 \omega_0 t + \left(\frac{1}{2\tau}\right)^2 \sin^2 \omega_0 t - \frac{\omega_0}{2\tau} \sin 2\omega_0 t\right]}{T}$$

where T is time period

$$= \frac{1}{2} ma^2 e^{-t/\tau}\left[\frac{1}{2}\omega_0^2 + \frac{1}{8\tau^2}\right]$$

because average of a sine or cosine is zero.

$e^{-t/\tau}$ is taken outside the integral, since it may be assumed that the amplitude of oscillation, *i.e.* $ae^{-t/2\tau}$ does not change in one cycle of motion.

Therefore, the average kinetic energy

$$\overline{T}_E = \frac{1}{4} ma^2 e^{-t/\tau}\left(\omega_0^2 + \frac{1}{4\tau^2}\right) \qquad \ldots (14)$$

We have already assumed that $\omega_0^2 >> \frac{1}{4\tau^2}$.

Therefore, average kinetic energy $= \frac{1}{2} ma^2 \omega_0^2 e^{-t/\tau}$ $\qquad \ldots (15)$

Potential energy $V = \frac{1}{2}Kx^2 = \frac{1}{2}K(ae^{-t/2\tau}\sin\omega_0 t)^2$

$$= \frac{1}{2}Ka^2 e^{-t/\tau}\sin^2\omega_0 t$$

$$= \frac{1}{2}m\omega_0^2 a^2 e^{-t/\tau}\sin^2\omega_0 t \qquad \text{(since } K/m =$$

The average potential energy is obtained by taking the average over one perio
motion and is given by

$$\bar{V} = \frac{1}{4}m\omega_0^2 a^2 e^{-t/\tau}, \left[\text{since } \frac{\int_0^T \sin^2\omega_0 t\, dt}{T} = \frac{1}{2}\right] \quad ...$$

Therefore total energy $= \bar{E} = \bar{T}_E + \bar{V}$

$$= \frac{1}{4}ma^2\omega_0^2 e^{-t/\tau} + \frac{1}{4}ma^2\omega_0^2 e^{-t/\tau}$$

$$= \frac{1}{2}ma^2\omega_0^2 e^{-t/\tau} \quad ...$$

• 4.13. RELAXATION TIME

Clearly total mechanical energy depends on relaxation time τ.

Total energy at $t = 0$, $E_0 = \frac{1}{2}ma^2\omega_0^2$

$$\frac{E}{E_0} = e^{-t/\tau} \Rightarrow E = E_0 e^{-t/\tau}$$

If $t = \tau$, $E = E_0\, e^{-1} = \frac{1}{e}E_0$.

That is relaxation time is defined as the time in which total mechani energy becomes 1/e times of its initial value.

The average power dissipation $P(t)$ may be defined as negative of the rate of cha
of energy.

$$P(t) = -\frac{d}{dt}(\bar{E}) = -\frac{d}{dt}\left(\frac{1}{2}ma^2\omega_0^2 e^{-t/\tau}\right) = \frac{1}{2\tau}ma^2\omega_0^2 e^{-t/\tau}$$

$$= \frac{\bar{E}}{\tau} \qquad \text{[from (17)} \quad ...$$

• 4.14. QUALITY FACTOR Q

The 'quality factor' of an oscillator may be defined as 2π times the ratio energy stored to the average energy loss per unit time period and it is deno by Q, i.e.,

$$Q = 2\pi \times \frac{\text{energy stored}}{\text{energy loss in time}}$$

$$= 2\pi \times \frac{\text{energy stored}}{\text{energy loss per unit time} \times \text{time period}}$$

$$= \frac{2\pi E}{P \times T} = \frac{E}{P/\omega} \qquad \left[\text{since } \frac{T}{2\pi} = \frac{1}{\omega}\right] \quad ...$$

Since quality factor is the ratio of the energies, it has no dimensions.

In the case of weak damping, we know $P = -\frac{E}{\tau}$ from eq. (18).

With this value of P the quality factor is given by from eq. (19).

$$Q = \frac{E}{\dfrac{(E/\tau)}{\omega}} = \omega\tau \qquad \dots (20)$$

$$= \frac{\omega m}{b}. \qquad \left(\text{since } \frac{1}{\tau} = \frac{b}{m}\right) \quad \dots (21)$$

From equation (21) it is clear that greater is the value of Q, smaller is the value of b, that is, for lightly damped oscillator, the value of Q should be high.

• 4.15. FORCED (OR DRIVEN) HARMONIC OSCILLATOR

The problem of forced motion of damped harmonic oscillator is of great importance. If the particle is also acted by an external force $F(t)$ such that $F(t) = F_0 \sin pt$, along with frictional (or damping) forces, the resulting oscillations are said to be driven (or forced) harmonic oscillations and the oscillating system is known as *"driven harmonic oscillator"*.

The differential equation of motion is

$$m\frac{d^2x}{dt^2} + \frac{dx}{dt} + Kx = F_0 \sin pt$$

or

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin pt, \qquad \dots (1)$$

Putting

$$\frac{b}{m} = \frac{1}{\tau}, \frac{K}{m} = \omega_0^2 \text{ and } \frac{F_0}{m} = f_0. \qquad \dots (2)$$

(where ω_0 represents natural frequency of the system in the absence of friction and applied force) equation (1) becomes

$$\frac{d^2x}{dt^2} + \frac{1}{\tau}\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt. \qquad \dots (3)$$

Let the solution of equation (3) be

$$x = a \sin (pt - \phi), \qquad \dots (4)$$

where a is the amplitude and ϕ the phase angle. In above equation p is not the natural frequency of the oscillator but it is *that of driving force* and ϕ represents the phase difference between the impressed force and the displacement of the oscillator. Both the driving force and the displacement execute simple harmonic motion. The angles between two maxima of the force and displacement is 2π radians or 360°. The phase ϕ represents the angle by which the displacement lags behind the force.

From equation (4), we have

$$\frac{dx}{dt} = ap\cos(pt - \phi) \text{ and } \frac{d^2x}{dt^2} = -ap^2 \sin(pt - \phi). \qquad \dots (5)$$

With these values equation of motion (3) may be written as

$$-ap^2 \sin(pt - \phi) + (1/\tau)ap\cos(pt - \phi) + \omega_0^2 a\sin(pt - \phi) = f_0 \sin pt$$

or

$$(\omega_0^2 - p^2)a\sin(pt - \phi) + (1/\tau)pa\cos(pt - \phi) = f_0 \sin pt. \qquad \dots (6)$$

We know

$$\left.\begin{aligned}\sin(pt - \phi) &= \sin pt\cos\phi - \cos pt\sin\phi \\ \cos(pt - \phi) &= \cos pt\cos\phi + \sin pt\sin\phi\end{aligned}\right\} \qquad \dots (7)$$

With these values equation (6) becomes

$$(\omega_0^2 - p^2)\, a\, (\sin pt \cos\phi - \cos pt \sin\phi)$$
$$+ (1/\tau)pa(\cos pt\cos\phi + \sin pt\sin\phi) = f_0 \sin pt,$$

$$\{(\omega_0^2 - p^2)\cos\phi + (1/\tau)p\sin\phi\}\, a\sin pt$$
$$+ \{-\omega_0^2 - p^2)\sin\phi + (1/\tau)p\cos\phi\}\, a\cos pt = f_0 \sin pt. \qquad \dots (8)$$

This equation holds for all times. Choosing time such that $\cos pt = 1$ then $\sin pt = 0$

or

$$-(\omega_0^2 - p^2)\sin\phi + (1/\tau)p\cos\phi = 0$$

or $$\tan\phi = \frac{p/\tau}{\omega_0^2 - p^2}$$

Then from equation (8); we have

$$a\{(\omega_0^2 - p^2)\cos\phi + (1/\tau)\, p\sin\phi\} = f_0$$

or $$a = \frac{f_0}{(\omega_0^2 - p^2)\cos\phi + (1/\tau)\, p\sin\phi} \quad \ldots (10)$$

from equation (9),

$$\left.\begin{aligned} \sin\phi &= \frac{p/\tau}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} \\ \cos\phi &= \frac{\omega_0^2 - p^2}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} \end{aligned}\right\} \quad \ldots (11)$$

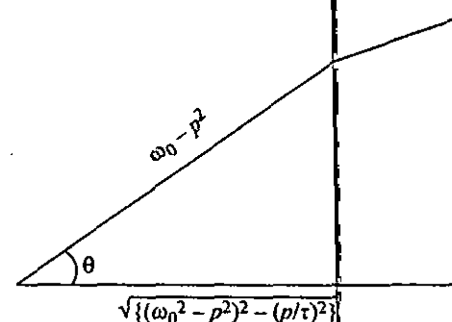


Fig. 9

With these values the amplitude of the motion is given, from equation (10), by

$$a = \frac{f_0}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} \quad \ldots$$

Putting the values of a and ϕ in equation (4), the solution of equation (3) is given

$$x = \frac{f_0}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} \sin\left(pt - \tan^{-1}\frac{p/\tau}{\omega_0^2 - p^2}\right) \quad \ldots$$

Now there arise three cases :

Case I. Low driving frequency *(i.e., when $p << \omega_0$).*

From equation (11), we see

$$\sin\phi \to 0 \text{ and } \cos\phi \to 1 \text{ if } \omega_0 >> p, \; i.e., \; \phi \to 0,$$

where *means that the driving force and the displacement are in the same phase.*

The amplitude of motion is given by

$$a = \frac{f_0}{\omega_0^2} = \frac{F_0/m}{K/m} = \frac{F_0}{K} \quad \ldots$$

Thus in this case the amplitude does not depend on the mass and the damping o oscillating system; but it depends only on the force constant K.

Now to see how the amplitude varies as the driving frequency p is gradu increased, we put the expression for a in slightly modified form as

$$a = \frac{f_0}{\sqrt{[(\omega_0^2 - p^2)^2 + (p/\tau)^2]}} = \frac{f_0}{\sqrt{\left[\left(\omega_0^2 - \frac{1}{2\tau^2} - p^2\right)^2 + \frac{\omega_0^2}{\tau^2} - \left(\frac{1}{4\tau^4}\right)\right]}} \quad \ldots$$

This equation shows that if the frequency of the driving force is gradually incre the amplitude a starts increasing. To find the *condition of maximum amplitude* differentiate the denominator of equation (11) with respect to p and put equal to *i.e.*,

$$\frac{d}{dp}\left[\sqrt{\{(\omega_0^2 - p^2)^2 + (p\tau)^2\}}\right] = 0$$

This gives

$$p^2 = \omega_0^2 - (1/2)\,\tau^2$$

This is also self evident from eqn. (15). Thus amplitude becomes maximum w angular frequency of driving force becomes equal to p_r given by

$$p_r = \sqrt{\{\omega_0^2 - ((1/2)\,\tau^2)\}} \quad \ldots$$

and the maximum amplitude is $a_{max} = \dfrac{f_0 \tau}{\sqrt{\{\omega_0^2 - ((1/4)\tau^2)\}}}$, ... (17)

Thus we see that ***for a certain frequency*** $\dfrac{p_r}{2\pi}$ ***of the driving force, the amplitude of the oscillating system becomes maximum.*** This phenomenon is called **resonance** and this frequency p_r is called the **resonant frequency.** For a driven harmonic oscillator the resonant frequency is $\dfrac{pr}{2\pi}$. The value of p_r is less than ω_0 or ω where ω_0 is natural frequency of oscillation in the absence of any damping and $\omega = \left[\sqrt{\{\omega_0^2 - (1/4\tau^2)\}}\right]$ is the frequency of free oscillations in the presence of damping.

If the damping is very low, then from eqn. (16) p_r ' ω_0 and so

$$a_{max} = \frac{f_0 \tau}{\omega_0} \quad \text{... (18)}$$

From this equation it is obvious that in the case of zero damping (b or $1/\tau = 0$), the amplitude should be infinity. But it is practically impossible since practically frictional force are never zero.

Case II. When $p = \omega_0$;

In this case $\tan\phi \to \infty$ or $\phi = \dfrac{\pi}{2}$

and the amplitude a is given by $a = \dfrac{f_0 \tau}{p}$

This is less than the amplitude given by equation (7). But for very low damping the resonant frequency is equal to the natural frequency of the oscillator and then amplitude is maximum.

Case III. High driving frequency (when $p >> \omega_0$).

In this case $\tan\phi = -\dfrac{1}{p\tau} \to 0$ or $\phi \to \pi$.

Also the amplitude is given by

$$a = \frac{f_0}{p^2} = \frac{F_0}{mp^2} \quad \text{... (19)}$$

Obviously the response (amplitude) is inversely proportional to the square of p. Consequently on increasing the driving frequency p, the response or amplitude decreases. In this case (as $\phi = \pi$) the displacement lags behind the driving frequency by an angle π.

Resonance

Fig. 10 respresents the variation of amplitude a with driving frequency p of various values of various damping constant K. Initially when the angular frequency p of the driving force is increasing from low frequency region, the amplitude a continuously increases and at a certain value p_r (which is nearly equal to ω_0 for small damping) the amplitude becomes maximum. This is the condition of resonance. Now on further increasing p the amplitude decreases gradually. The values of p corresponding to peaks of curves in fig. 10 represent the resonant frequencies. As is obvious from eqn. (16) $p_r = \sqrt{[\omega_0^2 - (1/2\tau^2)]}$,

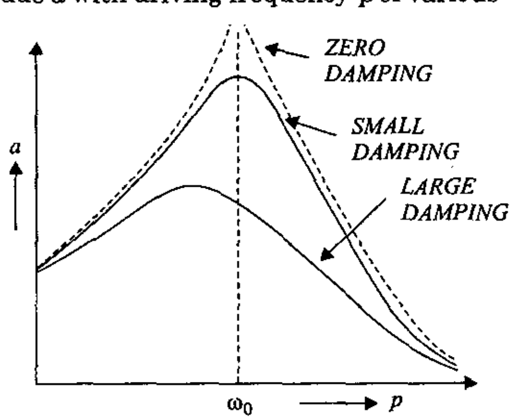


Fig. 10

the resonant frequency p_r is less than ω_0 for large damping; but is the case of lo- damping the resonant frequency is approximately equal to the natural frequency ω_0 the oscillator. From equation (17), we have $a_{max} = \frac{f_0\tau}{\sqrt{\{\omega_0 - (1/4\tau^2)\}}}$, therefore, it is als obvious that for low damping the height of the peaks of the curves become greater an when damping is zero (ideal case) the height of the peak rises to infinity.

