ANALYSIS

SC–114

Self Learning Material

Directorate of Distance Education

SWAMI VIVEKANAND SUBHARTI UNIVERSITY
MEERUT–250 005
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Study Material Assessment Committee, as per the SVSU ordinance No. VI (2)

Published by : Pragati Prakashan, 240 W.K. Road, Meerut
Tel. : 2643636, 2640642, 6544643, E-mail : pragatiprakashan@gmail.com

Typeset by : Pragati Laser Type Setters (Pvt.) Ltd., Meerut   Printed at : Arihant Electric Press, Meerut

EDITION : 2019

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B.Sc. (Part-II)
ANALYSIS (SC-114)

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CHAPTER II

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UNIT 1

CONTINUITY

STRUCTURE

- Continuity
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- Four Functional Limits
- Some Important Theorems
- Uniform Continuity
- Some Important Theorems
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LEARNING OBJECTIVES

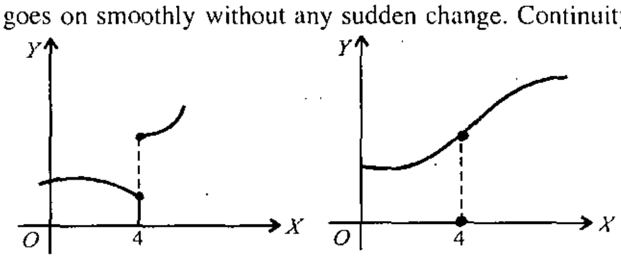
After going through this unit you will learn :

- What are continuity, discontinuity and uniform continuity ?
- How to check whether a function is continuous or not.

• 1.1. CONTINUITY

A continuous process is one that goes on smoothly without any sudden change. Continuity of a function can also be interpreted in a similar way. For better understanding, consider the following figures.

The graph of the function in fig. 1(a) has sudden cut at the point $x = 4$ whereas the graph of the function in fig. 1(b) proceeds smoothly. We say that the function of fig. 1(b) is continuous, while function of fig. (a) is not continuous.



(a) (b)

Fig. 1

Also, while defining $\lim_{x \to a} f(x)$, the function f may or may not be defined at $x = a$. Even if f is defined at $x = a$, $\lim_{x \to a} f(x)$ may or may not be equal to the value of the function at $x = a$. If $\lim_{x \to a} f(x) = f(a)$, then we say that f is continuous at $x = a$.

Systematic study of the continuous nature of various phenomena began at the close of the 17^{th} century. The french mathematician **G. W. Leibnitz** (17^{th} cent.) was a pioneer who first specified the two concepts underlying various physical phenomena of the universe. The first of these is calculus, which is the natural language of the continuity, and the second is combintional analysis which deals with the discrete or the discontinuous. The study of continuity of functions is the most important aspects of analysis and is based on the notion of limit.

Continuous Functions.

Continuity at a point. A function f, defined on some nbd of a point a, is said to be continuous at a if and only if any one of the following condition is saitsfied

(i) $\lim_{x \to a} f(x) = f(a)$

(ii) $f(a - 0) = f(a + 0) = f(a)$

(iii) Cauchy Definition of continuity for $\varepsilon > 0, \exists, \delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon, \text{ whenever } 0 < |x - a| < \delta.$$

The above all conditions are equivalent to each other, and being, simple, are of common use.

REMARKS

- Checking the continuity of a function from the smoothness of its graph is not a complete method. Consider the graph of the function $f(x) = x \sin \frac{1}{x}$, then we observe that it has no breaks in the nbd of $x = 0$. But this function is not continuous. Observe that the graph oscillate widly near zero.

Some More Definitions of Continuity.

(i) If $\lim_{x \to a+0} f(x) = f(a)$, then we say that f is continuous to the right of a (or right continuous at a).

(ii) If $\lim_{x \to a-0} f(x) = f(a)$, then we say that f is continuous to the left of a (or left continuous at a).

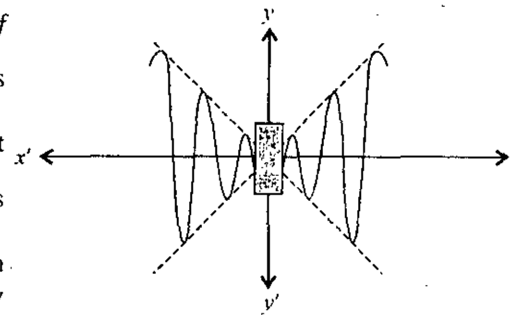


Fig. 2

(iii) A function f is said to be continuous in an open interval $]a, b[$ if it is continuous at every point of $]a, b[$.

(iv) A function f is said to be continuous in a closed interval $[a, b]$ if it is

(1) right continous at a

(2) continuous at every point of $]a, b[$

(3) left continuous at b.

(v) A funcntion f is said to be continuous in a semi-closed interval $[a, b[$ if it is

(1) right continuous at a

(2) continuous at every point of $]a, b[$

(vi) A function f is continuous in a semi-closed interval $]a, b]$ if it is

(1) continuous at every point of $]a, b[$

(2) left continuous at b.

(vii) A function f is said to be continuous at $a \in I$, iff $\lim_{x \to a} f(x)$ exists, is finite and is equal to $f(a)$, otherwise the function is said to discontinuous at $x = a$.

(viii) **Heine's definition of continuity :** The necessary and sufficient condition for a function f defined on an interval $I \subset \mathbf{R}$ to be continuous at a point of interval I is that for each sequence $< a_n >$ in I converges to a, the sequence $< f(a_n) >$ converges to $f(a)$.

Here, we have that f is said to be continuous iff

$$\lim_{n \to \infty} f(a_n) = f(a).$$

Graphical meaning of continuity of a function. Continuity of a function f at a point a graphically means that there is **no break** in the graph of the curve $y = f(x)$ at $x = a$ and given however small $\varepsilon > 0, \exists, \delta > 0$ such that the graph of $y = f(x)$ from $x = a - \delta$ to $a + \delta$ lies between the lines $y = f(a) - \varepsilon$ and $y = f(a) + \varepsilon$.

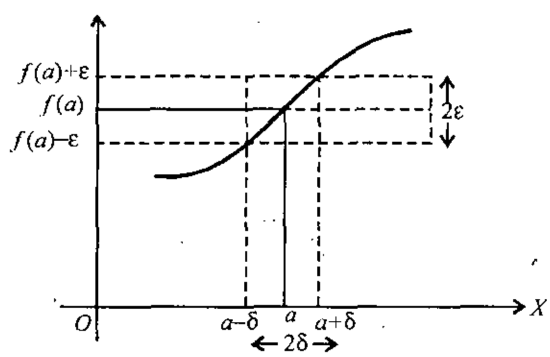


Fig. 3

Examples on Continuous Function.

(i) Every constant function $f : R \to C$ is continuous on R.

For $\varepsilon > 0, a \in \mathbf{R}, |x - a| < \varepsilon \Rightarrow |C - C| = 0 < \varepsilon$

(ii) The identity function $f : x \to x \in R$ is continuous on R.

For $\varepsilon > 0, \delta = \varepsilon$ and $|x - a| < \varepsilon \Rightarrow |x - a| < \varepsilon \ \forall \ a \in \mathbf{R}$.

(iii) The function $f : x \to x^n, n \in N$ is continuous on R

$\because$ For any $a \in \mathbf{R}$, $\lim_{x \to a} f(x) = a^n = f(a)$.

(iv) The polynomial function $f(x) = a_0 + a_1x + ... + a_nx^n$ is continuous on R.

$\because$ For any $a \in R$, $\lim_{x \to a} f(x) = f(a)$.

• 1.2. DISCONTINUITY

(1) A function f which is not continuous at a point a is said to be discontinuous at the point 'a', where 'a' is called the **point of discontinuity** of f or f is said to have a discontinuity at a.

(2) A function which is discontinuous even at a single point of an interval, is said to be discontinuous in that interval.

(3) A function f can be discontinuous at a point $x = a$, because of any one of the following reasons :

(i) $f(x)$ is not defined at $x = a$.

(ii) $\lim_{x \to a} f(x)$ does not exists.

(iii) $\lim_{x \to a} f(x)$ *and* $f(a)$ both exist but are not equal.

Types of Discontinuity.

(i) Removable discontinuity. *A function f is said to have a removable discontinuity at a point a if* $\lim_{x \to a} f(x)$ *exists but is not equal to the function value at a i.e.,*

$$f(a - 0) = f(a + 0) \neq f(a).$$

REMARK

➤ In the above case, a function f can be made continuous by assigning some suitable value to a, such that

$$\lim_{x \to a} f(x) = f(a).$$

For example. Suppose f is a function defined on]0, 1[as follows :

$$f(x) = \begin{cases} 2, & 0 < x < 1,\ x \neq \frac{1}{2} \\ 1, & x = \frac{1}{2} \end{cases}$$

Then, it is clear that f is continuous in]0, 1[except at the point $x = \frac{1}{2}$. At the point $x = \frac{1}{2}$, we have

$$f\left(\frac{1}{2} - 0\right) = f\left(\frac{1}{2} + 0\right) = 2$$

but
$$f\left(\frac{1}{2}\right) = 1$$

$\Rightarrow f$ has a removable discontinuity at $x = \frac{1}{2}$.

The discontinuity at $x = \frac{1}{2}$ may be removed by choosing $f\left(\frac{1}{2}\right) = 2$.

(ii) Discontinuity of first kind. *A function f is said to have a discontinuity of first kind at a point a, if both the limits* $f(a - 0)$ *and* $f(a + 0)$ *exist but are not equal. The point a is said to be a point of discontinuity from the left or from right according as*

$$f(a - 0) \neq f(a) = f(a + 0)$$

or
$$f(a - 0) = f(a) \neq f(a + 0).$$

For example. Consider a function f defined on]0, 1[as follows

$$f(x) = \begin{cases} 1/2, & 0 < x < 1/2 \\ 0, & x = \frac{1}{2} \\ -1/2, & 1/2 < x < 1 \end{cases}$$

Obviously, f is continuous over the open interval]0, 1/2[and]1/2, 1[

At the point $x = \frac{1}{2}$

$$f\left(\frac{1}{2}-0\right)=\lim_{h\to 0} f\left(\frac{1}{2}-h\right)=\frac{1}{2}\neq 0=f(1/2)$$

$$f\left(\frac{1}{2}+0\right)=\lim_{h\to 0} f\left(\frac{1}{2}+h\right)=-\frac{1}{2}\neq 0=f\left(\frac{1}{2}\right)$$

$\Rightarrow$ $$f\left(\frac{1}{2}-0\right)\neq f\left(\frac{1}{2}+0\right)$$

$\Rightarrow$ f has a discontinuity of the first kind at $x=\frac{1}{2}$.

(iii) Discontinuity of second kind. *A function f is said to have a discontinuity of second kind at a point a if none of the limit f(a − 0) and f(a + 0) exist at a. The point a is said to be a point of discontinuity of second kind from the left or from the right according as f(a − 0) or f(a + 0) does not exists.*

For example. Consider the function $f(x)=\cos\left(\frac{\pi}{x}\right)$ defined an $]-\infty,\infty[$. The graph of the function as given below :

Obviously, at the point $x = 0$, both the limits $\lim_{x\to 0-}\cos\left(\frac{\pi}{x}\right)$ and $\lim_{x\to 0+}\cos\left(\frac{\pi}{x}\right)$ do not exist.

Hence, $x = 0$ is a point of discontinuity of the second kind.

(iv) Mixed discontinuity. *A function f is said to have a mixed discontinuity at a point a if f has a discontinuity of second kind on one side of a and on the other side a discontinuity of first kind or may be continuous.*

For example. For the function $f(x)=e^{1/x}\sin\frac{1}{x}$

$\lim_{x\to 0-} f(x)=0$. $\lim_{x\to 0+} f(x)$ does not exists and the function is not defined at $x = 0$. Therefore, the function has a discontinuity of first kind from the left and a discontinuity of the second kind from the right at $x = 0$. Thus, the function has a mixed discontinuity at $x = 0$.

(v) Infinite discontinuity. *A function f is said to have an infinite discontinuity at $x = a$ if $f(a + 0)$ or $f(a - 0)$ is $+\infty$ or $-\infty$. If f has a discontinuity at a and is unbounded in every nbd of a, then f is said to have an infinite discontinuity at a.*

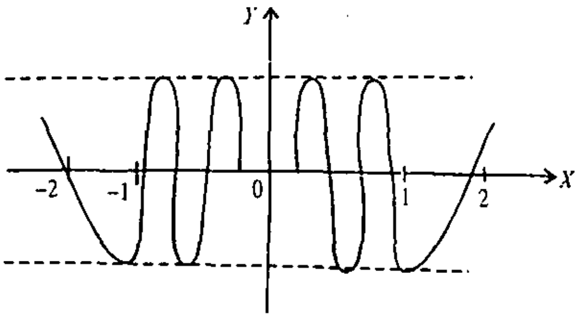


Fig. 4

For Example. Suppose $f(x)=\frac{1}{x}$ in $]-\infty,\infty[$.

It is clear that f is continuous on $]-\infty,\infty[$ except at $x = 0$. At $x = 0$, te limits do not exist but tends to infinity. So, $x = 0$ is a point of infinite discontinuity. Hence, a rectangular hyperbola is a curve with one point of infinite discontinuity.

Jump of a Function at a Point.

If f(a + 0) and f(a − 0) both exist, but not equal, then the jump in the function at x = a is defined as the non-negative difference f(a + 0) − f(a − 0).

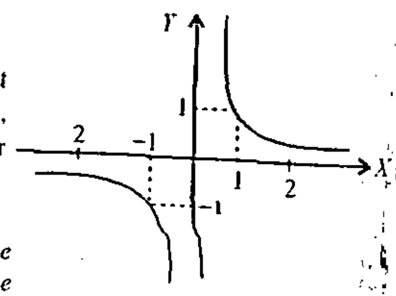


Fig. 5

REMARK

- A function having a finite number of jumps in a given interval is called piecewise continuous or sectionally continuous.

1.3. FOUR FUNCTIONAL LIMITS

Let us suppose the function $f(x)$ be defined on the closed interval $[a, b]$ and let $x_0 \in [a, b]$.

Let the upper and lower bounds of the function $f(x)$ in the right hand *nbd* $[x_0, x_0 + h]$ of x_0 denoted by M and m respectively where $M = M(h)$ and $m = m(h)$. Let the sequence of diminishing values $h_1, h_2, \ldots$ be assigned to h, which converges to zero, then $M(h_1), M(h_2), M(h_3) \ldots$ is a decreasing sequence and so it possesses a lower limit.

Similarly, the sequence $m(h_1), m(h_2), m(h_3) \ldots$ is an increasing sequence and have an upper limit. These lower and upper limits are respectively known as the upper and lower limits of the function $f(x)$ at $x = x_0$ on the right and are denoted by $\overline{f(x_0 + 0)}$ and $\underline{f(x_0 + 0)}$ respectively.

$$\therefore \qquad \overline{f(x_0 + 0)} = \lim_{h \to 0} M(h) \text{ and } \underline{f(x_0 + 0)} = \lim_{h \to 0} m(h).$$

If the right hand upper limits $\overline{f(x_0 + 0)}$ is equal to the right hand lower limit $\underline{f(x_0 + 0)}$, then their common value is known as the right hand limit of the function $f(x)$ at $x = x_0$ and is denoted by $f(x_0 + 0)$

i.e.,
$$f(x_0 + 0) = \overline{f(x_0 + 0)} = \underline{f(x_0 + 0)}.$$

Similarly, if we consider the left hand *nbd* $[x_0 - h, x_0]$, then the upper limit of $m(h)$ and the lower limit of $M(h)$ are respectively known as the **lower** and **upper limits** of the function $f(x)$ at $x = x_0$ on the left and are denoted by $\underline{f(x_0 - 0)}$ and $\overline{f(x_0 + 0)}$ respectively.

If the left hand upper limit $\overline{f(x_0 - 0)}$ is equal to the left-hand lower limit $\underline{f(x_0 - 0)}$, then their common value is known as the left hand limit of the function $f(x)$ at $x = x_0$ and is denoted by $f(x_0 - 0)$

i.e.,
$$f(x_0 - 0) = \overline{f(x_0 - 0)} = \underline{f(x_0 - 0)}.$$

REMARK

- The four numbers $\overline{f(x_0 + 0)}$, $\underline{f(x_0 + 0)}$, $\overline{f(x_0 - 0)}$ and $\underline{f(x_0 - 0)}$ are known as four functional limit of the function $f(x)$ at $x = x_0$.
- The four functional limits of the function $f(x)$ at $x = x_0$ are independent of the value of the function
- At $x = 0$, the functional limits are denoted by
$$\overline{f(+0)}, \underline{f(+0)}, \overline{f(-0)} \text{ and } \underline{f((-0)}.$$

SOLVED EXAMPLES

Example 1. *Show that* $f(x) = \dfrac{x^2 - 1}{x - 1}$ *is continuous for all values of x except* $x = 1$.

Solution. If $x \neq 1$, then $f(x) = (x + 1)$ is a polynomial.

$\Rightarrow f(x)$ is continuous for all values of $x \neq 1$.

($\because$ Every polynomial function is continuous)

If $x = 1, f(x)$ is of the form $\frac{0}{0}$, which is not defined and so the function $f(x)$ is discontinuous at $x = 1$.

Example 2. *Show that the function f(x) is defined by*

$$f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

is discontinuous at $x = 1$.

Solution. Here the value of $f(x)$ at $x = 1$ is 2

$\Rightarrow$ $f(1) = 2.$

Now, $\text{RHL} = f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} (1+h)^2 = 1$

Also $\text{LHL} = f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1-h)^2 = 1$

Therefore, we have

$$f(1+0) = f(1-0) \neq f(1)$$

$\Rightarrow$ $f(x)$ is not continuous at $x = 1$.

Example 3. *Examine whether or not the function*

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & \text{when } x \neq 0 \\ 2, & \text{when } x = 0 \end{cases}$$

is continuous at $x = 0$.

Solution. Given that $f(x) = 1$, when $x = 0$

$\Rightarrow$ $f(0) = 2.$

Now, $\text{RHL} = f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \left[\dfrac{\sin 2(0+h)}{(0+h)}\right]$

$$= 2 \qquad \left(\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right)$$

and $\text{LHL} = f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \left[\dfrac{\sin 2(0-h)}{(0-h)}\right]$

$= 2.$

Therefore, we have

$$f(0+0) = f(0-0) = f(0) = 2.$$

Hence, $f(x)$ is continuous at $x = 0$.

Example 4. *A function* $f(x)$ *is defined as follows*

$$f(x) = \begin{cases} (x^2/a) - a, & \text{when, } x < a \\ 0, & \text{when } x = a \\ a - (a^2/x), & \text{when } x > a \end{cases}$$

Prove that the function $f(x)$ *is continuous at* $x = a$.

Solution. Here, we have

$$\text{RHL} = f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \left[a - \frac{a^2}{(a+h)}\right]$$

$$\left[\text{By using } f(x) = a - \frac{a^2}{x} \text{ for } x > a\right]$$

$$= \left[a - \frac{a^2}{a}\right] = (a - a) = 0.$$

and $\text{LHL} = f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \left[\dfrac{(a-h)^2}{a} - a\right]$

$$\left[\text{By using } f(x) = \frac{x^2}{a} - a \text{ for } x < a\right]$$

$$= \frac{a^2}{a} - a \qquad \ldots(2)$$

$= 0.$

Also $f(x) = 0$ for $x = a$

$\Rightarrow$ $f(a) = 0.$ ...(3)

Now, from (1), (2) and (3), we have

$$f(a+0) = f(a-0) = f(a) = 0$$

$\Rightarrow f(x)$ is continuous at $x = a$.

Example 5. *A function f(x) is defined as follows*

$$f(x) = \begin{cases} 1 + x & \text{if } x \le 2 \\ 5 - x & \text{if } x \ge 2 \end{cases}$$

check the continuity of f(x) at x = 2.

Solution. Here, we have

$$f(2) = 1 + 2 \text{ or } 5 - 2 = 3. \quad ...(1)$$

Now, $$\text{RHL} = f(2 + 0) = \lim_{h \to 0} f(2 + h)$$

$$= \lim_{h \to 0} [5 - (2 + h)] = \lim_{h \to 0} [3 - h] = 3 \quad ...(2)$$

and $$\text{LHL} = f(2 - 0) = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} [1 + (2 - h)] = 3. \quad ...(3)$$

Now, from (1), (2) and (3), we have

$$f(2 + 0) = f(2) = f(2 - 0) = 3.$$

Hence, the function $f(x)$ is continuous at $x = 2$.