From above three cases we have seen that the value of ϕ increases from zero to π an always remains positive. This means that *the displacement of the oscillator always lag behind the driving force in phase.* The variation of the phase difference ϕ with frequenc is shown in fig. 10.

Sharpness of Resonance : From fig. 11 we have seen that at resonant frequenc the amplitude of the oscillating system becomes maximum. If the frequency of th driving force is increased or decreased the amplitude falls from the maximum value.

The term sharpness of resonance refers to the rate of fall of amplitude with the change in the driving frequency on either side of resonant frequency.

From fig. 11 we see that when damping is small, the amplitude falls off very rapidly on either side of the resonant frequency and then we say that the resonance is sharp. On the other hand when the damping is large, the amplitude falls off very slowly on either side of resonant frequency and then we say that the resonance is flat.

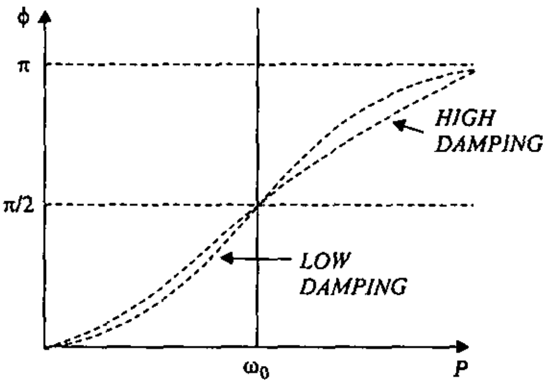


Fig. 11

Thus we may say that ***smaller is the damping, sharper is the resonance or larger is the damping, flatter is the resonance.***

Power Absorption : In the case of forced or driven harmonic vibrations the power is being absorbed by the system continuously due to applied driving force and none of it being dissipated.

From equation (13), we have

$$x = \frac{f_0}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} \sin(pt - \phi) \qquad \text{...(20)}$$

$$\therefore \quad \frac{dx}{dt} = \frac{f_0}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} \cos(pt - \phi)$$

The work done per unit time on the oscillating system by the driving force is given by

$$W = \text{force} \times \text{velocity} = F\frac{dx}{dt} = (F_0 \sin pt) \cdot \frac{dx}{dt}$$

$$\therefore \quad W = F_0 \sin pt \times \frac{f_0 p}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} \cos(pt - \phi)$$

$$= \frac{m f_0^2\, p}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} \sin pt \cos(pt - \phi) \quad \left[\text{since } f_0 = \frac{F_0}{m}\right]$$

$$= \frac{mf_0^2\, p}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} \sin pt\,(\cos pt \cos\phi + \sin pt \sin\phi)$$

$$= \frac{mf_0^2\, p}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} (\sin pt \cos pt \cos\phi + \sin^2 pt \sin\phi).$$

The time average of work done per unit time given by

$$P = \frac{\int_0^T W\,dt}{T} = \frac{mf_0^2\, p}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}}\left(\frac{1}{2}\sin\phi\right)$$

$$\left(\text{since time average of } \sin pt \cos pt \text{ vanishes and that of } \sin^2 pt = \frac{1}{2}\right)$$

Putting the value of $\sin\phi$ from equation (11), we have

$$P = \frac{1}{2}\frac{mf_0^2\, p}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} \times \frac{p/\tau}{\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}}$$

or

$$p = \frac{mf_0^2\, p}{\frac{1}{2\tau}\sqrt{\{(\omega_0^2 - p^2)^2 + (p/\tau)^2\}}} \quad \text{... (21)}$$

When $p = \omega_0$, the power absorption is called resonance power absorption and is given by

$$P_{resonance} = \frac{1}{2} mf_0{}^2 \tau. \quad \text{... (22)}$$

Band width of the oscillator : We have seen that the driven harmonic oscillator absorbs power continuously due to the applied driving force. The power absorbed is maximum at resonance (*i.e.* when the driving frequency p is equal to ω_0 the natural frequency of free oscillations for small damping). Fig. 20 represents the variation of p/p_{res} with frequency p of the driving force. The value of P/P_{res} decreases with change in resonant frequency. The two values of ω_1 and ω_2 of p for which the absorbed power becomes half of that at resonance are called *half power frequencies*. These frequencies in fact determine the range of the frequencies of the applied force to which the system effectively responds. The difference between these two half power frequencies is called the band width of the oscillator.

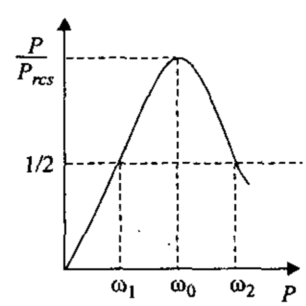


Fig. 20.

For a weakly damped oscillator if ω_1 and ω_2 are designated by $\omega - \Delta\omega$ and $\omega + \Delta\omega$, then applying the relation at half power frequencies

$$\frac{P}{P_{res}} = \frac{1}{2},$$

we may derive that

$$2\Delta\omega = \frac{b}{m} = \frac{1}{\tau} = \frac{\omega_0}{Q}.$$

i.e., the band width of a weakly damped oscillator is equal to its damping constant per unit mass. **The smaller is the damping, the smaller is the band width and therefore sharper is the frequency response.**

Also we see that band width of the oscillator is inversely proportional to the quality factor Q. Thus for larger Q, $2\Delta\omega$ is small and therefore the oscillator has a *sharp frequency response*. Thus the quality factor Q of a oscillator is a measure of the sharpness of the frequency response of the oscillator.

• 4.16. COMPOSITION OF TWO PERPENDICULAR SIMPLE HARMONIC MOTIONS (S.H.M'S) : (LISSAJOUS' FIGURES)

When two perpendicular simple harmonic motions act on a particle simultaneous— the resultant path of the particle is, in general, a closed curve and is called a **Lissajo figure**. The nature of the motion or the curve traced out depends on (i) the amplitud (ii) the ratio of frequencies or periods and (iii) the relative phase of the two compone motions. Let us consider the following cases :

(a) Composition of two perpendicular simple harmonic motions of sam frequencies (*i.e.* periods) : Let us consider a particle acted on by two simple harmor motions of same period, one along x-axis and the other along y-axis. The motions may represented respectively by

$$x = a \sin(\omega t + \phi) \qquad \ldots (1)$$

$$y = b \sin \omega t \qquad \ldots (2)$$

where a and b are respective amplitudes and ϕ is the phase difference between ther $f = \dfrac{\omega}{2\pi}$ is their common frequency.

The equation of resultant path of the particle is obtained by eliminating t betwee equations (1) and (2).

From (2) we have

$$\left.\begin{aligned} \sin \omega t &= \frac{y}{b} \\ \therefore \quad \cos \omega t &= \sqrt{(1 - \sin^2 \omega t)} = \sqrt{\left(1 - \frac{y^2}{b^2}\right)} \end{aligned}\right\} \qquad \ldots (3)$$

From (1), we have

$$x = a\,[\sin \omega t \cos \phi + \cos \omega t \sin \phi]$$

Substituting values of $\sin \omega t$ and $\cos \omega t$ from (3) in above equation, we get

$$\frac{x}{a} = \frac{y}{b} \cos \phi + \sqrt{\left(1 - \frac{y^2}{b^2}\right)} . \sin \phi$$

or
$$\frac{x}{a} - \frac{y}{b} \cos \phi = \sqrt{\left(1 - \frac{y^2}{b^2}\right)} \sin \phi$$

Squaring both sides, we get

$$\left(\frac{x}{a} - \frac{y}{b} \cos \phi\right)^2 = \left(1 - \frac{y^2}{b^2}\right) \sin^2 \phi$$

or
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \cos^2 \phi - \frac{2xy}{ab} \cos \phi = \sin^2 \phi - \frac{y^2}{b^2} \sin^2 \phi$$

or
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} (\cos^2 \phi + \sin^2 \phi) - \frac{2xy}{ab} \cos \phi = \sin^2 \phi$$

or
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \cos \phi = \sin^2 \phi \qquad \ldots (4)$$

This equation represents an **oblique ellipse**, which is the resultant path of the particle.

Particular cases :

(i) When $\phi = 0$ *i.e.*, when the two component vibrations are in same phase, we have

$$\sin \phi = 0 \quad \text{and} \quad \cos \phi = 1$$

then equation (4) becomes

$$\frac{x^2}{a^2} + \frac{b^2}{y^2} - \frac{2xy}{ab} = 0$$

or
$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

i.e.
$$\pm\left(\frac{x}{a} - \frac{y}{b}\right) = 0$$

i.e.
$$\pm\, y = \pm\,\frac{b}{a}\,x \qquad \ldots (5)$$

This represents two coincident straight lines passing through the origin, inclined to x-axis at an angle θ given by

$$\theta = \tan^{-1}\frac{b}{a} \qquad \ldots (6)$$

and lying in first and third quadrants represented by diagonal POQ [fig. 21 (a)]

(ii) When $\phi = \frac{\pi}{4}$, then

$\sin\phi = \cos\phi = \frac{1}{\sqrt{2}}$; so that equation (4) takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2}$$

which represents an oblique ellipse (fig. 21(b)).

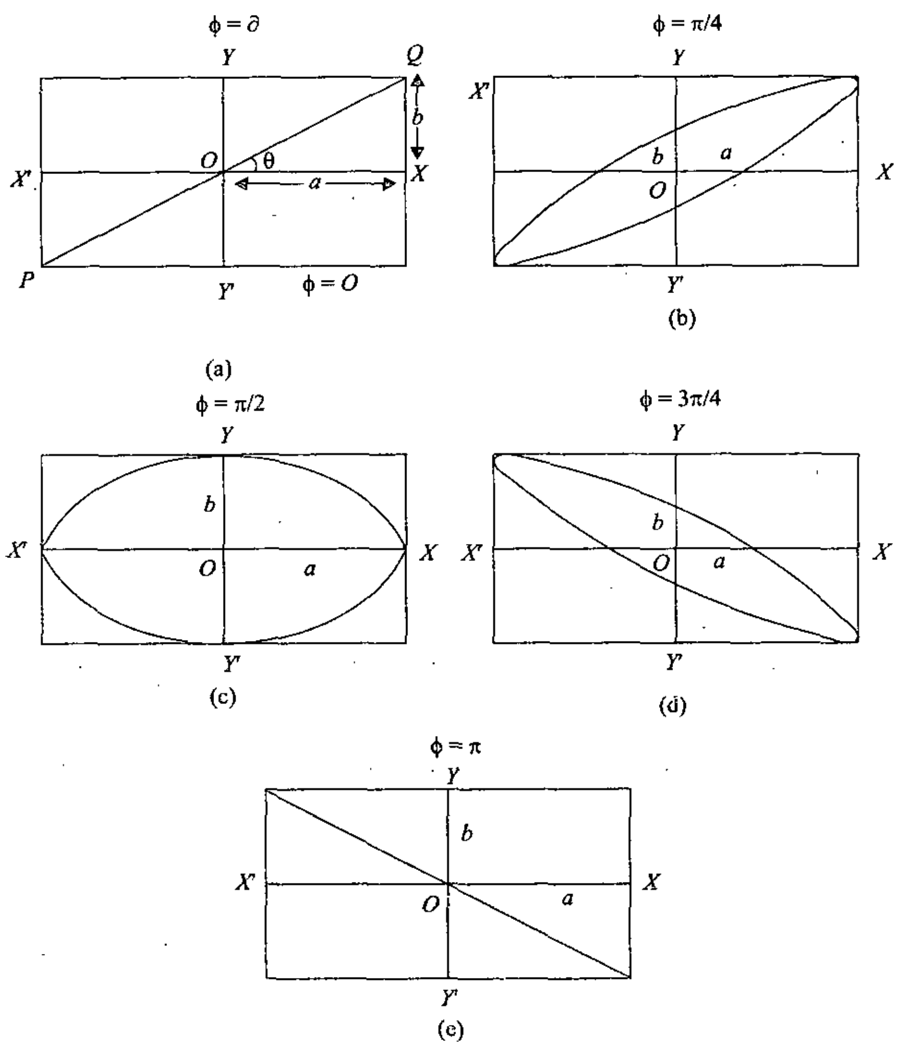


Fig. 21.

(iii) When $\phi = \frac{\pi}{2}$, we have $\sin \phi = 1$ and $\cos \phi = 0$; so that equation (4) takes t form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which represents a symmetrical ellipse whose major and minor axis coincident with t coordinate axis [fig. 21 (c)].

(iv) When $\phi = 3\pi/4$, we have

$$\sin \phi = \frac{1}{\sqrt{2}} \text{ and } \cos \phi = -\frac{1}{\sqrt{2}};$$

so that equation (4) then takes form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

This equation again represents an oblique ellipse [fig. 21(d)] which is resultant path the particle.

(v) When $\phi = \pi$, we have $\sin \phi = 0$ and $\cos \phi = -1$ then equation (4) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$$

or $$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$

or $$\pm\left(\frac{x}{a} + \frac{y}{b}\right) = 0$$

or $$\mp y = \pm \frac{b}{a} x$$

This equation again represents two coincident straight lines lying in second an fourth quadrants passing through the origin and inclined to x-axis at an angle θ give by

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

as shown in fig. 21 (e).

For values of $\phi = \frac{5\pi}{4}, \frac{3\pi}{2}$ and $\frac{7\pi}{4}$ we obtain the curves shown in fig. (d), (c) and (b respectively but they are described in reverse order. At $\phi = 2\pi$, the original pair c coincident straight lines is obtained as shown in fig. (a).

(b) Composition of two simple harmonic motions of periods in the ratio o 1 : 2 : Let the component vibrations be represented by

$$x = a \sin (2\omega t + \phi) \quad \text{... (1}$$
$$y = \sin \omega t \quad \text{... (2}$$

where a and b are respective amplitudes and ϕ is the phase angle by which x-vibration i initially ahead of y-vibration. The period of former is $\frac{2\pi}{2\omega}$ and that of latter is $\frac{2\pi}{\omega}$.