Example 6. *Test the following function for continuity at x = 0*

(*i*) $f(x) = x \sin \frac{1}{x}$, $x \ne 0, f(x) = 0$ *at* $x = 0$.

(*ii*) $f(x) = \frac{1}{1 - e^{-1/x}}$, $x \ne 0, f(x) = 0$ *at* $x = 0$.

Solution. (i) Here, we have

$$\text{LHL} = f(0 - 0) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} (-h) \sin\left(\frac{1}{-h}\right)$$

$$= \lim_{h \to 0} h \sin \frac{1}{h}$$

$= 0 \times$ a finite quantity lying between 1 and -1

$= 0$

and $$\text{RHL} = f(0 + 0) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} h \sin \frac{1}{h}$$

$= 0.$

Also, given that $f(0) = 0$

$\Rightarrow \quad f(0 + 0) = f(0 - 0) = f(0).$

Hence, the function $f(x)$ is continuous at $x = 0$.

(ii) Here we have

$$\text{LHL} = f(0 - 0) = \lim_{h \to 0} f(0 - h)$$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{1}{1 - e^{1/h}} = 0$$

and $$\text{RHL} = f(0 + 0) = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{1}{1 - e^{-1/h}} = 1$$

Also, $\quad f(0) = 0$

$\Rightarrow \quad f(0 + 0) \ne f(0 - 0) = f(0)$

Hence, $f(x)$ is discontinuous at $x = 0$ and this discontinuity is of first kind.

Example 7. *Discuss the continuity of the function f(x) defined by*

$$f(x) = \begin{cases} x^2 \ for & x < -2 \\ 4 \ for & -2 \le x \le 2 \\ x^2 \ for & x > 2. \end{cases}$$

Solution. Here, we shall check the continuity of $f(x)$ at $x = -2$ and 2.

At $x = -2$.

Here, we have $f(-2) = 4$

$$\text{LHL} = f(-2-0) = \lim_{h \to 0} f(-2-h) = \lim_{h \to 0} (-2-h)^2 = 4$$

and
$$\text{RHL} = f(-2+0) = \lim_{h \to 0} f(-2+h) = \lim_{h \to 0} 4 = 4$$

$$\Rightarrow \quad f(-2-0) = f(-2) = f(-2+0) = 4.$$

Hence, $f(x)$ is continuous at $x = -2$.

At $x = 2$.

Here, we have $f(2) = 4$

$$\text{RHL} = f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2+h)^2 = 4$$

$$\text{LHL} = f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} 4 = 4$$

$$\Rightarrow \quad f(2-0) = f(2) = f(2+0) = 4.$$

Hence, $f(x)$ is continuous at $x = 2$.

REMARK

➤ At $x = 0$, neither function value nor limit exist. Therefore, the function $f(x)$ has discontinuity of second kind.

Example 8. *Show that the function $f(x)$ defined on **R** by*

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$$

is discontinuous at every point of **R**.

Solution. Let us first suppose, x be rational. Then $f(x) = 1$. For each positive integer n, let x_n be an irrational number such that $|x_n - x| < \frac{1}{n}$. Then the sequence $< x_n >$ converges to x. Now, by definition $f(x_n) = 1 \ \forall \ n$

$$\Rightarrow \quad \lim_{n \to \infty} f(x_n) = -1 \ne f(x).$$

Hence, f is discontinuous at each rational point.

Now suppose x is an irrational number. Then $f(x) = 1$. For each positive integer n, let x_n be the rational number such that $|x_n - x| < \frac{1}{n}$. Then, the sequence $< x_n >$ converges to x. Now $f(x_n) = 1 \ \forall \ n$ so that

$$\lim_{n \to \infty} f(x_n) = 1 \ne f(x).$$

Therefore, f is discontinuous at each irrational point.

Hence, f is discontinuous at every point of **R**.

• TEST YOURSELF–1

1. Discuss the continuity of the following functions

(i) $f(x) = \cos\left(\frac{1}{x}\right)$, when $x \ne 0, f(0) = 0$

(ii) $f(x) = \frac{\sin x}{x}, x \ne 0, f(0) = 1$

(iii) $f(x) = \frac{1}{1 - e^{1/x}}$, when $x \ne 0$, and $f(0) = 0$

(iv) $f(x) = \dfrac{\sin^{-1} x}{x}$, $x \neq 0, f(0) = 1$

(v) $f(x) = \dfrac{e^{1/x} \sin (1/x)}{1 + e^{1/x}}$, $x \neq 0$ and $f(0) = 0$

(vi) $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$, $x \neq 0, f(0) = 0$

2. Examine the following function for continuity at $x = 0$ and $x = 1$

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ 1 & \text{if } 0 < x \leq 1 \\ 1/x & \text{if } x > 1. \end{cases}$$

3. A function f defined on [0, 1] is given by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f takes every values between 0 and 1, but it is continuous only at the point $x = \frac{1}{2}$.

4. Examine the continuity of the function

$$f(x) = \begin{cases} -x^2 & , \text{ if } x \leq 0 \\ 5x - 4 & , \text{ if } 0 < x \leq 1 \\ 4x^2 - 3x & , \text{ if } 1 < x < 2 \\ 3x + 4 & , \text{ if } x \geq 2 \end{cases}$$

at $x = 0$, 1 and 2.

5. Show that the function f defined by $f(x) = \dfrac{xe^{1/x}}{1 + e^{1/x}}$, $x \neq 0, f(0) = 1$ is not continuous at $x = 0$ and also show how the discontinuity can be removed.

ANSWERS

1. (i) Discontinuous at $x = 0$ (ii) Continuous at $x = 0$
   (iii) Discontinuous at $x = 0$ with ordinary discontinuity
   (iv) Continuous at $x = 0$
   (v) Discontinuity of the second kind at $x = 0$
5. Continuous at $x = 1, 2$, discontinuous at $x = 0$.

• SOME IMPORTANT THEOREMS

Theorem 1. *If f and g be two continuous functions at a point $a \in I$ then the function*

(i) $f + g$ *(ii)* cf

(iii) fg *(iv)* f/g $[g(a) \neq 0]$ *are also continuous.*

Proof. Since f and g are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a).$$

(i) By definition, we have

$$(f + g)(x) = f(x) + g(x) \ \forall x \in I.$$

$$\therefore \quad \lim_{x \to a} (f + g)(x) = \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$
$$= f(a) + g(a)$$
$$= (f + g)(a)$$

$\Rightarrow (f + g)$ is continuous.

(ii) By definition, we have

$$(cf)(x) = cf(x) \ \forall x \in I.$$

Therefore $$\lim_{x \to a} (cf)(x) = \lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$$
$$= cf(a)$$
$$= (cf)(a).$$

Hence, cf is continuous at $x = a$.

(iii) By definition, we have

$$(fg)(x) = f(x) \cdot g(x) \quad \forall x \in I.$$

Therefore, $\lim_{x \to a} (fg)(x) = \lim_{x \to a} [f(x) \cdot g(x)]$

$$= \left[\lim_{x \to a} f(x)\right] \cdot \left[\lim_{x \to a} g(x)\right]$$

$$= f(a) \cdot g(a)$$

$$= (fg)(a)$$

Hence, fg is continuous at $x = a$.

(iv) We have

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \forall\ x \in I,\ g(x) \neq 0.$$

Therefor, $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a).$

Hence, $\frac{f}{g}$ is continuous.

Theorem 2. *If f is continuous at $a \in I$, then $|f|$ is also continuous at a.*

Proof. Since f is continuous at $x = a$

$\Rightarrow$ $\lim_{x \to a} f(x) = f(a).$

We know that

$$|f|(x) = |f(x)|,\ x \in I$$

$\Rightarrow$ $\lim_{x \to a} |f|(x) = \lim_{x \to a} |f(x)| = \left|\lim_{x \to a} f(x)\right| = |f(a)| = |f|(a).$

Hence, $|f|$ is continuous.

Theorem 3. (Boundedness theorem). *If a function f is continuous in a closed interval $[a, b]$, then it is bounded in $[a, b]$.*

Proof. Let, if possible f be unbounded on I. Then for each $n \in \mathbf{N}, \exists\, x_n \in I$ such that $|f(x_n)| > n$. The bounded sequence $< x_n >$ in I has a subsequence $< x_{n_k} >$ such that it converges to a point $x_0 \in I$ ($\because$ every subsequence of a convergent sequence is convergent)

$\Rightarrow$ $< x_{n_k} > \to x_0$ and $|f(x_{n_k})| > n_k \quad \forall\ n_k \in \mathbf{N}$

$\Rightarrow$ $< f(x_{n_k}) >$ can not converge to $f(x_0)$

$\Rightarrow$ f is not continuous at x_0.

which is a contradiction.

This contradiction leads to the result that f is bounded on I.

Theorem 4. *If a function f is continuous on a closed and bounded interval $[a, b]$, then, it attains its bounds on $[a, b]$.*

Proof. Since, the function f is continuous on the closed and bounded interval $[a, b]$, therefore, it is bounded

$\Rightarrow$supremum M and infimum m of f exist in $[a, b]$.

To show, there exist two point $x_1, x_2 \in [a, b]$ such that

$$f(x_1) = m,\ f(x_2) = M.$$

Then, by definition of supremum

$$f(x) \leq M \quad \forall\, x \in [a, b].$$

Let, if possible $f(x) \neq M$ for any $x \in [a, b]$, then $f(x) < M \quad \forall\ x \in [a, b]$. Therefore,

$$M - f(x) > 0,\ \forall\ x \in [a, b].$$

Since, $f(x)$ is continuous on $[a, b]$ and M is constant, therefore $M - f(x)$ is continuous on $[a, b]$.