The equation of resultant path is obtained by eliminating t between equations (1 and (2).

From equation (2), we have

$$\left.\begin{aligned} \sin \omega t &= \frac{y}{b} \\ \therefore \cos \omega t = \sqrt{(1 - \sin^2 \omega t)} &= \sqrt{\left(1 - \frac{y^2}{b^2}\right)} \end{aligned}\right\} \quad \text{.. (3}$$

So that equation (1) may be expressed as

$$\frac{x}{a} = \sin 2\omega t \cos \phi + \cos 2\omega t \sin \theta$$

$$= 2 \sin \omega t \cos \omega t \cos \phi + (1 - 2 \sin^2 \omega t) \sin \phi$$

Substituting for sin ωt and cos ωt from (3), we get

$$\frac{x}{a} = \frac{2y}{b} \sqrt{\left(1 - \frac{y^2}{b^2}\right)} \cos \phi + \left(1 - \frac{2y^2}{b^2}\right) \sin \phi$$

or $$\left\{\frac{x}{a} - \left(1 - \frac{2y^2}{b^2}\right) \sin \phi\right\} = \frac{2y}{b} \sqrt{\left(1 - \frac{y^2}{b^2}\right)} \cos \phi$$

Squaring both sides, we get

$$\frac{x^2}{a^2} + \left(1 - \frac{2y^2}{b^2}\right)^2 \sin^2 \phi - 2 \frac{x}{a}\left(1 - \frac{2y^2}{b^2}\right) \sin \phi = \frac{4y^2}{b^2}\left(1 - \frac{y^2}{b^2}\right) \cos^2 \phi$$

or $$\frac{x^2}{a^2} + \sin^2 \phi + \frac{4y^4}{b^4} \sin^2 \phi - \frac{4y^2}{b^2} \sin^2 \phi - \frac{2x}{a} \sin \phi + \frac{4xy^2}{ab^2} \sin \phi$$

$$= \frac{4y^2}{b^2} \cos^2 \phi - \frac{4y^4}{b^4} \cos^2 \phi$$

or $$\frac{x^2}{a^2} + \sin^2 \phi - \frac{2x}{a} \sin \phi + \frac{4y^4}{b^4} (\sin^2 \phi + \cos^2 \phi)$$

$$- \frac{4y^2}{b^2} (\sin^2 \phi + \cos^2 \phi) + \frac{4xy^2}{ab^2} \sin \phi = 0$$

or $$\left(\frac{x}{a} - \sin \phi\right)^2 + \frac{4y^4}{b^4} - \frac{4y^2}{b^2} + \frac{4xy^2}{ab^2} \sin \phi = 0$$

or $$\left(\frac{x}{a} - \sin \phi\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a} \sin \phi - 1\right) = 0 \quad \ldots (4)$$

This is the general equation of a curve having two loops for any values of phase difference and amplitude.

Particular Cases :

(i) When $\phi = 0$ *i.e.*, when two component vibrations are in the same phase then $\sin \phi = 0$ so that equation (4) becomes

$$\frac{x^2}{a^2} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1\right) = 0 \quad \ldots (5)$$

This equation represents a figure of '8' shown in fig. 22 (a) and expresses the resultant path of the composite vibration.

(ii) When $\phi = \frac{\pi}{4}$, then $\sin \phi = \frac{1}{\sqrt{2}}$, so that equation (4) gives

$$\left(\frac{x}{a} - \frac{1}{\sqrt{2}}\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a} \cdot \frac{1}{\sqrt{2}} - 1\right) = 0 \quad \ldots (6)$$

This represents the curve shown in fig. (b).

(iii) When $\phi = \frac{\pi}{2}$, then $\sin \phi = 1$ so that equation (4) becomes

$$\left(\frac{x}{a}-1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2}+\frac{x}{a}-1\right) = 0$$

or
$$\left(\frac{x}{a}-1\right)^2 + \frac{4y^4}{b^4} + \frac{4y^2}{b^2}\left(\frac{x}{a}-1\right) = 0$$

i.e.
$$\left\{\left(\frac{x}{a}-1\right) + \frac{2y^2}{b^2}\right\}^2 = 0 \qquad \ldots (7)$$

This equation represents the two coincident parabolas, the equation of each being

$$\left(\frac{x}{a}-1\right) + \frac{2y^2}{b^2} = 0$$

or
$$\frac{2y^2}{b^2} = -\left(\frac{x}{a}-1\right)$$

or
$$y^2 = -\frac{b^2}{2a}(x-a) \qquad \ldots (8)$$

Obviously the composite vibration takes place along a parabolic path symmetrical about x-axis and shown in fig. 22 (c).

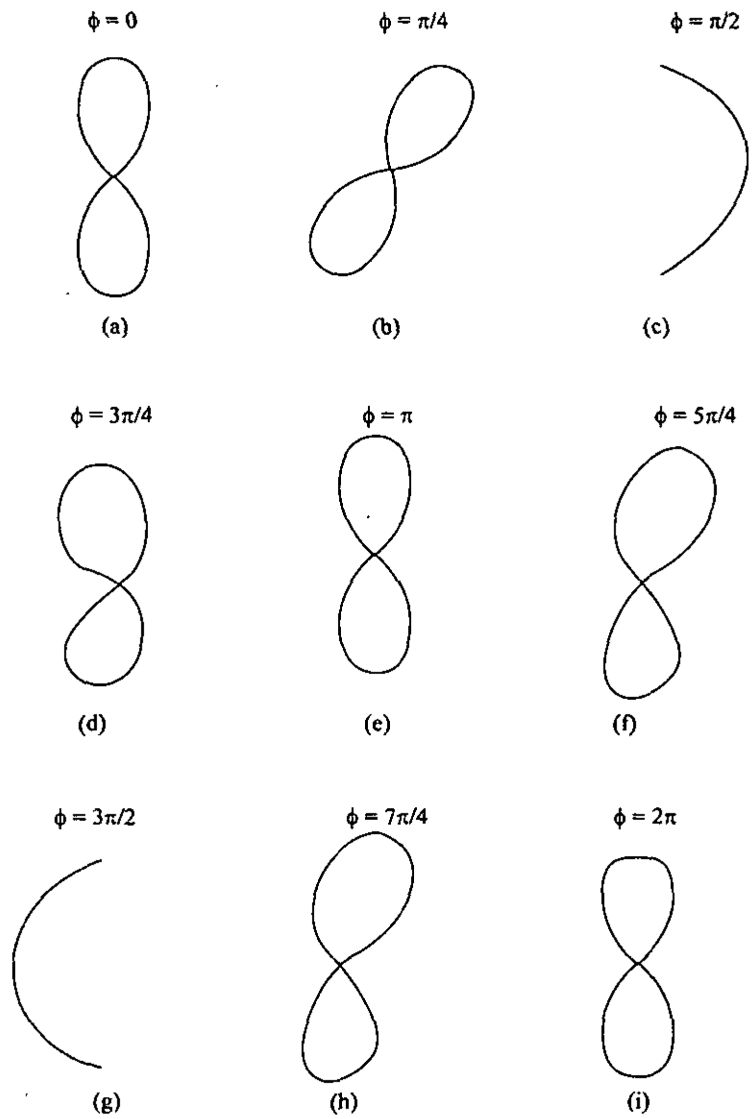


Fig. 22.

(iv) When $\phi = \frac{3\pi}{4}$, then equation (4) reduces to same form as in case (ii); the path of resultant motion is same as shown in fig. 22 (d).

(v) When $\phi = \pi$, then equation (4) again reduces to the one representing figure of 8 shown in fig. 22 (c) when $\phi = \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$ and 2π, the paths of resultant motion will be as shown in fig. 22 f, g, h and i respectively.

Uses of Lissajous Figures : The Lissajous figures have a number of practical applications :

(i) If the frequencies of two vibrating systems are in the whole number ratio as 1 : 1, 2 : 1, 3 : 1, the ratio may be read off at once from the steady figure produced by them.

Let n_1 and n_2 be the frequencies of two forks.

Let the fork of frequency n_1 be vibrating along x-axis and that of frequency n_2 be vibrating parallel to y-axis. Let t be the time during which the complete figure is traced once. During this time t, the number of vibrations parallel to x-axis $= n_1 t$ and that parallel to y-axis $= n_2 t$.

Therefore if rectangular axis are drawn across the fig. 23, the y-axis will be cut by the figure

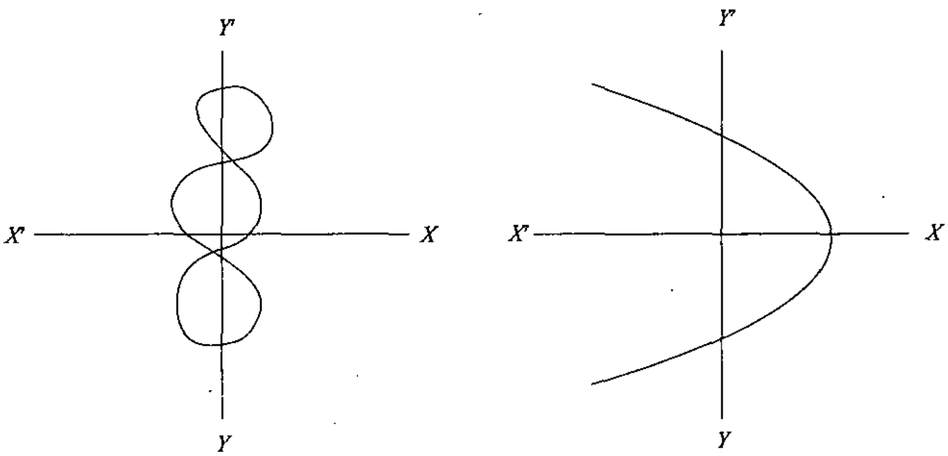


Fig. 23.

$$2n_1 t = p_1 \text{ times (say)} \qquad \dots (1)$$

and the x-axis will be cut

$$2n_2 t = p_2 \text{ times (say)} \qquad \dots (2)$$

From (1) and (2), we get

$$\frac{n_1}{n_2} = \frac{p_1}{p_2}$$

or
$$\frac{\text{frequency of } x\text{-vibration}}{\text{frequency of } y\text{-vibration}} = \frac{\text{number of times } y\text{-axis is cut}}{\text{number of times } x\text{-axis is cut}}$$

By counting the numbers p_1 and p_2, the ratio of frequencies of tuning forks is obtained.

In fig. $\quad p_1 = 6$ and $\quad p_2 = 2$

$$\therefore \quad \frac{n_2}{n_2} = \frac{6}{2} = \frac{3}{1}$$

If the figure obtained is a parabolic, it cuts y-axis twice and x-axis once; therefore ratio of frequencies.

$$\frac{n_1}{n_2} = \frac{p_1}{p_2} = \frac{2}{1}$$

(ii) If the frequencies of two forks are nearly equal (*i.e.* $n_1 = n_2$) : In this when one tuning fork makes one vibration, the other completes a little more or vibration.

Therefore the phase difference between them changes continuously and the fi passes slowly through all possible forms until the initial figure is repeated. Let duration in which the original figure reappears be t. In this duration fork has made more or less vibration than the other *i.e.*,

$$n_1 t - n_2 t = 1$$

$$\therefore \qquad n_1 - n_2 = \frac{1}{t}$$

Thus noting time t, the relative frequencies of forks may be determined.

(iii) Frequency nearly 2 : 1 *i.e.,* frequencies are very nearly an octave apa In this case a cycle of figures is traced which is repeated after a duration in which upper fork makes one vibration more or one vibration less than the double the num made by the lower. Let t be this interval then

$$n_1 t \sim 2n_2 t = 1$$

$$\therefore \qquad n_1 \sim 2n_2 = \frac{1}{t}$$

Thus noting the time t, relative frequencies of forks may be determined.

• SUMMARY

- The motion of a body is aid to be periodic motion if its motion is repeated identic after a fixed interval of time and this fixed interval of time is known as period **motion.**
- When a body moves to and fro repeatedly about its mean position in a defin interval of time then this motion is known as oscillatory or vibratory motion.
- When a particle moves to and fro repeatedly about its mean position under restoring force, which is always directed towards the mean position and whc magnitude at any instant is directly proportional to the displacement of the parti from the mean position at that instant, this motion of the particle is known **simple harmonic motion.**
- Equations of motion for a simple harmonic oscillator are :
  (i) Displaceemnt $(x) = a \sin(\omega t + \phi)$
  (ii) Velocity $(v) = 2\pi\sqrt{\frac{M}{K}}$
  (iii) Time period $(T) = 2\pi\sqrt{\frac{M}{K}}$
  (iv) Frequency $(f) = \frac{1}{2\pi}\sqrt{\frac{K}{M}}$
- Energy of particle executing simple harmonic motion

$$U = \frac{1}{2}Ka^2 \sin^2(\omega t + \phi)$$

$$\text{K.E.} = \frac{1}{2}Ka^2 \cos^2(\omega t + \phi)$$

  Total energy $(E) = 2\pi^2 n^2 ma^2$
- When a heavy particle is suspended by an inextensible, weightless and flexib string from a rigid support, then this system is known as simple pendulum.
- When a rigid body which is capable of oscillating freely in a vertical plane about fixed axis, passing through the body but not through its centre of gravity then th system is called compound pendulum and this fixed point is known as point suspension.

- The time period T of a compound pendulum is given by $T = 2\pi \sqrt{\dfrac{\dfrac{k^2}{l} + l}{g}}$.
- When one end of a very thin and long wire is clamped to a rigid support and the other end is attached to the centre of a heavy disc or sphere, then this arroungement is known as torision pendulum.
- The time period T of a Torsion Pendulum is given by $T = 2\pi\sqrt{\dfrac{I}{c}}$.
- When a particle is acted upon by two mutually perpendicular simple harmonic motions simultaneously then due to the effect of these motions the particle traces a curve. These curves are known as Lissajou's figures.

• TEST YOURSELF

1. Define simple harmonic motion. State its necessary and sufficient conditions.
2. What is the importance of the study of S.H.M. in physics?
3. The motion of a particle is given by
$$x = A \sin \omega t + B \cos \omega t$$
4. Show that the total energy of a particle executing SHM is directly proportional to the square of its amplitude.
5. For a forced harmonic oscillator, establish relation between quality factor and sharpness of resonance.
6. Write differential equation of forced and damped harmonic oscillation and explain each term. Define half life time and relaxation time in case of damped harmonic motion.
7. What do you mean by relaxation time in damped oscillator?
8. What do you mean by the sharpness of resonance? Discuss.
9. Define quality factor in the case of damped harmonic oscillator. Comment on the statement "smaller is the damping, the larger will be the relaxation time and larger the quality factor".
10. What are Lissajous figures and what are their uses?
11. Discuss relaxation time and quality factor.
12. Define quality factor and relaxation time for a damped harmonic oscillations.
13. The necessary and sufficient condition for a particle executing SHM is
    (a) Constant period
    (b) Constant acceleration
    (c) Proportionality between velocity and displacement from mean position.
    (d) Proportionality between restoring force and displacement from mean position.
14. The differential equation of a particle given by
$$\frac{d^2x}{dt^2} + \alpha x = 0$$
    (where x is displacement, t is time and α is a constant) represents
    (a) Oscillatory and SHM   (b) Non-oscillatory and SHM
    (c) Uniform circular motion   (d) Straight line motion
15. The motion of a particle given by
$$y = A \sin(\omega t + \theta), \text{ represents}$$
    (a) Oscillatory but not SHM   (b) Oscillatory and SHM
    (c) Neither oscillatory nor SHM   (d) Uniform circular motion
16. The displacement of the motion of a particle is represented by the equation
$$y = 0.4\left(\cos^2 \frac{\pi t}{2} - \sin^2 \frac{\pi t}{2}\right)$$
    The motion of the particle is
    (a) Oscillatory but not S.H.M   (b) S.H.M. with amplitude 0.4

(c) S.H.M. with amplitude $0.4\sqrt{2}$
(d) S.H.M. with amplitude $\sqrt{0.8}$

17. A particle executes S.H.M. The K.E. of particle is maximum at
(a) Maximum displacement
(b) Minimum displacement
(c) Both minimum and maximum displacement
(d) Nowhere.

18. The potential energy of a particle with displacement x is $U(x)$. The motion is simple harmonic. If K is a positive constant, then
(a) $U = Kx^{1/2}$ (b) $U = -Kx^2$.
(c) $U = 2Kx$ (d) $U = Kx^2$

19. The graph between length and time period of simple pendulum is a :
(a) straight line (b) circle
(c) parabola (d) hyperbola

20. Sharper is the resonance, the bandwidth is
(a) Larger (b) Smaller
(c) Moderate (d) None of these

21. A simple pendulum has a frequency f on earth. If it is taken at moon and vibra its frequency would be (g at moon is $\frac{1}{6}$ th of that earth)
(a) f (b) $6f$
(c) $\sqrt{6}\,f$ (d) $f/\sqrt{6}$

22. The velocity and acceleration of a particle performing simple harmonic motion have a steady phase relationship. The acceleration shows a phase lead over the velocity of :
(a) $+\pi$ (b) $+\pi/2$
(c) $-\pi/2$ (d) $-\pi$

23. The relation between quality factor Q and relaxation time τ of an oscillator is
(a) $Q = \frac{\omega}{\tau}$ (b) $Q = \frac{\tau}{\omega}$
(c) $Q = \omega\tau$ (d) $Q = 1/\omega\tau$

24. A heavily damped oscillator is
(a) oscillatory (b) non-oscillatory
(c) periodic (d) may be oscillatory or non-oscillatory

25. An electrical oscillator having inductance L, resistance R and capacitance C will oscillatory if
(a) $R > 2\sqrt{\frac{L}{C}}$ (b) $R = 2\sqrt{\frac{L}{C}}$
(c) $R < 2\sqrt{\frac{L}{C}}$ (d) $R > \frac{1}{\sqrt{LC}}$

26. In a driven harmonic oscillator the displacement lags behind the driven force by
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) between 0 and $\frac{\pi}{2}$ (d) between 0 and π

ANSWERS

13. (d) **14.** (a) **15.** (b) **16.** (b) **17.** (b) **18.** (d) **19.** (c) **20.** (b) **21.** (d) **22.** (b)
23. (c) **24.** (b) **25.** (c) **26.** (b) **27.** (d).

UNIT 5

WAVE MOTION

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- Wave motion, Mechanics Wave and Mechanical Wave
- Relation between preamery wave speed and wave length
- Particle velocity and wave velocity
- Varicle displacement britch velcoity and calculation
- Calculation of velocity of sound pressure temperature, humidity etc.

5.1. WAVE MOTION

When we throw a stone in the calm water of a pond, we note that a disturbance is produced at the place where stone strikes the water. This disturbance advances in the same form and reaches the edge of pond. In the same manner if we tie one end of a rope to a hook and move the other end up and down, we note that a disturbance is produced in the rope and advances in the same form with a definite speed. *Such a disturbance is called a mechanical wave and the process of its propagation is called wave motion.* ***Thus a mechanical wave is a disturbance produced in a material medium and propagating with a definite speed in the medium without changing its form.***

5.2. PROGRESSIVE WAVE

If we produce disturbance in a medium continuously, the particles of medium vibrate continuously. In this condition the disturbance produced in a medium is called a progressive wave train. If we place corks at different places on the surface of water and produce waves in water by a vibrator then after some time all the corks begi vibrate up and down about their normal positions. The cork nearest to the vibr starts vibrating first, the farthest cork starts vibrating last. If we see all the cork same instant, then all corks appear vibrating in the same form in different states; s one will be in the mean position, the some will be going up, while some going down the language of physics different corks will be in different phases of their vibrati Thus it is clear that ***when a progressive wave train propagates in the medi then at any instant all the particles of the medium vibrate in the same way the phase of vibrations is different for different particles.***

• 5.3. CHARACTERISTICS OF MEDIUM FOR MECHANICAL WAVES

For propagation of mechanical waves the medium must have the follow characteristics :

1. Elasticity : The medium must be elastic, so that the particles may have tende to return to their original positions after disturbance.

2. Inertia : The medium must have inertia, so that the particles of medium r store energy.

3. Low Damping : The medium must have very low damping, so that the wave r propagate in forward direction. Wood has larger friction or damping, due to which wave can not propagate forward. That is way the electric bell is made of iron and no wood.

• 5.4. TYPES OF MECHANICAL WAVES

Mechanical waves are of two types :

1. Transverse Waves : ***If on propagation of mechanical wave in a mediu the particles of medium vibrate perpendicular to the direction of propagati of the wave, then the wave is said to be transverse wave.*** The waves on the surf of water, the waves in a stretched string are the examples of transverse waves. If we one end of a rope by a hook and move the other end up and down, then the transve waves are propagated along its length (Fig. 1).

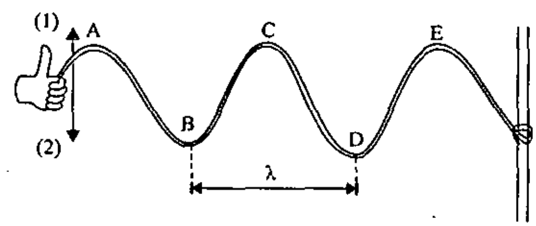


Fig. 1.

In a transverse wave the position of maximum displacement in upward direction called *crest* and the position of maximum displacement along downward direction called **trough.** These states of crest and trough continue to advance in the direction motion of the wave. In fig. A, C, E are crests and B, D. are troughs. The distance betwe two successive crests (or troughs) is called the wavelength (λ) of the transverse wave.

The transverse waves may be produced in a material medium only if it has rigidi As gases and air possess no rigidity, hence the transverse waves can not be produced gases or in air. In liquids the transverse waves can be formed only on the surface but r in interior.

Longitudinal Waves : ***If on propagation of mechanical wave in a mediu the particles of the medium vibrate along the direction of propagation of wa then the wave is said to be longitudinal wave.*** The examples of longitudinal wave (i) Sound waves in air are longitudinal (ii) If one end of a long spiral spring is clamp and the other end is moved forward and backward along the length of spring, the coils spring vibrate along the length of spring and the disturbance appears to be advanci along the length. The waves produced in the spring are longitudinal waves (fig. 2).

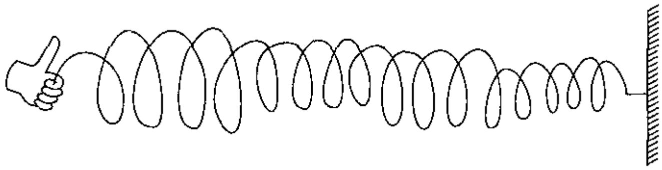


Fig. 2.

The longitudinal waves propagate in the form of compressions and rarefactions. The positions where the turns of the spring appear to be closer are called ***compressions,*** while the position where the turns of the spring appear to be farther are called ***rareactions***. The states of compressions and rarefactions continue to propagate along the direction of motion of the wave. The distance between successive compressions (or rarefactions) is called the wavelength of longitudinal wave. The longitudinal waves can be produced in all types of media : solid, liquid and gases. The waves produced in air (or gases) are always longitudinal. In a liquid the longitudinal waves propagate in its interior but not on its surfaces.

When a longitudinal wave propagates in a medium, then at the positions of compressions, the particles of the medium are closer than their normal state while at the positions of rarefactions, they are farther than their normal states. Therefore, at the positions of compressions the density and pressure are greater than those in the normal state while at the positions of rarefactions, the pressure and density are less than those in normal state. Thus in longitudinal waves the density and pressure vary along the direction of propagation of waves.

Difference between transverse and longitudinal waves

S.N.	Transverse Waves	Longitudinal Waves
1.	In these waves the particles of medium vibrate perpendicular to the direction of propagations of wave.	In these waves the particles of the medium vibrate along the direction of propagation of wave.
2.	These waves propagate in the form of crests and toughts.	These waves propagate in the form of compressions and rarefactions.
3.	These waves can be produced in the interior of solids and on the surface of liquids.	These waves can be produced in all types of media; solid, liquid and gas
4.	In these waves there are no variations of pressure and density along the direction of propagation of wave.	In these waves the pressure and density vary along the direction of propagation of wave, being maximum at the positions of compressions and minimum at the positions of rarefactions.

• 5.5. SOME DEFINITIONS REGARDING WAVES

The waves which transmit energy from one place to another are called ***progressive waves.*** The progressive waves may be longitudinal or transverse. When a longitudinal or transverse wave propagates in a medium, all the particles of the medium vibrate in the same manner but the phase of vibration changes from particle to particle. Some definitions regarding waves are given below.

Amplitude : The maximum displacement of any vibrating particle on either side of its equilibrium position is called the amplitude. It is denoted by 'a'

Time period : The time taken by a particle of medium in completing one vibration is called the time period.

It is denoted by 'T'.

3. Frequency : The number of vibrations completed by a vibrating particle of the medium is called the frequency. It is denoted by 'n'.

From definition it is clear, that $n = \frac{1}{T}$.

4. Wave-speed : The distance transversed by the wave per second is called wave-speed. It is denoted by 'v'.

5. Phase : *The phase of a vibrating particle at any instant represents the position direction of motion of the vibrating particle at that instant* : If the two particles of medium, at any instant, be at equal distance from the equilibrium position in the s direction, then they are said to be in the same phase. In Fig. 3, the particles O, D an are in same phase. Similarly particles A and E are in same phase, and C and G a the same phase.

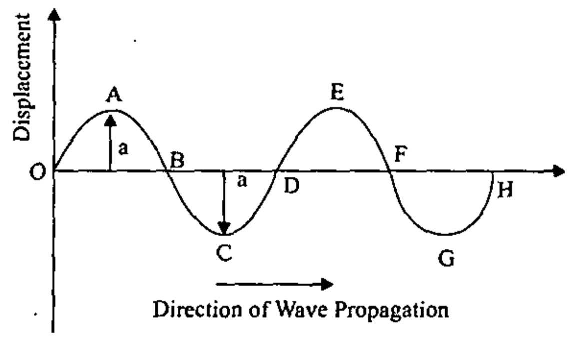


Fig. 3.

The difference between the phases of two particles or the two positions of the sa particles is called the phase difference. In general, the phase difference is represen in terms of an angle. In fig. 3, particle O is taken as reference position of zero phase. phase difference between O and A is $\pi/2$, that between O and B is π, that betw O and C is $\frac{3\pi}{2}$, while the phase difference between O and D is 2π. Obviously if the ph difference between the two vibrating particles be 0, 2π, 4π ...(*i.e.*, even multiple of π), particles are said to be in the same phase. Similarly if the phase difference betw their positions be π, 3π, 5π,... (or odd multiple of π), the particles are said to be opposite phase.

6. Wavelength : ***The distance traversed by wave in one time-period is cal the wavelength.*** Alternatively the distance between two nearest particles of medi vibrating in the same phase is called the wavelength. It is denoted by 'λ'. In a transve wave the distance between two successive crests (or roughs) is equal to waveleng while in a longitudinal wave the distance between two successive positions of maxim compressions (or maximum rarefactions) is equal to the wavelength.

5.6. RELATION BETWEEN FREQUENCY, WAVE-SPEED AND WAVELENGTH

By definition, the distance transversed in one time-period T seconds = waveleng (λ)

$\therefore$ Distance transversed in one second $= \frac{\lambda}{T}$

But distance transversed in one second = wave speed v.

$$\therefore \quad \frac{\lambda}{T} = v \text{ or } \lambda = vT \quad \ldots$$

Substituting value of T from the relation between frequency (n) and time period

i.e., $T = \frac{1}{n}$ in (1), we get $\lambda = \frac{v}{n}$

or
$$v = n\lambda \quad \ldots$$

This relation holds for all type of waves.