Also $M - f(x) \neq 0$ for any $x \in [a, b]$

$\Rightarrow$ $\frac{1}{M - f(x)}$ is continuous on $[a, b]$

$\Rightarrow \quad \dfrac{1}{M-f(x)}$ is bounded on $[a, b]$

$\Rightarrow \quad \exists$ a number $K > 0$ such that

$$\frac{1}{M-f(x)} \leq K, \quad \forall\, x \in [a, b]$$

$$\Rightarrow \quad M - f(x) \geq \frac{1}{K}, \quad \forall\, x \in [a, b]$$

$$\Rightarrow \quad f(x) \leq M - \frac{1}{K} \quad \forall\, x \in [a, b]$$

$\Rightarrow \quad M - \dfrac{1}{K}$ is an upper bound if f on $[a, b]$ such that

$$M - \frac{1}{K} < M = \sup f(x)$$

which is *a* contradiction

$\Rightarrow \quad \exists$ a point $x_2 \in [a, b]$ such that

$$M = f(x_2).$$

Similarly, we can show that if $m = \inf f(x)$ $\exists$ a point x_1 such that

$$m = f(x_1).$$

Theorem 5. *If a function f is continuous in $[a, b]$ and $f(a), f(b)$ have opposite signs, then there is at least one value of x for which $f(x)$ vanishes.*

Proof. Since, the function $f(x)$ have opposite signs for a and b

i.e., $\quad f(a) < 0$ and $f(b) > 0$.

Let us define

$$S = [x : x \in [a, b], f(x) < 0].$$

Now, since $f(a) < 0$, therefore $a \in S \Rightarrow S \neq \phi$.

Let $\quad u = \sup S$.

Now, to show $a < u < b$ and $f(u) = 0$.

First, we shall show that $u \neq a$. Since $f(a) < 0$ and f is continuous at a,

$\Rightarrow \exists$ a number δ_1, such that $f(x) < 0 \quad \forall\, x \in\,]a, a + \delta_1[$.

$\Rightarrow [a, a + \delta_1] \subset S$

$\Rightarrow \sup S$ must be greater than or equal to $a + \delta_1$

Therefore, $\quad u \geq a + \delta_1 \Rightarrow u \neq a$.

Now, to show $\boldsymbol{u \neq b}$

Since, $f(b) > 0 \Rightarrow \exists\, \delta_2$ such that $f(x) > 0 \quad \forall\, x \in [b - \delta_2, b]$

$\Rightarrow \quad]b - \delta_2, b\,[\subset S$

$\Rightarrow \quad u = \sup S \leq b - \delta_2 < b$

$\Rightarrow \quad u \neq b.$

Now, we shall show that $f(u) \not> 0$. Since $a < u < b$. Therefore, if $f(u) > 0$. Then we can find a number $\delta_3 > 0$, such that

$$f(x) > 0 \text{ for } u - \delta_3 < x < u + \delta_3.$$

Also, $u = \sup S$. Therefore, $\exists\, x_1 \in S : u - \delta_3 < x_1 \leq u$

$$\Rightarrow \quad f(x) > 0.$$

Also $\quad x_1 \in S \Rightarrow f(x_1) < 0$

which is a contradiction

$$\Rightarrow \quad f(u) \not> 0.$$

Now, we shall show that $f(u) \not< 0$. If $f(u) < 0$, then we can find a positive number δ_4 such that

$$u + \delta_4 < b \text{ and } f(x) < 0 \text{ for } u - \delta_4 < x < u + \delta_4.$$

If x_2 is any other point such that $u < x_2 < u + \delta_4$. Then $f(x_2) < 0$. But this is a contradiction to the fact that u is the supremum of S consequently $f(u) \not< 0$

Hence, $f(u) = 0$.

Theorem 6. (Intermediate value theorem). *Let f be a function continuous on the closed and bounded interval [a, b]. If K be any real number between f(a) and f(b), then there exists a real number c between a and b (a < c < b) such that*

$$f(c) = K.$$

Proof. Let us suppose

$$f(a) < K < f(b). \qquad ...(1)$$

Define a fucntion g such that

$$g(x) = f(x) - K \; ; x \in [a, b]. \qquad ...(2)$$

Now, since f is continuous on $[a, b]$ and K is constant, g is continuous on $[a, b]$. ...(3)

From (1), we have that K lies between $f(a)$ and $f(b)$. Therefore, either

$$f(a) < K < f(b) \text{ or } f(b) < K < f(a).$$

From (2)

$$g(a) = f(a) - K < 0$$
$$g(b) = f(b) - K > 0$$
$$\Rightarrow \quad g(a) \,.\, g(b) < 0.$$

Now, from (3) and (4) there exists a point $c \in \,]a, b[$ such that

$$g(c) = 0$$
$$\Rightarrow \quad f(c) - K = 0$$
$$\Rightarrow \quad f(c) = K.$$

Hence, there exists a point c such that $a < c < b$ and $f(c) = K$.

• 1.5. UNIFORM CONTINUITY

Since, we know that if a function $f(x)$ is continuous in the closed interval I, then for a given positive number ε, $\exists$ a positive number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ for } |x - a| < \delta, a \in I.$$

Here, we observe that the number δ depends besides ε, on the point a as it is a function of a. In general, δ is different at different points in I.

For this, let us consider the figure 9, where PQ, divided into equal parts, each of length ε.

The corresponding subdivision of $I = [a, b]$ is such that δ is not the same for all points x in $[a, b]$.

Therefore, if we can find a positive number δ_0 such that for a chosen ε, $|f(x) - f(a)| < \varepsilon$ for $|x - a| < \delta_0$ where the number δ_0 is independent of the point a, then the function $f(x)$ is said to be uniformly continuous on $[a, b]$.

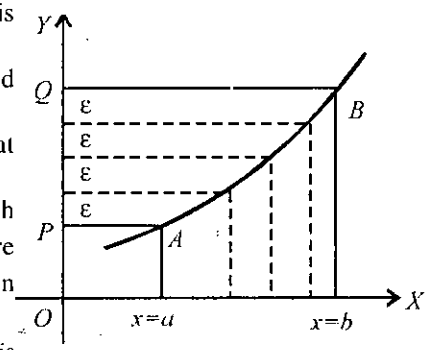


Fig. 9

Defintion. *A function f(x) defined on an interval I is said to be uniformly continuous in I if to each* $\varepsilon > 0 \; \exists$ *a positive number* $\delta > 0$, *(depending upon ε) but independent of* $x \in I$ *such that*

$$|f(x_2) - f(x_1)| < \varepsilon, \text{ whenever } |x_2 - x_1| < \delta$$

where $x_1, x_2 \in I$.

• 1.6. SOME IMPORTANT THEOREMS

Theorem 1. *If a function f is uniformly continuous on an interval I, then it is continuous on I.*

Proof. Let us suppose that f is uniformly continuous on I

$$\Rightarrow \quad \text{given } \varepsilon > 0, \exists \, \delta > 0 \text{ such that}$$

$$|f(x_2) - f(x_1)| < \varepsilon, \text{ whenever } |x_2 - x_1| < \delta, \; \forall \, x_1, x_2 \in I$$

Let $x_1 \in I$ and $x_2 \in x$, then we have

$$|f(x) - f(x_1)| < \varepsilon, \text{ whenver } 0 < |x - x_1| < \delta$$

$\Rightarrow$ $f(x)$ is continuous at $x_1 \in I$.

Since, x_1 is arbitrary, consequently $f(x)$ is continuous on I.

Theorem 2. *If a function $f(x)$ is continuous on an closed and bounded interval $I = [a, b]$, then it is uniformly continuous on $[a, b]$.*

Proof. Since f is given to be continuous in the interval $[a, b]$.

Let $\varepsilon > 0$ be given $\Rightarrow [a, b]$ can be divided into a finite number of subintervals such that $|f(x_2) - f(x_1)| < \frac{\varepsilon}{2}$, where x_1, x_2 are any two points of the same subinterval.

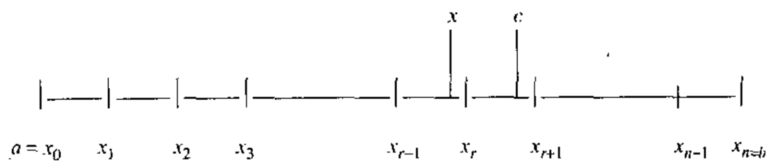
Let us divide the whole interval $[a, b]$ into n sub-intervals, say

$[x_0 = a, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n = b]$

$\Rightarrow |f(x') - f(x'')| < \frac{\varepsilon}{2}$, where x', x'' belongs to the same subinterval...(1)

Let $\delta = \min \{\delta_1, \delta_2 \ldots \delta_r \ldots \delta_n\}$ where δ_r denotes the length of the r^{th} subinterval *i.e.*,

$\delta_r = |x_r - x_{r-1}|$



Let x and c be any two points of $[a, b]$ such that

$$|x - c| < \delta.$$

Since $\delta > 0$, less than the length of each subinterval. Therefore, following two cases may arise:

Case (i) When x and c belongs to same interval :

$$\Rightarrow \qquad |f(x) - f(c)| < \frac{\varepsilon}{2}, \text{ when } |x - c| < \delta$$

where $x, c \in [a, b]$

$\Rightarrow$ function f is uniformly continuous in $[a, b]$.

Case (ii) When x and c belongs to the two consecutive sub-intervals say $x_{r-1} < x < x_r < c < x_{r+1}$.

Now, consider

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f(x_r) + f(x_r) - f(c)| \\ &\leq |f(x) - f(x_r)| + |f(x_r) - f(c)| \qquad \text{(By triangle inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \text{ when } |x - c| < \delta \\ &< \varepsilon, \text{ when } |x - c| < \delta. \end{aligned}$$

$\therefore$ Given $\varepsilon > 0, \exists\, \delta > 0$ such that

$|f(x) - f(c)| < \varepsilon$ where x and c are any two points of $[a, b]$ such that $|x - c| < \delta$

$\Rightarrow$ f is uniformly continuous on $[a, b]$.

Hence,

f is continuous on a closed and bounded interval $[a, b]$

$\Rightarrow$ f is uniformly continuous on $[a, b]$.

SOLVED EXAMPLE

Example 1. *Show that the function $f(x) = x^2 + 3x, x \in [-1, 1]$ is uniformly continuous in $[-1, 1]$.*

Solution. Let $\varepsilon > 0$ be given .