• 5.7. EQUATION OF A PLANE-PROGRESSIVE WAVE

If, on propagation of wave in a medium, the particles of medium execute simple harmonic motion, then the wave is said to be a ***simple harmonic progressive wave*** and, moreover, if the amplitude of such a wave remains unchanged, then it is said to be **simple harmonic plane progressive wave.**

Suppose a plane *progressive wave is propagating* in a medium along positive X-axis (*i.e.*, from left to right) (Fig. 4). In fig. (a) the positions of particles *O, A, B, C, D*... are shown. As the wave propagates, all the particles of the medium begin to vibrate to and fro about their mean positions. In fig. (b) the instantaneous positions of these particles are shown. The curve joining these positions represents the progressive wave.

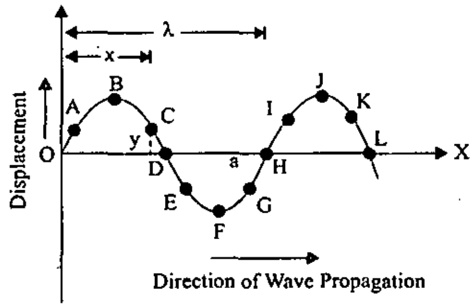


Fig. 4.

Let the particle begin to vibrate from *origin O at time $t = 0$. If y is the displacement* of the particle at time t, then equation of particle executing simple harmonic motion about O is

$$y = A \sin \omega t \qquad \text{... (1)}$$

where A is amplitude and ω is angular velocity. If n is frequency of wave, then $\omega = 2\pi n$. As the advancing wave reaches the other particles $A, B, C,$... (beyond particle O), these particles begin to vibrate. If v is the speed of wave and C is a particle at a distance x from O, then the time taken by wave to reach point C is $\frac{x}{v}$ seconds, therefore the particle will start vibrating $\frac{x}{v}$ seconds after particle O.

Therefore the displacement of particle C at any time t will be the same which wave of particle O at time $\left(t - \frac{x}{v}\right)$. The displacement of particle O at time $\left(t - \frac{x}{v}\right)$ can be obtained by substituting $\left(t - \frac{x}{v}\right)$ in place of t in equation (1). Thus the displacement of particle C at a distance x from origin O at any time t is given by

$$y = A \sin \omega\left(t - \frac{x}{v}\right) \qquad \text{... (2)}$$

If T is time-period and λ the wavelength of light, then $\omega = \frac{2\pi}{T}$

$$\therefore \qquad y = A \sin \frac{2\pi}{T}\left(t - \frac{x}{v}\right)$$

$$= A \sin 2\pi\left(\frac{t}{T} - \frac{x}{vT}\right)$$

But $\qquad vT = \lambda$

$$\therefore \qquad y = A \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \qquad \text{... (3)}$$

This equation may also be expressed as

$$y = A \sin \frac{2\pi}{\lambda}\left(\frac{\lambda}{T} - x\right)$$

But $\lambda / T = n\lambda = v$

$$\therefore \qquad y = A \sin \frac{2\pi}{\lambda}(vt - x) \qquad \text{... (4)}$$

Equation (2) may also be expressed as

$$y = A \sin\left(\omega t - \frac{\omega}{v} x\right)$$

But $$\frac{\omega}{k} = \frac{2\pi v}{2\pi / \lambda} = v\lambda = v,$$

or $$\boxed{u = -v \frac{\partial y}{\partial x}.} \quad \ldots (4)$$

Thus particle velocity = wave velocity × strain

$\partial y / \partial x$ measures the slope of the medium at the point x. **Thus the particle velocity at x is equal to the wave velocity multiplied by the slope of the medium at x.**

• 5.11. DIFFERENTIAL EQUATION OF WAVE MOTION

Differentiating eq. (2), above, we get

$$\frac{\partial^2 y}{\partial t^2} = -a\omega^2 \{\sin(\omega t - kx)\} \quad \ldots (5)$$

Again, differentiating eq. (3), we get

$$\frac{\partial^2 y}{\partial x^2} = -ak^2 \sin\{(\omega t - kx)\} \quad \ldots (6)$$

From eq. (5) and (6), we get

$$\frac{\partial^2 y / \partial t^2}{\partial^2 y / \partial x^2} = \frac{\omega^2}{k^2} = v^2$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$

This is differential equation of wave motion and is also called classical wave equation.

Student Activity

1. Give the differential equation of wave motion.

____

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• 5.12. ENERGY IN A PROGRESSIVE WAVE

When a progressive wave propagates in a medium; it sets new particles into vibrations due to elasticity of medium. If v is the speed of wave, then the length v of medium is set into vibrations per second. As a result the energy is continually supplied to particles of medium by the incoming wave. Thus the progression of a wave is accompanied by the transfer of energy from one part of the medium to the other. As medium has inertia and elasticity, therefore the energy of wave is partly kinetic and partly potential. We propose to find the energy density of the wave (*i.e.*, energy per unit volume).

Suppose a wave is propagating along positive direction of X-axis and the instantaneous displacement of the particle from mean position is y.

The equation of wave is

$$y = A \sin(\omega t - kx) \quad \ldots (1)$$

$$\Rightarrow \quad \frac{\partial y}{\partial t} = A\omega \cos(\omega t - kx) \quad \ldots (2)$$

and $$\frac{\partial^2 y}{\partial t^2} = -A\omega^2 \sin(\omega t - kx)$$
$$= -\omega^2 y \quad \ldots (3)$$

Consider a cylindrical region of medium of unit cross-sectional area. Consider a small thickness dx at distance x from the source.

If ρ is the density of the medium, then the mass of the layer $= \rho\, dx$.

The kinetic energy of layer $\left(=\frac{1}{2}mv^2\right)$,

$$T = \frac{1}{2}(\rho\ dx)\left(\frac{\partial y}{\partial t}\right)^2$$

Using (2), we get

$\therefore$ kinetic energy of layer

$$T = \frac{1}{2}(\rho\ dx)\ \{A\ \omega \cos(\omega t - kx)\}^2$$

$$= \frac{1}{2}\rho\ dx\ A^2\ \omega^2\ \cos^2(\omega t - kx) \qquad \dots (4)$$

The force acting on the layer,

$$F = (\rho dx)\frac{d^2y}{dt^2}$$

$$= \rho\ dx\ (-\omega^2 y) \qquad \text{[using (3)]}$$

$$= -\omega^2\ \rho\ dx\ .\ y$$

The work done in displacing the layer through y from its equilibrium position is stored as the potential energy of the layer.

$\therefore$ Potential energy of layer,

$$U = -\int_0^y F\ dy$$

$$= -\int_0^y (-\omega^2\ \rho\ dx\ y)\ dy$$

$$= \omega^2\ \rho\ dx \int_0^y y\ dy = \omega^2\ \rho\ dx\ \frac{y^2}{2}$$

Substituting value of y from (1), we get

$$U = \frac{1}{2}\omega^2\ \rho\ dx\ A^2\ \sin^2(\omega t - kx)$$

$\therefore$ Total energy of layer $= KE + PE = T + U$

$$= \frac{1}{2}\omega^2\ \rho\ dx\ A^2$$

Volume of layer $= 1 \times dx = dx$

$\therefore$ Energy of wave per unit volume or **energy density**

$$u = \frac{1}{2}\omega^2\ \rho\ A^2$$

If n is the frequency of wave, then $\omega = 2\pi n$

$$\therefore \qquad u_e = \frac{1}{2}(2\pi n)^2\ \rho\ A^2 = 2\pi^2\ n^2\ A^2\ \rho.$$

Clearly the energy density of the wave does not depend on x and t.

Intensity of wave (or flux density energy current.) *The intensity of wave is defined as the total energy of the wave passing per unit cross-sectional area per second.*

The distance traversed by wave in one second $= v$.

$\therefore$ Volume of cylindrical region of unit cross-sectional area $= 1 \times v = v$

$\therefore$ Energy of wave crossing unit cross-sectional area per second

$$= u_e v = 2\pi^2 n^2 A^2 \rho v$$

i.e., **Intensity of wave,**

$$\boxed{I = 2\pi^2\ n^2\ A^2\ \rho v}$$

• 5.13. PRESSURE VARIATION IN LONGITUDINAL WAVES

Waves in a fluid are always longitudinal. Now when longitudinal wave-train (suc as sound) travels through a medium the particles of the medium oscillate to and fr about their equilibrium positions in the direction of propagation of the wave. The phas of oscillation varies from particle to particle. The distance between the particles so alte that at any instant the particles are alternately crowded together and spread out. A the positions of crowding, the pressure and density are maximum. Such positions ar called condensations while at the positions of spreading out the pressure and densit are minimum; such positions are called rarefactions. Therefore pressure varies fror particle to particle in the medium.

Let us consider a cylindrical tube of air of unit cross-section with its length along th axis of X. Let A and B (Fig. 7) be two planes at right angles to the axis of the tube a distance x and $x + \delta x$ respectively from the origin.

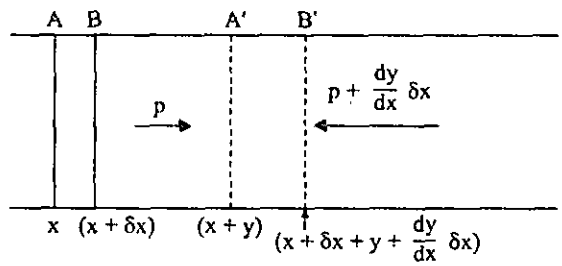

Fig. 7.

Let a plane longitudinal wave pass along the axis of the tube. Suppose at some particular instant when the wave is passing, the displaced positions of the planes are A' and B' respectively. Let y be the displacement of the plane A. Then the rate of change of displacement with distance is (dy/dx), so that the displacement of the plane B is $y + \left(\frac{dy}{dx}\right)\delta x$. Thus the new position of the plane A' and B', are $(x + y)$ and $\left(x + \delta x + y + \frac{dy}{dx}\delta x\right)$ respectively.

The area of cross-section of the tube considered is unity. Therefore, the initial volume of air between the planes A and B

$$= (x + \delta x) - x = \delta x.$$

The final volume of air between the planes (A' and B')

$$= \left(x + \delta x + y + \frac{dy}{dx}\delta x\right) - (x + y) = \left(\delta x + \frac{dy}{dx}\delta x\right)$$

$\therefore$ $$\text{Increase in volume} = \left(\delta x + \frac{dy}{dx}\delta x\right) - \delta x = \frac{dy}{dx}\delta x.$$

$\therefore$ $$\text{Volume strain} = \frac{\text{increase in volume}}{\text{initial volume}}$$

$$= \frac{\frac{dy}{dx}\delta x}{\delta x} = \frac{dy}{dx}. \quad \ldots(1)$$

The wave has thus created a volume strain in the air between the planes. Hence there is a variation of pressure from point to point along the tube.

Now, let p be the pressure-excess above normal on the plane A (now at A'). On B (now at B') the pressure-excess will be $p + \frac{dp}{dx}\delta x$. These two pressure-excess will act on the sample of air between A' and B' in the directions shown in figure. These are equivalent to :

(i) Equal and opposite pressure-excesses p, p, which provide the stress on the sample of air between A' and B' and produce the volume strain (dy/dx) in the sample.

(ii) A resultant pressure-excess $\left(\frac{dp}{dx}\right)\delta x$ in the direction from B' and A' which provides the restoring force causing the sample to come back to its initial position.

Let E be the bulk modulus of air. Then we have, by Hooke's law

$$E = \frac{\text{stress}}{\text{strain}} = -\frac{p}{dy/dx}$$

negative sign showing that when pressure increases, the volume decreases.

$$\therefore \qquad p = -E\frac{dy}{dx}. \qquad \ldots (1)$$

This is the required expression.

$\frac{dy}{dx}$ represents the slope of the displacement curve for the wave. Hence in a longitudinal wave the slope of the displacement curve at any point, $\frac{dy}{dx}$, measures the pressure-change (compression or rarefaction) at that point. When the slope $\frac{dy}{dx}$ is negative, p is positive (compression); where $\frac{dy}{dx}$ is positive, p is negative (rarefaction).

Relation between particle velocity and excess pressure : Relation between particle velocity u and wave velocity v is given by

$$u = -v\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{u}{v} \qquad \ldots (2)$$

Substituting this value in eqn. (2), we get

$$p = \frac{Eu}{v} \qquad \ldots (3)$$

This is the relation between particle velocity (u) and excess pressure (p).

• 5.14. CALCULATION OF VELOCITY OF SOUND IN AIR

First of all the velocity of sound in air was calculated by Newton assuming that the propagation of sound waves in a gas takes place under isothermal conditions *i.e.*, the temperature during compressions and rarefactions in the sound waves remains constant and hence Boyle's law holds good.

Let a given mass of gas have pressure P and volume V. Now if the pressure is changed by dP, the volume will change by amount dV. Therefore according to Boyle's law

$$PV = \text{constant}$$

Differentiating $PdV + VdP = 0$

$$P = -V\frac{dP}{dV}$$

$$= -\frac{dP}{dV/V} = \frac{\text{Stress}}{\text{Volume strain}} = E_T$$

E_T being isothermal elasticity.

$\therefore$ velocity of sound in air, $v = \sqrt{\left(\frac{E_T}{\rho}\right)} = \sqrt{\left(\frac{P}{\rho}\right)}$

Now at 0°C, $\qquad P = 0.76 \times 13.6 \times 10^3 \times 9.8 \text{ N/m}^2 = 1.01 \times 10^5 \text{ N/m}^2$

and $\qquad \rho = 1.29 \text{ kg/m}^3$

$\therefore$ velocity of sound at 0°C,

$$v_0 = \sqrt{\left(\frac{1.01 \times 10^5}{1.29}\right)}$$

$$= 280 \text{ m/s}$$

which is very much less than the experimental value which is 332 metre per second. The error comes out to be 16%. Newton could not explain the reason of this discrepancy.

Laplace's correction :

The discrepancy between the calculated and observed values of the velocity of sound was explained by Laplace in 1816. According to him the compressions and refractions in sound waves occur so rapidly that the gain of heat due to compression and loss in heat due to refraction cannot be communicated to the surroundings by radiation and conduction and hence the temperature of the medium does not remain constant due to which the Boyle's law is not applicable.

Therefore, changes occurring in the medium due to propagation of sound waves are adiabatic and not isothermal.

For an adiabatic change,

$$PV^{\gamma} = \text{constant},$$

where γ is the ratio of specific heat at constant pressure to specific heat at constant volume, i.e., $\gamma = C_P / C_v$.
Differentiating, we get

$$P\gamma V^{\gamma-1}\, dV + V^{\gamma}\, dP = 0$$

or $$\gamma P = -\frac{VdP}{dV} = \frac{dP}{-\frac{dV}{V}} = \frac{\text{increase in pressure}}{\text{decrease in volume per unit volume}}$$

$$= \frac{\text{stress}}{\text{strain}} = E_S$$

where E_S is adiabatic elasticity of gas.

Therefore according to Laplace the velocity of sound waves in gases is given by

$$v = \sqrt{\left(\frac{\gamma P}{\rho}\right)}$$

... (12)

For air, $\gamma = 1.41$

Therefore velocity of sound in air using same data is given by

$$v = 280\sqrt{(1.41)} = 332.5 \text{ metre/second.}$$

which is in close agreement with the experimental value.

Student Activity

1. How will you calculate the velocity of sound in air.

• 5.15. EFFECT OF VARIOUS FACTORS (PRESSURE, TEMPERATURE, HUMIDITY ETC.) ON VELOCITY OF SOUND

1. Effect of Pressure : Speed of sound in air

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

From this formula it appears that the velocity of sound should increase with increase of pressure, but it is not so in practice. The reason for this is as below :

Consider 1.g-mole of air pressure P, volume V and absolute temperature T. According to gas-equations

$$PV = RT \qquad \text{... (2)}$$

where $R =$ universal gas constant. If M is molecular weight and d is density of gas, then

$$\text{Volume } V = \frac{\text{mass}}{\text{density}} = \frac{M}{\rho}$$

From (2) $$P\frac{M}{\rho} = RT \quad \text{or} \quad \frac{P}{\rho} = \frac{RT}{M} \qquad \text{... (3)}$$

Therefore, at a given temperature (T) and for given mass (m) of a gas,

$$\frac{P}{\rho} = \text{constant} \qquad \text{... (4)}$$

That is at a given temperature (T) and given mass (m) of a gas, if P changes, then ρ also changes in such a way that $\frac{P}{\rho}$ remains constant. Thus from formula $v = \sqrt{\frac{\gamma P}{\rho}}$, we conclude that if ***the temperature of the gas remains constant, there is no effect of pressure change on the speed of sound in the gas.***

2. Effect of Temperature : From equation (3)

$$\frac{P}{\rho} = \frac{RT}{M}$$

Substituting this value in equation (1), we get

$$v = \sqrt{\frac{\gamma RT}{M}} \qquad \text{... (5)}$$

i.e., $v \propto \sqrt{T}$

Clearly ***the speed of sound in a gas is directly proportional to the square root of absolute temperature.*** If absolute temperature is increased to four times, the speed of sound will become double of its initial value.

If temperature-change is small, then speed of sound may be calculated as follows :

From equation (5), speed of sound at 0°C or 273 K.

$$v_0 = \sqrt{\left[\frac{\gamma R \,.\, 273}{M}\right]} \qquad \text{... (6)}$$

and at $t°\,C$ or $T = (273 + t)\,K$

$$v_1 = \sqrt{\left[\frac{\gamma R\,(23 + t)}{M}\right]} \qquad \text{... (7)}$$

Dividing equation (7) by (6), we get

$$\frac{v_1}{v_0} = \sqrt{\frac{273 + t}{273}} = \left(1 + \frac{t}{273}\right)^{1/2}$$

$$= 1 + \frac{1}{2} \cdot \frac{t}{273} + \ldots \qquad \text{(using Binomial theorem)}$$

$$= 1 + \frac{t}{546} \qquad \text{(if } t \text{ is very small)}$$

$$\therefore \quad \boxed{v_t = v_0\left(1 + \frac{t}{546}\right)} \qquad \text{... (8)}$$

This equation holds for all gases.

For air : If we take the speed of sound at NTP in air = 332 m/s then

$$v_t = v_0 + \frac{332}{546}t = v_0 + 0.61t \text{ m / s} \qquad \text{... (9)}$$

From this it is clear that ***the speed of sound in air increases by 0.61 m/s per degree celcius rise in temperature.***

3. Effect of Molecular Weight : From equation (5), it is clear that at a constant temperature

$$v \propto \frac{1}{\sqrt{M}}$$

i.e., the ***speed of sound is inversely proportional to square root of molecular weight of the gas.*** The molecular weight of hydrogen is 2 and that of oxygen is 32 *i.e.*, molecular weight of oxygen is 16 times that of hydrogen, so the speed of soun oxygen is one-fourth that of sound in hydrogen.

4. Effect of humidity : The density of humid air (*i.e.*, air containing water-vapo is less than that of dry air, therefore if we take same values of γ and P for humid and airs, then for formula $v = \sqrt{\dfrac{\gamma P}{\rho}}$, it is clear that the speed of sound in humid air wil greater than that in dry air. That is the speed of sound increases with increas humidity in air. That is why the sirens of factories and the whistle of the train are he upto longer distances in rainy season as compared in summer.

5. Effect of motion of air : As the propagation of sound is due to particle medium, therefore the speed of sound is also affected by the motion of particles of ai v is the actual speed of sound and w that of air, then the net speed of sound in direction of motion of air is $(v + w)$ and in opposite direction it will be $(v - w)$.

6. Effect of frequency : There is no effect of frequency of sound waves on tl speed. Sound waves of different frequencies travel in air with the same speed. If speed of sound were dependent of frequency, then we could not have enjoyed orchest

Effect of Amplitudes : There is no effect of amplitude of vibrating particles on speed of sound.

Student Activity

1. What are the major factors affecting velocity of sound.

• 5.16. STATIONARY WAVES

When two progressive waves of same frequency and same amplitude tra in a bounded medium with same speed in opposite directions along a sa straight line, then their superposition results in a new type of wave, wh appears stationary in a medium. This wave is called the stationary wave.

Examples : Like progressive waves, stationary waves may be of both typ transverse and longitudinal.

(i) When a wave is sent along a string, it is reflected from the end of the string. incident and reflected waves superpose to form transverse stationary waves in string. The transverse stationary waves are produced in the strings of sitar, vio gitar, piano etc.

(ii) When a wave is sent in an air column of a pipe, it is reflected from the end of pipe. The incident and reflected waves, superpose to form longitudinal station waves in air-column. Longitudinal stationary waves are formed in the air column flute, bigule, bina, whistle etc.

No transfer of energy takes place in a stationary wave : As the station waves are produced due to superposition of two identical waves (incident and reflect travelling with the same speed in opposite directions. Therefore at any point of medium the amount of energy flowing in one direction due to incident wave is same that due to reflected wave in opposite direction. Consequently there is no net flov energy in a medium by means of stationary wave.

Nodes and Antinodes : In a stationary wave there are some points of medi where the resultant displacement is zero. Such points are called nodes. On the ot hand there are some other points of the medium where the resultant displacem (positive or negative) is maximum. Such points are called antinode.

In a stationary wave the nodes and antinodes occur alternatively at equal distanc

Formation of Stationary Waves : To understand the formation of station waves suppose two identical transverse waves (1) and (2) are travelling in a boun medium in opposite directions with the same speed along the same straight line.

time period of each wave is T and wavelength is λ. Fig. 8 (i) represents the initial state (at $t = 0$) of the two waves. At this instant the crests and troughs of both the waves are exactly at the same positions. The resultant displacement at any point of the medium is the algebraic sum of displacements of two individual waves. In fig. 8. resultant displacement is shown by black continuous curve. At this instant points the resultant displacement at points N_1, N_2, N_3 is zero while the points A_1 and A_2 represent the positions of maximum displacement.

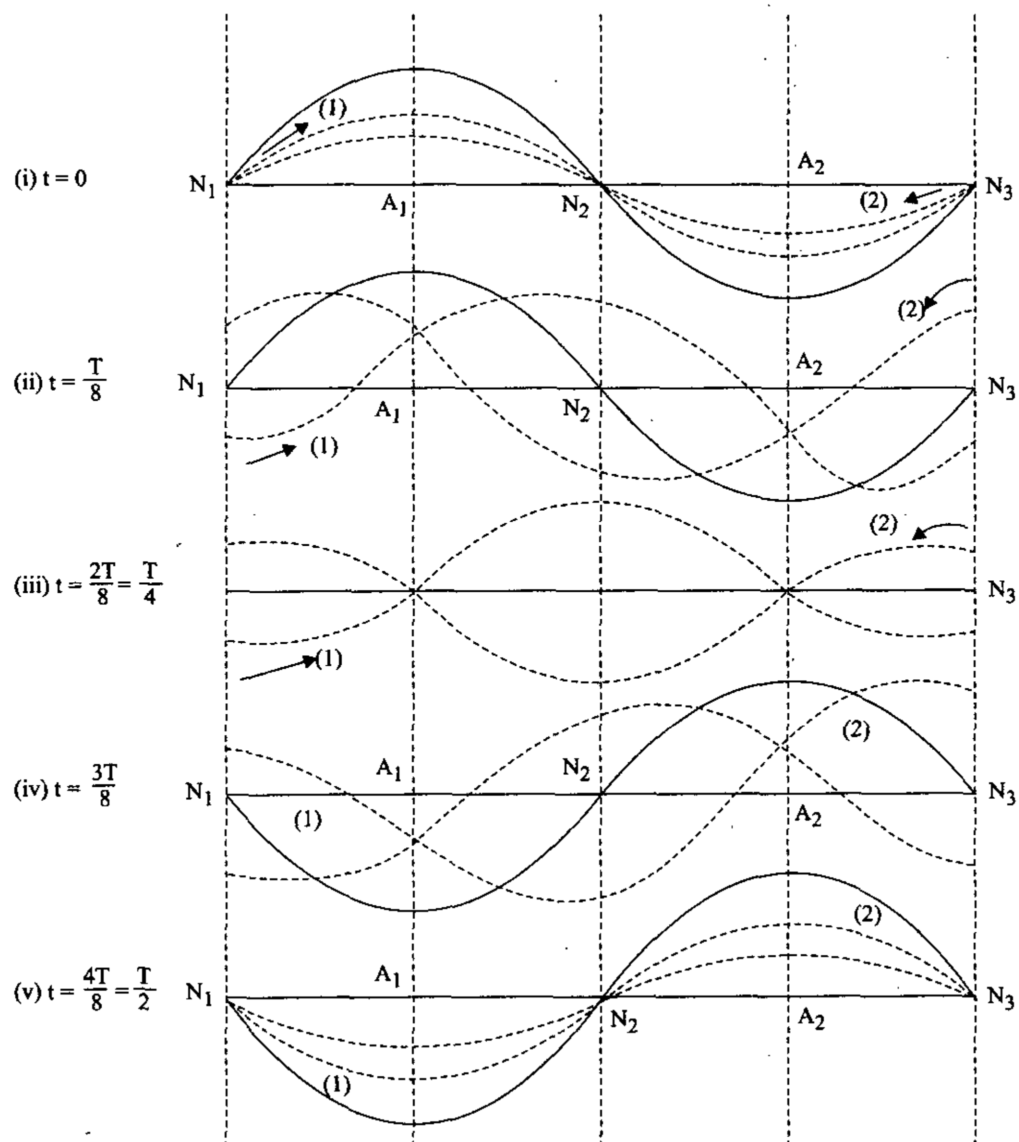

Fig. 8.

Fig. 8 (ii) represents the positions of particles of medium at $t = \frac{T}{8}$. At this instant the wave (1) advances $\frac{\lambda}{8}$ towards right and the wave (2) advances $\frac{\lambda}{8}$ towards left. The resultant displacement is again represented by black continuous curve. At this instant too the displacement of points N_1, N_2, N_3 is zero while that of A_1, A_2 is maximum.

Fig. 8 (iii) represents the state of the particles of medium at $t = \frac{2T}{8} = \frac{T}{4}$. In this state each wave has advanced a distance $\frac{\lambda}{4}$ along its own direction of advance. At this instant, the two waves produce equal and opposite displacements at each point of medium. Hence the net displacement at each point in zero. As a result the net displacement is shown by a **straight line**.

Fig. 8 (iv) represents the state of the particles at $t = \frac{3T}{8}$. In this state each wav
travelled a distance of $\frac{3\lambda}{8}$ along its own direction of advance. Now the displaceme
particles of medium becomes opposite. The displacement of particles between N_1
N_2 is negative, while between N_2 and N_3 is positive. Still the displacements are ze
N_1, N_2, N_3 and maximum at A_1 and A_2.

Fig. 8 (v) represents the state of particles at $t = \frac{4T}{8} = \frac{T}{2}$. In this state each wav
travelled a distance $\frac{\lambda}{2}$ in its own direction of propagation. In this state the resu
wave is reciprocal to that at $t = 0$. Still the displacements of N_1, N_2, N_3 are zero v
those of A_1, A_2 are maximum.

In the same way if the curve are plotted at $t = \frac{5T}{8}, \frac{6T}{8}, \frac{7T}{8}$ and $\frac{8T}{8} = T$, then the
similar to $\frac{3T}{8}, \frac{2T}{8}, \frac{T}{8}$ and $t = 0$ respectively *i.e.*, the displacements of particles
similar in opposite order and $t = T$, the state of $t = 0$ is repeated. This cycle is repe
again and again.