Let $x_1, x_2 \in [-1, 1] \Rightarrow |f(x_2) - f(x_1)| = |(x_2^2 + 3x_2) - (x_1^2 + 3x_1)|$

$$= |(x_2^2 - x_1^2) + 3(x_2 - x_1)|$$

$$= |(x_2 - x_1)(x_2 + x_1 + 3)|$$

$$= |\ [illegible]\ |\,|x_2 + x_1 + 3|$$

$$\leq |\ [illegible]\ | + |x_1| + 3)$$

$$\leq 5\,|x_2 - x_1| \qquad [\because\ x_1, x_2 \in [-1, 1] \Rightarrow |x_1| \leq 1 \text{ and } |x_2| \leq 1]$$

$$\Rightarrow\ |f(x_2) - f(x_1)| < \varepsilon \text{ for } |x_2 - x_1| < \frac{\varepsilon}{5}.$$

Thus for any $\varepsilon < 0, \exists\, \delta = \frac{\varepsilon}{5} > 0$ such that

$$|f(x_2) - f(x_1)| < \varepsilon, \text{ whenever } |x_2 - x_1| < \delta,\ \forall\, x_1, x_2 \in [-1, 1].$$

Hence, $f(x)$ is uniformly continuous in $[-1, 1]$.

Example 2. *Show that the function f defined by $f(x) = x^3$ is uniformly continuous on* $[-2, 2]$.

Solution. In order to show that the function f is uniformly continuous we have to prove that for a given $\varepsilon > 0, \exists\, \delta > 0$ such that

$$|f(x2) - f(x1)| < \varepsilon, \text{ when } 0 < |x2 - x1| < \delta \text{ where } x1, x2 \in [-2, 2]$$

Consider

$$|f(x2) - f(x1)| = |x2^3 - x1^3|$$

$$= |(x_2 - x_1)(x_2^2 + x_1^2 + x_1x_2)|$$

$$\leq |x_2 - x_1|\,[|x_1^2| + |x_2^2| + |x_1x_2|]$$

$$\leq 12\,|x_2 - x_1| \qquad (\because\ x_1, x_2 \in [-2, 2] \Rightarrow |x_1| \leq 2\ |x_2| \leq 2)$$

$$\therefore\quad |f(x_2) - f(x_1)| < \varepsilon \text{ whenever } |x_2 - x_1| < \varepsilon/12.$$

Therefore, given $\varepsilon > 0, \exists\, \delta = (\varepsilon/12)$ such that

$$|f(x_2) - f(x_1)| < \varepsilon \text{ whenever } |x_2 - x_1| < \delta,\ x_1, x_2\ \varepsilon\ [-2, 2].$$

Hence, f is uniformly continuous on $[-2, 2]$.

Example 3. *Show that the function f defined by*

$$f(x) = \frac{1}{x},\quad \forall\, x \in\ ]0, 1]$$

is not uniformly continuous in $]0, 1]$.

Solution. In order to show that the function f is uniformly continuous in $]0, 1]$ we have to prove that for a given $\varepsilon > 0, \exists\, \delta > 0$, independent of the choice of x, ($x \in\ ]0, 1]$) such that

$$|f(x) - f(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon \text{ whenever } 0 < |x - c| < \delta$$

i.e.,
$$|x - c| < \delta \Rightarrow \left|\frac{c - x}{cx}\right| < \varepsilon$$

i.e.,
$$x \in\ ]c - \delta, c + \delta[\ \Rightarrow \left|\frac{c - x}{cx}\right| < \varepsilon. \qquad ...(1)$$

Let us take $c = \delta$, then $]c - \delta, c + \delta[\ =\ ]0, 2\delta[$.

Since, the condition (1) must hold $\forall\, x \in\ ]0, 2\delta[$.

$$\therefore\qquad \text{as } x \to 0, \frac{\delta - x}{\delta x} \to \infty \text{ and } x \in\ ]0, 2\delta[$$

i.e., if we choose x close to zero, then condition (1) does not hold.

$$\Rightarrow\qquad f(x) = \frac{1}{x} \text{ is not uniformly continuous in }]0, 1].$$

Example 4. *Show that the function f defined on* $\mathbf{R}^+$ *as*

$$f(x) = \sin\frac{1}{x},\quad \forall\, x > 0$$

is continuous, but not uniformly continuous on $\mathbf{R}^+$.

Solution. Let $a \in \mathbf{R}^+$.

We have

$$\text{LHL} = f(a - 0) = \lim_{h \to 0} f(a - h) = \lim_{h \to 0} \sin\frac{1}{a - h} = \sin\frac{1}{a}$$

$$\text{RHL} = f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \sin \frac{1}{a+h} = \sin \frac{1}{a}$$

$$f(a) = \sin \frac{1}{a}$$

$\Rightarrow$ $$f(a+0) = f(a) = f(a-0)$$

$\Rightarrow$ f is continuous at a.

Since, a is arbitrary point in $\mathbf{R}^+$.

Therefore, f is continuous on $\mathbf{R}^+$.

Now, to show f is not uniformly continuous on $\mathbf{R}^+$.

Let δ be any positive number. Take

$$x_1 = \frac{1}{n\pi},\ x_2 = \frac{1}{n\pi + \pi/2} = \frac{2}{(2n+1)(\pi)} \text{ where } n \in Z^+$$

such that $$x_1 - x_2 = \frac{1}{n\pi} - \frac{2}{(2n+1).\pi} < \delta.$$

Now, $|x_1 - x_2| < \delta$ but

$$|f(x_1) - f(x_2)| = \left| \sin n\pi - \sin \frac{1}{2}(2n+1)\pi \right| = 1 > \varepsilon$$

which shows that for this choice of ε, we can not find a $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ for } |x_1 - x_2| < \delta \quad \forall\ x_1, x_2 \in \mathbf{R}^+.$$

Hence, f is not uniformly continuous on $\mathbf{R}^+$.

• SUMMARY

- **Cauchy definition of continuity :** A function f is said to continuous at $x = a$ if for given $\varepsilon > 0$ there exists a $\delta > 0$ such that

  $$|f(x) - f(a)| < \varepsilon \quad \text{whenever } |x - a| < \delta$$

- **Discontinuous function :** If a function f is not continuous at $x = a$, then it is discontinuous at $x = a$.

  (i) Removable discontinuity : A function f is said to have removable discontinuity at $x = a$ if $\lim_{x \to a} f(x)$ exists but it is not equal to $f(a)$.

  i.e., $$f(a-0) = f(a+0) \neq f(a).$$

  (ii) Discontinuity of first kind : A function f is said to have discontinuity of first kind at $x = a$ if both $f(a-0)$ and $f(a+0)$ exist but not equal to each other.

  (iii) Discontinuity of second kind : A function f is said to have discontinuity of second kind at $x = a$ if none of $f(a-0)$ and $f(a+0)$ exist.

  (iv) Mixed discontinuity : A function f is said to have mixed discontinuity at $x = a$ if it is discontinuous of first kind on one side of a and discontinuous of second kind on other side of a.

- **Uniform continuity :** A function f defined on an interval I is said to be uniformly continuous in I if for given $\varepsilon > 0$, $\delta > 0$ (depending on ε not on x) such that

  $$|f(x) - f(y)| < \varepsilon \quad \text{whenever } |x - y| < \delta.$$

• STUDENT ACTIVITY

1. Show that the function f on R defined by $f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$ is discontinuous at every point of R.

2. Show that the function f on $[-2, 2]$ defined by $f(x) = x^3$ is uniformly continuous on $[-2, 2]$.

• TEST YOURSELF

1. Let $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$. Show that f is not uniformly continuous on **R**.
2. Show that the function x^2 and x^3 are not uniformly continuous on $[0, \infty[$.
3. In each of the following cases, show that f is continuous but not uniformly continuous on their respective intervals.

   (i) $f(x) = \sin \frac{1}{x}$, $\forall\, x \in\,]0, 1[$ (ii) $f(x) = \frac{1}{2x}$, $\forall\, x \in [-1, 0[$

   (iii) $f(x) = \frac{1}{1-x}$, $\forall\, x \in\,]0, 1[$ (iv) $f(x) = e^x$, $\forall\, x \in [0, \infty[$.
4. If $f(x + y) = f(x) \cdot f(y)$, $\forall\, x, y \in \mathbf{R}$, show that f is continuous on **R** if and only if f is continuous at least one point of **R**. If f is continuous at some point $a \in \mathbf{R}$, prove that f is uniformly continuous on every bounded subset of **R**.
5. Show that the function f defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

   is uniformly continuous in $[-1, 1]$.

Fill in the Blanks :

1. A function $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) =$ ....
2. A function is said is have .... if $f(a+0) = f(a-0) \neq f(a)$.
3. If $f(a+0) \neq f(a-0)$ then $f(x)$ is said to have a discontinuity of ....
4. If f is continuous then $|f|$ is ....
5. Every uniformly continuous function is ....

True or False :

Write 'T' for true and 'F' for false :

1. Every continuous function inclosed interval is bounded. *(T/F)*
2. Every continuous function in open interval is bounded. *(T/F)*
3. For $\lim_{x \to a} f(x)$ do exists, the function $f(x)$ must be defined at $x = a$. *(T/F)*
4. The limit of products is equal to the product of the limit. *(T/F)*
5. $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \frac{2}{3}$. *(T/F)*

Multiple Choice Questions :

Choose the most appropriate one :

1. If $\lim_{x \to a} f(x) = l$ and $f(x) \geq 0$, then :
   (a) $l = 0$ (b) $l \leq 0$ (c) $l \geq 0$ (d) None of these.
2. If $\lim_{x \to a} f(x) = l$, then $\lim_{x \to a} |f(x)| =$:
   (a) l (b) $|l|$ (c) 0 (d) 1.
3. If $\lim_{x \to \infty} f(x) = l$ and $\lim_{x \to \infty} g(x)$ does not exists, then :
   (a) $\lim_{x \to \infty} f(x) \cdot g(x)$ does not exists
   (b) $\lim_{x \to \infty} f(x) \cdot g(x)$ exists necessarily
   (c) $\lim_{x \to \infty} f(x) \cdot g(x)$ may or may not exists
   (d) None of these.
4. $\lim_{x \to 2} \frac{|x-2|}{x-2} =$:
   (a) 0 (b) 1 (c) 2 (d) Does not exist.

ANSWERS

Fill in the Blanks :

1. $f(a)$ 2. Removable discontinuity 3. First kind 4. Continuous 5. Continuous

True or False :

1. T 2. F 3. F 4. T 5. T

Multiple Choice Questions :

1. (c) 2. (b) 3. (c) 4. (d) 5. (a)

UNIT 2

DIFFERENTIABILITY

STRUCTURE

- Derivative of a Function
- Continuity and Differentiability
- Algebra of Derivatives
- Rolle's Theorem
- Lagrange's Mean Value Theorem
- Cauchy's Mean Value Theorem
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to obtain the derivative of a function ?
- How to check the differentiability of a function ?
- What is Roille's Theorem ?
- What is Lagrange's Mean Value Theorem ?
- What is Cauchy's Mean Value Theorem ?
- How to apply these theorems ?

2.1. DERIVATIVE OF A FUNCTION

If a function $f(x)$ is defined on nbd of a point a and

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

exists (finitely), then the function $f(x)$ is said to be differentiable at a and this limit is called derviative of the function $f(x)$ at a.

Symbolically, this derivative, is denoted by $f'(a)$ and is full read as the derivative of $f(x)$ at $x = a$ with respect to the variable x. The process of evaluating $f'(a)$ is called differentiation.

If the above limit exists infinitely even then we shall admit it as the derivative at a. But the admission does not seems to serve any fruitful purpose in our discussions. Therefore the case when the limit exists infinitely is excluded.

Left hand derivative. The left hand derivative (**regressive derivative**) of f at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}, \; h > 0$$

and, is denoted by $Lf'(a)$.

Right hand derivative. The right hand derivative (**progressive derivative**) of f at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}, \; h > 0.$$

It is denoted by $Rf'(a)$.

The derivative $f'(a)$ exists when $Lf'(a) = Rf'(a)$.

Differentiability in an Interval.

(i) A function $f :]a, b[\to R$ is said to be differentiable in $]a, b[$ iff it is differentiable at every point of $]a, b[$.

(ii) A function $f : [a, b] \to R$ is said to be differentiable in $[a, b]$ iff $Rf'(a)$ and $Lf'(b)$ exists and f is differentiable at every point of $]a, b[$.

(iii) Let f be a function whose domain is an interval I. If I_1 be the set of all those points x of I at which f is differentiable *i.e.* $f'(x)$ exists and if $I_1 \neq \phi$, we get another function f' with domain I_1. It is called the first derivative of f. Similarly $2^{nd}, 3^{rd}, \ldots, n^{th}$ derivative of f are defined and one denoted by $f'', f''', \ldots, f^n$ respectively of course, in order that $f^n(x)$ may be defined, it is necessary (though not sufficient) that $f^{n-1}(x)$ may be defined for all x in some open interval containing a.

• 2.2. CONTINUITY AND DIFFERENTIABILITY

A necessary condition for the existence of a finite derivative. *Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.*

Proof. Let f be differentiable at a. Then $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists and equal to $f'(a)$. Now we may write

$$f(x) - f(a) = \frac{f(x) - f(a)}{(x - a)}(x - a) \qquad \text{(If } x \neq a\text{)}$$

Now, taking limit as $x \to a$, we get

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \left\{\frac{f(x) - f(a)}{(x - a)}(x - a)\right\}$$

$$= \lim_{x \to a} \left\{\frac{f(x) - f(a)}{x - a}\right\} . \lim (x - a)$$

($\because$ limit of the product of two function is equal to product of their limits)

$$= f'(a) . 0$$

$$= 0$$

so that $$\lim_{x \to a} f(x) = f(a).$$

Hence, f is continuous at $x = a$. Thus continuity is a necessary condition for differentiability.

• 2.3. ALGEBRA OF DERIVATIVES

Theorem 1. *Let functions f and g be defined on an interval I. If f and g are differentiable at $x = a \in I$, then $f \pm g$ is also differentiable and*

$$(f \pm g)'(a) = f'(a) \pm g'(a).$$

Proof. Since, the functions f and g are differentiable at a, therefore

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) \qquad \ldots(1)$$

and $$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a) \qquad \ldots(2)$$

Now, consider $\lim\limits_{x \to a} \dfrac{(f \pm g)(x) - (f \pm g)(a)}{x - a}$

$$= \lim_{x \to a} \frac{[f(x) \pm g(x)] - [f(a) \pm g(a)]}{x - a}$$

$$= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \pm \frac{g(x) - g(a)}{x - a}\right]$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \pm \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

$$= f'(a) \pm g'(a).$$

Hence $f \pm g$ is differentiable at a and

$$(f \pm g)'(a) = f'(a) \pm g'(a).$$

Theorem 2. *Let a function $f(x)$ be differentiable at a point a and $c \in R$, then the function cf is also differentiable at a and*

$$(cf)'(a) = cf'(a).$$

Proof. By the definition of the derivative of a function at $x = a$, we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Now, consider

$$\lim_{x \to a} \frac{(cf)(x) - (cf)(a)}{x - a} = \lim_{x \to a} \frac{c f(x) - c f(a)}{x - a}$$

$$= \lim_{x \to a} \left\{ c \left(\frac{f(x) - f(a)}{x - a} \right) \right\}$$

$$= c \lim_{x \to a} \left\{ \frac{f(x) - f(a)}{x - a} \right\}$$

$$= cf'(a).$$

Hence, cf is differentiable at a and $(cf)'(a) = cf'(a)$.

Theorem 3. *Let, the functions f and g be defined on an interval I. If f and g are differentiable at $a \in I$, then $f \cdot g$ is also differentiable and*

$$(fg)'(a) = f'(a)\, g(a) + f(a)\, g'(a)$$

Proof. Since, f and g are differentiable at a, we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) \qquad ...(1)$$

and

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a). \qquad ...(2)$$

Consider $\displaystyle \lim_{x \to a} \frac{(fg)(x) - (fg)(a)}{x - a} = \lim_{x \to a} \frac{f(x)\, g(x) - f(a)\, g(a)}{x - a}$

$$= \lim_{x \to a} \frac{f(x)\, g(x) - f(a)\, g(x) + f(a)\, g(x) - f(a)\, g(a)}{x - a}$$

$$= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \cdot g(x) + f(a) \cdot \frac{g(x) - g(a)}{x - a} \right]$$

$$= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right] \lim_{x \to a} g(x) + f(a) \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

$$= f'(a)\, g(a) + f(a)\, g'(a).$$

[By applying the theorem on limits of sums and products and using the fact $\lim_{x \to a} g(x) = g(a)$]

Hence, fg is differentiable at a and

$$(fg)'(a) = f'(a)\, g(a) + f(a)\, g'(a).$$

Theorem 4. *If a function f is differentiable at $x = a$ and $f(a) \neq 0$, then the function $\frac{1}{f}$ is differentiable at a and*

$$\left(\frac{1}{f} \right)'(a) = -\frac{f'(a)}{[f(a)]^2}.$$

Proof. Since f is differentiable at a, therefore, it is also continuous at $x = a$.

Also, since $f(a) \neq 0$.

Consider

$$\frac{\frac{1}{f(x)} - \frac{1}{f(a)}}{x - a} = -\left[\frac{f(x) - f(a)}{x - a} \right] \cdot \frac{1}{f(x)} \cdot \frac{1}{f(a)}. \qquad ...(1)$$

Since f is differentiable at $x = a$, therefore,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a). \qquad ...(2)$$

Also, f is continuous at $x = a$, therefore

$$\lim_{x \to a} f(x) = f(a) \neq 0. \qquad ...(3)$$

By applying the theorem on the limits of a product to (1), and using (2) and (3), we find that

$$\frac{\lim_{x \to a} \frac{1}{f(x)} - \frac{1}{f(a)}}{x - a} \text{ exists and equal to } -\frac{f'(a)}{[f(a)]^2}.$$

Theorem 5. *Let f and g be defined on an interval I. If f and g be differentiable at $a \in I$, and if $g(a) \neq 0$, then the function f/g is also differentiable at a.*

Proof. Let $F = f/g$. Then, we have

$$F(x) - F(a) = (f/g)(x) - (f/g)(a)$$

$$= \frac{f(x)}{g(x)} - \frac{f(a)}{g(a)} = \frac{1}{g(x)\,g(a)}[f(x)\,g(a) - f(a)\,g(x)]$$

$$= \frac{1}{g(x)\,g(a)}[f(x)\,g(a) - f(a)\,g(a) + f(a)\,g(a) - f(a)\,g(x)].$$

Therefore $$\lim_{x \to a} \frac{F(x) - F(a)}{x - a} = \lim_{x \to a} \frac{1}{g(x)\,g(a)} \times \left[\left\{\frac{f(x) - f(a)}{x - a}\right\} g(a) - f(a)\left\{\frac{g(x) - g(a)}{x - a}\right\}\right]$$

or $$F'(a) = \frac{1}{g(a)\,g(a)}[f'(a)\,g(a) - f(a)\,g'(a)]$$

$$\Rightarrow \quad \left(\frac{f}{g}\right)'(a) = \frac{f'(a)\,g(a) - f(a)\,g'(a)}{[g(a)]^2}.$$

Theorem 6. *Let f and g be functions such that the range of f is contained in the domain of g. If f is differentiable at a and g is differentiable at f(a), then g o f is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a)) \,.\, f'(a)$$

(*This is known as* **Chain rule**).

Proof. Since, the range of f contained in the domain of g, therefore, $g \circ f$ has the same domain as that of f.

Now, let $y = f(x)$ and $y_0 = f(a)$.

Since, f is differentiable at a, we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

or $$f(x) - f(a) = (x - a)\,[f'(a) + A(x)]. \qquad ...(1)$$

where $A(x) \to 0$) as $x \to a$.

Further since g is differentiable at y_0, we have

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = g'(y_0)$$

or $$g(y) - g(y_0) = (y - y_0)\,[g'(y_0) + B(y)] \qquad ...(2)$$

where $B(y) \to 0$ as $y \to y_0$.

Now $$(g \circ f)(x) - (g \circ f)(a) = g(f(x)) - g(f(a)) = g(y) - g(y_0)$$

$$= (y - y_0)\,[g'(y_0) + B(y)] \qquad \text{[By (2)]}$$

$$= [f(x) - f(a)]\,[g'(y_0) + B(y)]$$

$$= (x - a)\,[f'(a) + A(x)]\,[g'(y_0) + B(y)], \qquad \text{[By (1)]}$$

Thus if $x \neq a$, then

$$\frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = [g'(y_0) + B(y)]\,[f'(a) + A(x)]. \qquad ...(3)$$

Also f being differentiable at a is continuous at a and hence $x \to a, f(x) \to f(a)$ *i.e.* $y \to y_0$.