Thus we note that in the resultant wave the positions of particles of minimum
maximum displacements are definite and they do not propagate in the medium. Th
why these waves are called **stationary waves.** It is to be noted that the part
N_1, N_2, N_3 of the medium are always at rest while A_1, A_2 are always in the sta
maximum displacement (positive or negative) relative to other particles. The p
like A_1, A_2 are called **antinodes.** This explains the formation of transverse statio
waves. The longitudinal stationary waves are formed in the same manner. It ma
noted that the **nodes** are the points where the crests or troughs or compression
rarefactions of two opposite waves meet [Fig. (a)]. Therefore

	N	N
(a)	$\rightarrow \bullet \leftarrow$	$\leftarrow \bullet \rightarrow$
(b)	$\rightarrow \bullet \rightarrow$	$\leftarrow \bullet \leftarrow$
	A	A

although the displacement at these points is maximum or zero, but the change
strain and pressure are maximum. Thus ***nodes are the points where displaceme minimum, strain is maximum and (in longitudinal waves) the changes in pressure density are maximum.***

On the other hand the antinodes are the points where the crest of one wave
trough of the other or the compression of one wave and rarefaction of other me
produce maximum displacement [fig. (b)]. Consequently the strain is zero an
changes occur in pressure and density. Thus ***the antinodes are the points w displacement is maximum, strain is minimum and no changes in pressure density (in longitudinal waves) take place.***

Equation of stationary waves : Suppose a progressive wave of amplitud
wavelength λ and period of vibration T is propagating with velocity v in a li
bounded medium. The equation of this wave is

$$y_1 = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

When this wave strikes with the boundary of bounded medium, it is reflected
the reflected wave propagates along negative X-axis. The equation of reflected wave

$$y_2 = \pm\, a \sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

(+) sign is taken if the boundary of the medium be free and (–) sign is taken if
boundary of medium be rigid.

Case (i). Let the boundary of medium be free : Then taking (+) sign in equation (2).

$$y_2 = a \sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right) \qquad ...(3)$$

As both waves propagate along same straight line in opposite directions therefore by Young's principle of superposition.

$$y = y_1 + y_2 = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + a \sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

$$= a\left\{\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + a \sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)\right\}$$

using trigonometric relation

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}, \text{ we get}$$

$$y = a\left[2 \sin \frac{2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)}{2} \cos \frac{2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) - 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)}{2}\right]$$

$$= 2a \sin \frac{2\pi\, t}{T} \cos\left(-\frac{2\pi}{\lambda}\right)$$

As $\cos(-\theta) = \cos\theta$, we get

$$y = 2a \sin \frac{2\pi t}{T} \cos \frac{2\pi x}{\lambda}$$

or

$$y = 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi t}{T} \qquad ...(4)$$

This is the equation of **stationary wave**. Clearly the resultant displacement depends on distance x and time t.

Changes with respect to distance x :

The amplitude of resultant wave at given time t,

$$A = 2a \cos \frac{2\pi x}{\lambda} \qquad ...(5)$$

Intensity of wave,

$$I = A^2 = 4a^2 \cos^2 \frac{2\pi x}{\lambda} \qquad ..(6)$$

Position of Antinodes : For maximum intensity of wave

$$\cos^2 \frac{2\pi x}{\lambda} = 1 \quad \text{or} \quad \cos \frac{2\pi x}{\lambda} = \pm 1$$

This gives $\frac{2\pi x}{\lambda} = 0, \ \pi \ 2\pi, \ 3\pi, \ 4\pi, ...$

$$= r\,\pi\,(r = 0, 1, 2, 3, ...)$$

$$\therefore \quad x = 0, \frac{\lambda}{2}, \ \lambda, \ \frac{3\lambda}{2} \ \ 2\lambda, ... \qquad ...(7)$$

For these values of x, the intensity or amplitude, of resultant wave is maximum. Hence these points represents **antinodes.**

Positions of Nodes. For minimum intensity

$$\cos^2 \frac{2\pi x}{\lambda} = 0 \quad \text{or} \quad \cos \frac{2\pi x}{\lambda} = 0$$

This gives $\frac{2\pi x}{\lambda} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, ...$

$$= (2r - 1)\frac{\pi}{2}; (r = 1, 2, 3, ...)$$

$$x = \frac{\lambda}{2\pi}(2r-1)\frac{\pi}{2} = (2r-1)\frac{\lambda}{4};\ (r = 1, 2, 3, 4, \ldots)$$

i.e.,
$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \frac{7\lambda}{4}, \ldots$$

For these values of x the intensity (or amplitude) of resultant wave is zero. There these points represent **nodes.**

From equations (7) and (8) it is clear that

(i) At free end an antinode is formed

(ii) The nodes and antinodes are formed alternatively at equal distances.

(iii) The distance between any two consecutive nodes (or antinodes) $\frac{\lambda}{2}$.

(iv) The distance between a node and nearest antinode is $\lambda/4$.

Changes with respect to time : From equations (4) it is clear that at give changes with respect to time are according to equation $\sin\frac{2\pi t}{T}$.

For maximum intensity, $\sin\frac{2\pi t}{T} = \pm 1$

or
$$\frac{2\pi t}{t} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots = (2r-1)\frac{\pi}{2}, r = 1, 2, 3, \ldots$$

$\therefore$
$$t = \frac{T}{2\pi}(2r-1)\frac{\pi}{2} = (2r-1)\frac{T}{4}, r = 1, 2, 3, 4, \ldots$$
$$= \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}, \frac{7T}{4}, \ldots$$

At these instants the intensity (or amplitude) of a given particle of the medium maximum. Clearly each particle of medium (excluding nodes) is in a position maximum displacement twice in one period of vibration.

For minimum intensity; $\sin\frac{2\pi t}{T} = 0$

or
$$\frac{2\pi t}{T} = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots = r\pi, (r = 0, 1, 2, 3 \ldots)$$

$\therefore$
$$t = \frac{T}{2\pi}.r\pi = r\frac{T}{2}\ (r = 0, 1, 2, 3, \ldots)$$

or
$$t = 0, \frac{T}{2}, T, \frac{3T}{2}, 2T, \ldots$$

At these instants the resultant intensity or amplitude are minimum (zero). Cle ***each particle of medium passes its equilibrium position twice in each peri***

Case (ii). When boundary of medium is rigid : If incident wave is

$$y_1 = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

If the wave is reflected from rigid boundary, the equation of reflected wave is

$$y_2 = -a\sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

The equation of resultant wave will be

$$y = y_1 + y_2 = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) - a\sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$
$$= 2a\sin\left\{\frac{2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) - 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)}{2}\right\}\cos\left\{\frac{2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)}{2}\right\}$$

$$y = -2a\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi t}{T}$$

In this case, the resultant amplitude at distance x will be,

$$A = -2a \sin \frac{2\pi x}{\lambda}$$

Intensity $$I = A^2 = 4a^2 \sin^2 \frac{2\pi x}{\lambda}$$

For **maximum intensity,**

$$\sin^2 \frac{2\pi x}{\lambda} = 1$$

or $$\sin \frac{2\pi x}{\lambda} = \pm 1$$

or $$\frac{2\pi x}{\lambda} = (2r-1)\frac{\pi}{2}, r = 1, 2, 3 \ldots$$

$\therefore$ $$x = (2r-1)\frac{\lambda}{4}; \quad (r = 1, 2, 3, \ldots)$$

$$= \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$$

These are the positions of **antinodes.**

For minimum intensity,

$$\sin^2 \frac{2\pi x}{\lambda} = 0$$

$$\sin \frac{2\pi x}{\lambda} = 0$$

or $$\frac{2\pi x}{\lambda} = r\pi \qquad \text{where } r = 0, 1, 2, \ldots$$

$\therefore$ $$x = r\frac{\lambda}{2} = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2} \ldots$$

These are the positions of **nodes.**

Clearly (i) A node is always formed at rigid end

(ii) The nodes and antinodes are formed alternatively at equal distances.

(ii) The distance between two successive nodes (or antinodes) is $\frac{\lambda}{2}$.

Characteristics of Stationary Waves

1. The stationary waves are produced always in the bounded medium.

2. These waves do not propagate in the medium but are limited to their places. These waves do not transmit energy in the medium too, therefore they are called ***stationary waves.***

3. Some points in stationary wave do not leave their positions at all, but are always fixed in their positions. These points are called ***nodes.*** These points are located at equal distances. If the stationary waves are longitudinal, then the changes in pressure and density at nodes are maximum relative to other points.

4. There are some points in stationary wave, which have maximum displacement. These are called ***antinodes.*** If the stationary waves are longitudinal then there are no changes in pressure and density at antinodes.

5. In a stationary wave the nodes and antinodes are formed alternatively. There is always a node at fixed end but an antinode at free end. The distance between two consecutive nodes or antinodes is $\frac{\lambda}{2}$, while the distance between a node and nearest antinode is $\lambda/4$.

6. All particles (except nodes) of medium execute simple harmonic motion, wi same period of vibration but different amplitudes. The amplitude goes on increasi from nodes towards antinodes and becomes maximum at antinode.

7. All the particles between two successive nodes vibrate in the same phase. Th pass through their positions of maximum displacements and mean positio simultaneously.

8. At any instant the particles on either side of a node vibrate in opposite phase *i.* the phase difference between them is 180°.

9. All the particles of the medium pass through their mean positions simultaneous twice in each period *i.e.*, the stationary wave takes the form a straight line twice in o period.

10. In the longitudinal waves, the nodes are found alternatively in the states maximum compression and maximum rarefaction twice in each period.

• 5.17. ENERGY OF STATIONARY WAVE

A stationary wave is represented by

$$y = 2a\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi t}{T}$$

Putting $T = \frac{\lambda}{v}$, we get

$$y = 2a\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi\, vt}{\lambda}.$$

$$\therefore \qquad \frac{dy}{dt} = \frac{4\pi va}{\lambda}\cos\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}. \qquad \ldots($$

The kinetic energy per unit volume

$$= \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2$$

$$= \frac{1}{2}\rho\left(\frac{4\pi va}{\lambda}\right)^2\cos^2\frac{2\pi x}{\lambda}\cos^2\frac{2\pi vt}{\lambda}$$

$$= 8\rho\pi^2 n^2 a^2\cos^2\frac{2\pi x}{\lambda}\cos^2\frac{2\pi vt}{\lambda} \qquad \ldots($$

The equation shows that energy depends upon x.

The total energy per unit volume = maximum kinetic energy

$$= 8\rho\pi^2 n^2 a^2\cos^2\frac{2\pi x}{\lambda} \qquad \ldots($$

Integrating between limits x and $x + \lambda$, we get total energy for a complete wave

$$= \int_x^{x+\lambda} 8\rho\pi^2 n^2 a^2\cos^2\frac{2\pi x}{\lambda}\,dx$$

$$= 4\rho\pi^2 n^2 a^2\int_x^{x+\lambda}\left(1 + \cos\frac{4\pi x}{\lambda}\right)dx, \qquad \ldots($$

$$\therefore \qquad E = 4\rho\pi^2 n^2 a^2 \,.\, \lambda$$

Comparing this with that of total energy of a progressive wave *i.e.*, $E = 2\rho\pi^2 n^2 a^2$ we see that the total energy of stationary waves is twice the total energy of progressiv waves. It suggests that stationary waves are formed due to superposition of two simila waves travelling in opposite directions. Total energy is the sum of individual wave since it is a scalar quantity. Also the energy currents due to the two waves are equal an opposite. ***Therefore, net flow of energy in any direction is zero.***

The potential energy = total energy − kinetic energy

$$= 8\rho\pi^2 n^2 a^2\cos^2\frac{2\pi x}{\lambda}\left(1 - \cos^2\frac{2\pi vt}{\lambda}\right)$$

$$\text{P.E.} = 8\rho\pi^2 n^2 a^2 \cos^2 \frac{2\pi x}{\lambda} \sin^2 \frac{2\pi v t}{\lambda}. \qquad ...(5)$$

The kinetic energy per unit wavelength is obtained by integrating expression (2) with respect to x between limits x and $x + \lambda$; we find, kinetic energy per unit wavelength

$$= \int_x^{x+\lambda} 8\rho^2 n^2 a^2 \cos^2 \frac{2\pi x}{\lambda} \cos^2 \frac{2\pi v t}{\lambda}\, dx$$

$$= 4\rho\pi^2 n^2 a^2 \cos^2 \frac{2\pi v t}{\lambda}. \qquad ...(6)$$

Potential energy per unit wavelength is obtained by integrating equations (5) between same limits, which gives

$$\text{P.E. per unit wavelength} = 4\rho\pi^2 n^2 a^2 \sin^2 \frac{2\pi v t}{\lambda}. \qquad ...(7)$$

Thus the distribution of kinetic and potential energy is independent of position, but depends upon time.

• 5.18. PRINCIPLE OF SUPERPOSITION OF WAVES

This principle states that "***when two (or more) waves travel simultaneously in an elastic medium, the resultant displacement of any particle at any instant is equal to the vector sum of displacements of that particle corresponding to the separate waves at that instant.***"

The principle of superposition means that of the many waves, each one moves independently as if the others were not present at all; and that their individual shapes and other characteristics are not changed due to the presence of one another. This is seen in daily life. When we listen to an orchestra, we receive a complex sound due to the superposition of many sound waves of different characteristics produced by different musical instruments. Still we can recognize the separate sounds of different instruments. Our radio antenna is open to the waves of different frequencies transmitted simultaneously by the different radio-stations. But when we adjust the radio to a particular station, we receive its programme as if the other stations were silent.

Derivation of the Principle of Superposition : Suppose y_1 is the displacement of a point x at time t due to a wave described by a function $y_1(x, t)$. It must be a solution of the general differential equation of wave motion, $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$.

That is

$$\frac{\partial^2}{\partial x^2} y_1(x, t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y_1(x, t). \qquad ...(1)$$

Similarly, let y_2 be the displacement of the same point x at the same time t due to another wave described by $y_2(x, t)$. This must also be a solution of the differential equaiton. That is,

$$\frac{\partial^2}{\partial x^2} y_2(x, t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y_2(x, t). \qquad ...(2)$$

Adding eq. (1) and (2), we get

$$\frac{\partial^2}{\partial x^2} [y_1(x, t) + y_2(x, t)] = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} [y_1(x, t) + y_2(x, t)]$$

or

$$\frac{\partial^2}{\partial x^2} y(x, t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y(x, t), \qquad ...(3)$$

where $y(x, t)$ is the sum of the two functions :

$$y(x, t) = y_1(x, t) + y_2(x, t) \qquad ...(4)$$

Eq. (3) shows that the sum of the two wave functions described by eq. (4) also satisfies the general differential equation as such separate wave function does. Hence $y(x, t)$ is a proper function to describe the displacement of the point x at time t. Eq. (4) shows that the displacement y of the point x at time t is equal to the sum of the displacements y_1 and y_2 of the point due to the separate waves at the same time. Thus the principle of superposition is a consequence of the *form* of the differential equation of wave motion.

Limitation : The principle of superposition holds for those waves only whose equations of motion are linear (*i.e.*, obey Hooke's law). Thus it does not hold for shock waves (created by explosions) and water-waves.

Phenomena arising from Superposition of Waves : The superposition of waves gives rise to various phenomena like interference, beats and stationary waves.

Interference : When two harmonic wave-trains of the same frequency and having a constant phase relation travel simultaneously through a medium, the resultant wave intensity at any point is different from the sum of the intensities due to the separate wave-trains. This is the phenomenon the phase difference between the superposing waves and the resulting amplitude (and hence intensity) may be greater or less than that of any single wave. Thus there is a distribution of intensity in space. This distribution is known as "interference pattern."

Beats : The principle of superposition results in another type of interference, which we may call 'interference in time.' It occurs when two wavetrains of slightly different frequencies travel through the same region. The amplitude of the resultant wave at any point is not constant but varies in time. In case of sound waves the varying amplitude gives rise to variations in loudness at the same position which are called 'beats'.

Stationary (or standing) Waves : When two harmonic wave-trains of the same frequency travel along the same line in *opposite directions*, their superposition gives a pattern having alternately nodal points of zero displacement and antinodal points of maximum displacement. This is known as 'stationary-wave pattern' having no energy-transmission.

Velocity of wave-groups : The principle of superposition is used to find the velocity of a group of waves which is very important concept in quantum physics.

• 5.19. WAVE VELOCITY (OR PHASE VELOCITY) AND GROUP VELOCITY

Phase Velocity : When a single monochromatic wave (wave of a single frequency or wavelength) travels through a medium, its velocity of advancement in the medium is called the **'wave velocity'**. For example, a plane harmonic wave travelling along + x direction is given by

$$y = a \sin(\omega t - kx)$$

where a is the amplitude, $\omega (= 2\pi n)$ is the angular frequency and $k (= 2\pi/\lambda)$ is the propagation constant of the wave. By definition, the ratio of angular frequency ω to the propagation constant k is the wave velocity v. That is,*

$$v = \frac{\omega}{k}.$$

$(\omega t - kx)$ is the phase of the wave-motion. Therefore, the planes of constant phase (wavefronts) are defined by

$$\omega t - kx = \text{constant}.$$

Differentiating with respect to time, we get

$$\omega - k\frac{dx}{dt} = 0 \quad \text{or} \quad \frac{dx}{dt} = \frac{\omega}{k}$$

$^*v = n\lambda = \frac{\omega}{2\pi} \times \frac{2\pi}{k} = \frac{\omega}{k}.$

which is the wave velocity v. Thus *the wave velocity is the velocity with which planes of constant phase advance through the medium.* Hence the wave velocity is also called the **'phase velocity'.**

Group Velocity : In practice, we come across pulses rather than monochromatic waves. A pulse consists of a number of waves differing slightly from one another in frequency. A superposition of these waves is called a 'wave packet' or a 'wave group'. When such a group travels in a dispersive medium, the phase velocities of its *different* components are different. The observed velocity is, however, the velocity with which the maximum amplitude of the group advances in the medium. This is called the **'group velocity'**. Thus the group velocity is the velocity with which the energy in the group is transmitted. The individual waves travel "inside" the group with their phase velocities.

In fig. 9 (a) are shown two plane harmonic waves of equal amplitudes but slightly different frequencies travelling from left to right. The dotted curve represents the waves of lower frequency and is travelling faster. At a certain instant the two waves are in phase at the point P. Therefore, at this instant, the maximum amplitude of the group formed by them also lies at P (Fig. 9 b). At a later instant the maximum will be built up a little to the left of P. That is, the maximum will move to the left with time relative to the waves. As a result, the group velocity will be lower than the wave velocities.

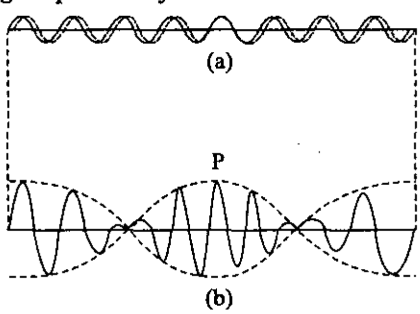

Fig. 9.

(b) Superposition of a number of Harmonic waves : A simple harmonic wave of frequency $\nu = \frac{\omega}{2\pi}$ and wavelength $\lambda = \frac{2\pi}{k}$ is represented by the equation

$$y = a \sin (\omega t - kx)$$

or in complex form

$$y = ae^{(i\,kx - \omega t)} \qquad \text{...(1)}$$

The velocity of a single wave is called the phase velocity given by

$$v = \frac{\omega}{k} \qquad \text{...(2)}$$

If N such waves superimpose, they produce a wave packet, which represents a constructive interference in small region of space. The maximum amplitude of this packet advances with a velocity called the group velocity.

A group of waves or wave packet is represented by

$$y(x, t) = \sum_{K=-\infty}^{+\infty} a(k_r)\, e^{\,i\,(k_r x - \omega_r t)}\, \delta k \qquad \text{...(3)}$$

where (k_r) is propagation constant and ω_r is angular frequency of rth wave. In a dispersive medium, the connection between ω and k is not linear (*e.g.*, waves in water or light waves in glass). In such a case the velocities of individual components of y depend on the wavelength. Due to this relative phase change of the component-waves in time, during propagation, the shape of the wave packet changes. If the range of values of k is limited, it is possible to assign and average velocity to the wave-packet. On this assumption the amplitude $a(k)$ is negligible except when k lies in a small interval ΔK, then the wave-packet can be expressed as

$$y(x, t) = \sum_{\Delta K} a(k)\, e^{\,i(kx - \omega t)}\, \delta k \qquad \ldots(4)$$

where the summation extends only for a significant interval.

If values of k are continuous within a small range ΔK, then equation (4) may be expressed as

$$y(x, t) = \int_{\Delta K} a(k)\, e^{\,i(kx - \omega t)}\, dx \qquad \ldots(5)$$

If we assume that ω varies slowly with k, a good approximation is

$$\omega \approx \omega_0 + \left(\frac{d\omega}{dk}\right)_{k = k_0} (k - k_0) \qquad \ldots(6)$$

in which $\omega_0 = \omega(k_0)$ and k_0 is some fixed value of k within ΔK. To this approximation equation (6) may be expressed as

$$y(x, t)' \; e^{i(k_0 x - \omega_0 t)} \int_{\Delta K} a(k)\, e^{\,i\left\{x - \left(\frac{d\omega}{dk}\right) t\,(k - k_0)\right\}}\, dk \qquad \ldots(7)$$

This equation represents a wave of wavelength $\frac{2\pi}{k_0}$ and frequency $\frac{\omega_0}{2\pi}$, which is modulated by the integral appearing as a factor. The factor depends on x and t only in the combination $x - \left(\frac{d\omega}{dk}\right) t$ and thus represents a wavepacket which moves with the group velocity

$$v_g = \frac{d\omega}{dk} \qquad \ldots(8)$$

(c) Relation between Group Velocity and Phase Velocity : Since $\omega = kv$, where v is wave (phase) velocity, the group velocity is given by

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(kv)$$

$$= v + k\frac{dv}{dk}.$$

Now, $k = \frac{2\pi}{\lambda}$, where λ is wavelength. Therefore,

$$v_g = v + \frac{2\pi}{\lambda}\,\frac{dv}{d\left(\frac{2\pi}{\lambda}\right)}$$

$$= v + \frac{1}{\lambda}\,\frac{dv}{d\left(\frac{1}{\lambda}\right)}$$

But $d\left(\frac{1}{\lambda}\right) = -\frac{1}{\lambda^2}\, d\lambda$. Therefore

$$v_g = v - \lambda\frac{dv}{d\lambda}.$$

This is the relation between group velocity v_g and wave velocity v in a dispersive medium.*

Normal and Anomalous Dispersion : Usually $dv/d\lambda$ is positive, so that the group velocity v_g is smaller than the wave velocity v. This is called 'normal dispersion'. But 'anomalous dispersion' can arise where $\frac{dv}{d\lambda}$ is negative, so that v_g is greater than v. An electrical conductor is anomalously dispersive to electromagnetic wave.

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*A dispersive medium is one in which the wave velocity is frequency-dependent (ω/k not constant)

Non-dispersive Medium : In a non-dispersive medium ($v = \omega / k$ = constant) we have $\frac{dv}{d\lambda} = 0$, so that

$$v_g = v,$$

that is, the velocities are identical. For light waves in free space the group velocity is same as the wave velocity.

Sound waves in gases are also nondispersive. This is a fortunate circumstance. If sounds of different frequencies travelled at different speeds through the air it would result in chaos and aural anguish. We could never listen to an orchestra.

• SUMMARY

1. A mechanical wave is a disturbance produced in material medium.
2. When the progressive wave propagates in the medium then at any instant all the particles of the medium vibrates in the medium in the same way.
3. For propagation of mechanical waves medium should have elasticy, inertia and low damping.
4. Mechanical waves are of two types
   (i) transverse wave
   (ii) longitudinal wave
5. Relation between frequency, wave speed and wavelength is $v = n\lambda$.
6. Relation between particle velocity and wave velocity
$$u = -v\frac{\partial y}{\partial x}.$$
7. Differential equation of wave motion
$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$
8. Intensity of wave
$$I = 2\pi^2 n^2 A^2 \rho v$$
9. Speed of sound in air is $v = \frac{\sqrt{\gamma P}}{\rho}$
10. There is no transfer of energy takes place in a stationary wave.
11. Relation between group velocity and phase velocity.
$$v_g = v - \lambda\frac{dv}{d\lambda}$$

• TEST YOURSELF

1. What is wave motion?
2. What is a mechanical wave?
3. What are the essential characteristics of medium for wave propagation?
4. Name the type of mechanical waves.
5. Define a transverse wave.
6. Define a longitudinal wave.
7. Write a relation between velocity, frequency and wavelength for a progressive wave.
8. Write the relation between phase difference and path difference.
9. Write the two forms of a plane progressive wave.
10. Write the differential equation of wave-motion.
11. Write an expression for speed of longitudinal waves in gases.

12. Write Newton's formula for the speed of sound in gases.
13. Write Laplace's formula for the speed of sound in gases.
14. The sound waves produced in a gas are always
    (a) transverse
    (b) longitudinal
    (c) stationary
    (d) electromagnetic
15. The equation of a transverse wave is given by $y = 20 \sin \pi (0.02\,x - 2\,t)$ where x and y are in cm and t in seconds.
    The wavelength of wave (in cm) will be
    (a) 50
    (b) 100
    (c) 200
    (d) 5
16. A transverse wave is described by the equation $y = y_0\ 2\pi\left(ft - \dfrac{x}{\lambda}\right)$
    The maximum particle velocity is four times the wave velocity. If
    (a) $\lambda = \pi y_0 / 4$
    (b) $\lambda = \pi y_0 / 2$
    (c) $\lambda = \pi y_0$
    (d) $\lambda = 2\pi y_0$
17. The ratio of intensities of two waves of same frequency is 1 : 16. The ratio of their amplitudes will be
    (a) 1 : 16
    (b) 1 : 4
    (c) 4 : 1
    (d) 1 : 8
18. Velocity of sound waves in air is 330 m/s. For a particular sound in air, a path difference of 40 cm is equivalent to a phase difference of $1.6\,\pi$. The frequency of this wave is
    (a) 150 Hz
    (b) 165 Hz
    (c) 330 Hz
    (d) 660 Hz
19. When a sound wave of frequency 300 Hz passes through a medium the maximum displacement of a particle of the medium is 0.1 cm. The maximum velocity of the particle is equal to
    (a) $60\,\pi$ cm/s
    (b) $30\,\pi$ m/s
    (c) 30 cm/s
    (d) 60 cm/s

20. When a wave travels in a medium, the particle displacements are given by

$$y(x, t) = 0.03 \sin \pi (2t - 0.01x)$$

where x and y are in metres and t in seconds, the wavelength of the wave is

(a) 10 m

(b) 20 m

(c) 100 m

(d) 200 m

21. The distance between any two successive compression zones, in a medium in which a sound wave of 50 cm wavelength is travelling, is

(a) 12.5 cm

(b) 25 cm

(c) 37.5 cm

(d) 200 cm

22. The classical wave equation is

(a) $y = a \sin \omega t$

(b) $y = a \sin (\omega t - kx)$

(c) $y = a \sin \omega t \cos kx$

(d) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$

23. Ultrasonic, infrasonic and audio waves travel through a medium with speeds v_u, v_i and v_a respectively then,

(a) $v_u = v_i = v_a$

(b) $v_u > v_a > v_i$

(c) $v_u < v_a < v_i$

(d) $v_a \le v_u$ and $v_u' \; v_i$

24. The speed of sound in air is 332 m/s at NTP. The speed of sound in hydrogen at NTP will be

(a) 5312 m/s

(b) 2546 m/s

(c) 1328 m/s

(d) 664 m/s

25. Differential equation of wave motion is

(a) $\frac{d^2 y}{dx^2} = v^2 \frac{d^2 y}{dt^2}$

(b) $\frac{dy}{dx} = v^2 \frac{d^2 y}{dt^2}$

(c) $\frac{d^2 y}{dt^2} = v^2 \frac{d^2 y}{dx^2}$

(d) $\frac{d^2 y}{dx^2} = - v^2 \frac{d^2 y}{dt^2}$

26. The speed of longitudinal wave travelling in a gas of pressure P and density d is

(a) $v = \sqrt{\frac{P}{d}}$ (b) $v = \sqrt{\frac{\gamma P}{d}}$

(c) $v = \sqrt{\gamma P d}$ (d) $v = \sqrt{\frac{d}{p}}$

ANSWERS

14. (b) **15.** (b) **16.** (a) **17.** (b) **18.** (d) **19.** (a) **20.** (d) **21.** (d) **22.** (d) **23.** (a) **24.** (c
25. (c) **26.** (b)

B.Sc. PCM-101

सर्वे भवन्तु सुखिनः सर्वे सन्तु निरामयाः ।
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