$\Rightarrow B(y) \to 0$ as $x \to a$ and $A(x) \to 0$ as $x \to a$.

Now, taking the limit as $x \to a$, we get from (3)

$$\lim_{x \to a} \frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = g'(y_0)\,f'(x_0).$$

Hence, the function is differentiable at a and

$$(g \circ f)'(a) = g'(f(a))\,f'(a).$$

Theorem 7. (Derivative of the inverse function). *If f is differentiable at $x = a$ and is one-one function defined on interval I with $f'(a) \neq 0$, then the inverse of the function f is differentiable at $f(a)$ and its derivative at $f(a)$ is $\frac{1}{f'(a)}$.*

Proof. Let the domain of f be X and let its range be Y.

If g be the inverse of f, then g is a function with domain Y and range X such that $f(x) = y \Leftrightarrow g(y) = x$.

Now, let us suppose $y = f(x)$ and $y_0 = f(a)$.

Since, f is differentiable at a, we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

or $$f(x) - f(a) = (x - a)\,[f'(a) + A(x)] \qquad \ldots(1)$$

where $A(x) \to 0$ as $x \to a$. Further, we have

$$g(y) - g(y_0) = x - a, \qquad \text{[by definition of } g]$$

$$\therefore \quad \frac{g(y) - g(y_0)}{y - y_0} = \frac{x - a}{y - y_0} = \frac{x - a}{f(x) - f(a)} = \frac{1}{f'(a) + A(x)}. \qquad \text{[By (1)]}$$

It can be easily seen that if $y \to y_0$, then $x \to a$.

In fact, f being differentiable at a, it is also continuous at a, which implies that $g = f^{-1}$ is continuous at $f(a) = y_0$ and consequently

$$g(y) \to g(y_0) \text{ as } y \to y_0 \text{ i.e. } x \to a \text{ as } y \to y_0,$$

so that $A(x) \to 0$ as $y \to y_0$.

$$\therefore \quad \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{f'(a) + A(x)} = \frac{1}{f'(a)}$$

or $$g'(y_0) = \frac{1}{f'(a)} \text{ or } g'(f'(a)) = \frac{1}{f'(a)}.$$

Theorem 8. (Darboux's Theorem or Intermediate Value Theorem). *If f is finitely differentiable in a closed interval $[a, b]$ and $f'(a)$, $f'(b)$ are of opposite sign, then there exists at least one point $c \in\ ]a, b[$ such that $f'(c) = 0$.*

Proof. Let us suppose that $f'(a) > 0$ and $f'(b) < 0$, then there exists intervals $]a, a + h[$ and $]b - h, b[$, $h > 0$, such that

$$f(x) > f(a), \ \forall\ x \in\ ]a, a + h[\qquad \ldots(1)$$
$$f(x) > f(b), \ \forall\ x \in\ [b - h, b[. \qquad \ldots(2)$$

Now, since f is finitely differentiable, then it is continuous in $[a, b]$ and hence it is bounded on $[a, b]$ and attains its supremum and infimum at least once in $[a, b]$. [$\because$ A continuous function attains its supremum and infimum at least once in $[a, b]$].

Thus if M is the supremum of f in $[a, b]$, then there exists $c \in [a, b]$ such that $f(c) = M$. It is clear from (1) and (2) that the upper bound is not attained at the end points a and b so that $c \in\ ]a, b[$.

Now we shall prove $f'(c) = 0$.

If $f'(c) > 0$, then there exists an interval $]c, c + h]$, $h > 0$, such that $f(x) > f(c) = M$, $\forall\ x \in\ ]c, c + h[$, which is not possible, since M is the supremum of the function $f(x)$ in $[a, b]$.

If $f'(c) < 0$ then there exists an interval $[c - h, c[$, $h > 0$ such that $f(x) > f(c) = M$, $\forall\ x \in [c - h, c[$, which is not possible.

Hence, we conclude that $f'(c) = 0$.

SOLVED EXAMPLES

Example 1. *Prove that the function $f(x) = |x| + |x - 1|$ is not differentiable at $x = 0$ and $x = 1$.*

Solution. Here, we observe that

(i) $|x| = -x$ and $|x - 1| = 1 - x$ when $x < 0$.
(ii) $|x| = x$ and $|x - 1| = 1 - x$, when $0 \leq x \leq 1$.
(iii) $|x| = x$ and $|x - 1| = x - 1$ when $x > 1$.

Hence, the given function can be rewritten as

$$f(x) = -x + 1 - x \quad = 1 - 2x, \quad x < 0$$
$$= x + 1 - x \quad = 1 \quad , \quad 0 \le x \le 1$$
$$= x + x - 1 \quad = 2x - 1, \quad x > 1.$$

Now, firstly we check the differentiability of $f(x)$ at $x = 0$.

We have $$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{1-1}{h}$$
$$= 0$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$
$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$
$$= \lim_{h \to 0} \frac{1 - 2(-h) - 1}{-h}$$
$$= \lim_{h \to 0} \frac{2h}{-h} = -2.$$

Thus $Rf'(0) \neq Lf'(0)$. Therefore the given function is not differentiable at $x = 0$.

Now, we check the differentiability of $f(x)$ at $x = 1$.

We have $$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{[2(1+h) - 1] - 1}{h}$$
$$= \lim_{h \to 0} \frac{2 + 2h - 2}{h}$$
$$= 2$$

and $$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$
$$= \lim_{h \to 0} \frac{1-1}{-h}$$
$$= 0.$$

Thus $Rf'(1) \neq Lf'(1)$. Therefore the given function is **not** differentiable at $x = 1$.

Example 2. *Prove that the function $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$, where $|x|$ is the absolute value of x.*

Solution. Firstly, we check the continuity of the function $f(x)$ at $x = 0$.

We have $$f(0) = |0| = 0$$
$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$
$$= \lim_{h \to 0} |h| = \lim_{h \to 0} h = 0$$
$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$
$$= \lim_{h \to 0} |-h| = \lim_{h \to 0} h = 0.$$

$\therefore$ $$f(0+0) = f(0) = f(0-0).$$

Hence, $f(x)$ is continuous at $x = 0$.

Now, we check the differentiability of the function $f(x)$ at $x = 0$.

We have, $$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{|h| - 0}{h} = 1$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{|-h| - 0}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$

$$\Rightarrow \quad Rf'(0) \neq Lf'(0).$$

Hence, the function $f(x)$ is not differentiable at $x = 0$.

Example 3. *If*

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$$

then, show that $f(x)$ is continuous and differentiable everywhere.

Solution. We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} (0+h)^2 \sin \frac{1}{0+h}$$

$$= \lim_{h \to 0} h^2 \sin \frac{1}{h} = 0$$

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} (0-h)^2 \sin \frac{1}{0-h}$$

$$= - \lim_{h \to 0} h^2 \sin \frac{1}{h} = 0$$

$$f(0) = 0$$

$$\Rightarrow \quad f(0+0) = f(0) = (0-0)$$

Hence, the function is continuous at $x = 0$.

Now $$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^2 \sin\left(-\frac{1}{h}\right) - 0}{-h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

$$\Rightarrow \quad Rf'(0) = Lf'(0)$$

Hence, $f(x)$ is differentiable at $x = 0$.

Example 4. *A function f is defined as follows :*

$$f(x) = 1 + x \quad \text{if} \quad x \leq 2$$
$$= 5 - x \quad \text{if} \quad x > 2.$$

Test the character of the function at $x = 2$ as regards its differentiability.

Solution. Here

$$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{5 - (2+h) - 3}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

and $$Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \frac{1 + (2-h) - 3}{-h}$$

$$= \lim_{h \to 0} \frac{-h}{-h} = 1.$$

$$\Rightarrow \quad Rf'(2) \neq Lf'(2)$$

Hence, the function $f(x)$ is not differentiable at $x = 2$.

Example 5. *Examine the following curve for differentiability at $x = 1$*

$$f(x) = \begin{cases} x^2 & , \text{ for } x \le 0 \\ 1 & , \text{ for } 0 < x \le 1 \\ 1/x & , \text{ for } x > 1. \end{cases}$$

Solution. Here,

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h \to 0} \frac{1 - 1 - h}{h(1+h)}$$

$$= \lim_{h \to 0} \frac{-1}{1+h} = -1$$

Now $$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{1-1}{-h} = \lim_{h \to 0} 0 = 0.$$

$\Rightarrow$ $Lf'(1) \neq Rf'(1)$.

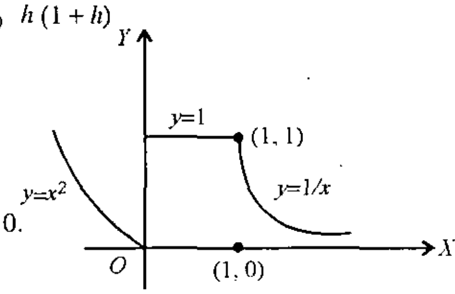


Fig. 1

Hence, $f(x)$ is not differentiable at $x = 1$.

The graph of the function consist of the following curves

(i) $y = x^2$ for $x \le 0$, (parabola)

(ii) $y = 1$ for $0 < x \le 1$, (straight line)

(iii) $y = 1/x$ for $x > 1$, (rectangular hyperbola).

• TEST YOURSELF–1

1. Let $f(x) = \begin{cases} -1 & , -2 \le x \le 0 \\ x - 1 & , 0 < x \le 2. \end{cases}$

   Test the differentiability.

2. Determine the set of all points where the function $f(x) = \dfrac{x}{1 + |x|}$ is differentiable.
3. Show that $f(x) = |x - 1|$, $0 \le x \le 2$ is not differentiable at $x = 1$.
4. Show that $f(x) = \begin{cases} -x & , \text{ when } x < 0 \\ x & , \text{ when } x \ge 0 \end{cases}$ is not differentiable at $x = 0$.
5. Show that the function $f(x) = \begin{cases} 2 + x, & \text{if } x \ge 0 \\ 2 - x, & \text{if } x < 0 \end{cases}$ is not differentiable at $x = 0$.
6. Show that the function $f(x) = |x - 1| + 2|x - 2| + 3|x - 3|$ is not differentiable at the points 1, 2 and 3.
7. Show that the function $f(x) = \begin{cases} x & , 0 \le x < 1 \\ 2 - x & , x \ge 1 \end{cases}$ is differentiable at $x = 1$.

ANSWER

1. Not differentiable **2.** Differentiable in $]-\infty, \infty[$

• 2.4. ROLLE'S THEOREM

Statement. *If a function f defined on* $[a, b]$ *is such that it is*

(i) *continuous in the closed interval* $[a, b]$,

(ii) *differentiable in the open interval* $]a, b[$,

(iii) $f(a) = f(b)$,

then there exists at least one value of x, say c, $(a < c < b)$ *such that*

$$f'(c) = 0.$$

Proof. Since, the function $f(x)$ is continuous on $[a, b]$

$\Rightarrow$ $f(x)$ is bounded [$\because$ Every continuous function is bounded]

$\Rightarrow$ $f(x)$ attains its bounds [$\because$ A function, which is continuous on a closed bounded interval $[a, b]$, then it attains its bound on $[a, b]$]

Let M and m are the supremum and infimum of $f(x)$ respectively.

Now there are two possibilites

(i) $M = m$ (ii) $M \neq m$.

(i) If $M = m$, then obviously $f(x)$ is a constant function, and therefore its derivative is zero, *i.e.*

$$f'(x) = 0 \quad \forall\, x \in\,]a, b[.$$

(ii) If $M \neq m$, then at least one of the numbers M and m must be different from the equal values $f(a)$ and $f(b)$.

Let us assume $M \neq f(a)$.

Now, since, every continuous function on a closed interval attains its supremum, therefore, there exists a real number c in $[a, b]$ such that $f(c) = M$. Also since $f(a) \neq m \neq f(b)$.

Therefore $c \neq a$ and $c \neq b$, this implies that $c \in\,]a, b[$.

Now, $f(c)$ is the supremum of f on $[a, b]$

$$\therefore \quad f(x) \leq f(c) \quad \forall\, x \in [a, b] \qquad \ldots (1)$$

(By the definition of supremum)

In particular,

$$f(c - h) \leq f(c) \quad h > 0.$$

$$\Rightarrow \quad \frac{f(c - h) - f(c)}{-h} \geq 0 \qquad \ldots(2)$$

Since $f'(x)$ exists at each point of $]a, b[$, and hence, $f'(c)$ exists.

Hence, from (2)

$$Lf'(c) \geq 0. \qquad \ldots(3)$$

Similarly from (1)

$$f(c + h) \leq f(c), \quad h > 0.$$

Then by the same arguments

$$Rf'(c) \leq 0. \qquad \ldots(4)$$

Since $f(x)$ is differentiable in $]a, b[\, \Rightarrow f'(c)$ exist

$$\Rightarrow \quad Lf'(c) = f'(c) = Rf'(c).$$

Now from (3), (4) and (5)

$$f'(c) = 0.$$

Similarly we can consider the case

$$M = f(a) \neq m.$$

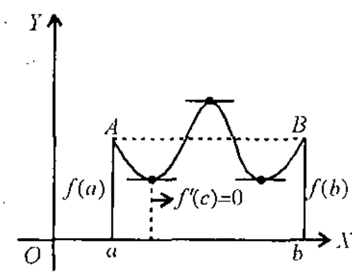


Fig. 2

Geometrical Interpretation of Rolle's Theorem.

Geometrically, Rolle's theorem means that if the curve $y = f(x)$ is continuous from $x = a$ to $x = b$, has a definite tangent at each point of $]a, b[$ and the ordinates at the extremities are equal, then there exists at least one point between a and b at which the tangent is parallel to x-axis.

• 2.5. LAGRANGE'S VALUE THEOREM

Let f be a function defined on $[a, b]$ such that

(i) *f is continuous on $[a, b]$*

(ii) *f is differentiable on $]a, b[$.*

Then, there exists a real number $c \in\,]a, b[$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. Let us define a function $F(x)$ such that

$$F(x) = f(x) + Ax \quad \forall\, x \in [a, b] \qquad \ldots(1)$$

where A is a constant to be suitably chosen such that

$$F(a) = F(b).$$

Now

(i) Since, f is continuous on $[a, b]$ and Ax is continuous on $[a, b]$ therefore, F is continuous on $[a, b]$ [$\because$ sum of two continuous functions is again continuous]

(ii) Similarly F is differentiable on $[a, b]$

(iii) $F(a) = F(b) \Rightarrow -A = \dfrac{f(b)-f(a)}{b-a}$. ...(2)

Hence, we find that F satisfy all the conditions of Rolle's Theorem on $[a, b]$ and consequently, there exists a real number $c \in]a, b[$ such that $F'(c) = 0$, this gives

$$f'(c) + A = 0$$

$$\Rightarrow \quad -A = f'(c). \quad ...(3)$$

Now, from (2) and (3), we have

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Geometrical Interpretation of Lagrange's Mean Value Theorem.

If the curve $y = f(x)$ is continuous from $x = a$ and $x = b$ and has *a* definite tangent at each point on the curve between $x = a$ and $x = b$, then, geometrically, the first mean value theorem means that there is at least one point between $x = a$ and $x = b$ on the curve where the tangent to the curve parallel to the chord joining the points $(a, f(a))$ and $(b, f(b))$.

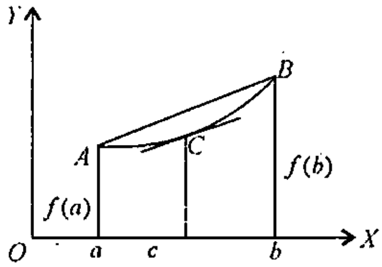


Fig. 3

Let ACB be the graph of the function $y = f(x)$ then the co-ordinate of the points A and B are given by $(a, f(a))$ and $(b, f(b))$ respectively. If the chord AB makes an angle θ with the x-axis, then

$$\tan\theta = \frac{f(b)-f(a)}{b-a} = f'(c), \quad \text{where } a < c < b.$$

Important Deduction from the First Mean Value Theorem :

Theorem 1. *If a function $f(x)$ satisfies the conditions of mean value theorem then*

(*i*) $f'(x) = 0 \;\; \forall x \in]a, b[\;\; \Rightarrow f$ *is constant on* $[a, b]$,

(*ii*) $f'(x) > 0 \;\; \forall x \in]a, b[\;\; \Rightarrow f$ *is strictly increasing on* $[a, b]$,

and (*iii*) $f'(x) < 0 \;\; \forall x \in]a, b[\;\; \Rightarrow f$ *is strictly decreasing on* $[a, b]$.

Proof. (i) Let x_1, x_2 (where $x_1 > x_2$) be any two distinict points of $[a, b]$, then by Lagrange's mean value theorem,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) = 0, \;\; x_1 < c < x_2 \quad ...(1)$$

$$\Rightarrow \quad f(x_2) = f(x_1).$$

$\Rightarrow$ function keeps the same value. Therefore $f(x)$ is constant on $[a, b]$.

(ii) From (1), we have

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \text{ for some } c \in]x_1, x_2[.$$

But $\quad f'(c) > 0 \quad$ [$\because f'(x) > 0 \;\; \forall x \in [a, b]$]

$\Rightarrow \quad f(x_2) - f(x_1) > 0$

$\Rightarrow \quad f(x_2) > f(x_1)$

Thus $\quad x_2 > x_1 \Rightarrow f(x_2) > f(x_1) \;\; \forall x_1, x_2 \in [a, b]$.

Therefore, f is strictly increasing on $[a, b]$.

(iii) Same as (ii).

• 2.6. CAUCHY'S MEAN VALUE THEOREM

Theorem 2. *Let f and g be two functions defined on $[a, b]$ such that*

(*i*) *f and g are continuous on* $[a, b]$,

(*ii*) *f and g are differentiable on* $]a, b[$,

and (*iii*) $g'(x) \neq 0$ *for any point of* $]a, b[$.

Then, there exists a real number $c \in]a, b[$, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Let us define a function

$$F(x) = f(x) + A \cdot g(x) \qquad ...(1)$$

where A is a constant, to be suitably chosen such that

$$F(a) = F(b). \qquad ...(2)$$

Now, the function F is the sum of two continuous and differentiable functions. Therefore

(i) F is continuous on $[a, b]$,

(ii) F is differentiable on $]a, b[$,

and **(iii)** $F(a) = F(b)$.

Then, by Rolle's theorem, there must exists a real number c between a and b such that

$$F'(c) = 0.$$

Here $\quad F'(x) = f'(x) + Ag'(x)$

$\quad F'(c) = 0 \quad \Rightarrow f'(c) + Ag'(c) = 0$

$\Rightarrow \quad -A = \dfrac{f'(c)}{g'(c)}.$

Now $\quad F(a) = F(b) \Rightarrow f(a) + Ag(a) = f(b) + Ag(b)$

$\Rightarrow \quad -A = \dfrac{f(b)-f(a)}{g(b)-g(a)}.$

From (3) and (4), we have

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Geometrical Interpretation of Cauchy's Mean Value Theorem.

(1) Under suitable conditions, Cauchy's mean value theorem geometrically means that there is an ordinate $x = c$ between $x = a$ and $x = b$, such that the tangents at the points where $x = c$ cut the graphs of the function $f(x)$ and $\dfrac{f(b)-f(a)}{g(b)-f(a)} g(x)$ are mutually parallel.

(2) The ratio of the mean rates of increase of two functions in an interval is equal to the ratio of the actual rates of increase of the functions at some point within the interval.

SOLVED EXAMPLES

Based on Rolle's, Lagrange's and Cauchy's Mean Value Theorem :

Example 1. *Discuss the applicability of Rolle's theorem in the interval* $[-1, 1]$ *to the function* $f(x) = |x|$.

Solution. Here, we have $\quad f(x) = |x|$

$$\begin{rcases} f(-1) = 1 \\ f(1) = 1 \end{rcases} \Rightarrow f(1) = f(-1).$$

Now, the function $f(x)$ is continuous throughout the closed interval $[-1, 1]$ but $f(x)$ is not differentiable at $x = 0 \in]-1, 1[$. Hence, Rolle's theorem is not satisfied (due to the second condition).

Example 2. *Discuss the applicability of Rolle's theorem to*

$$f(x) = log\left[\frac{x^2+ab}{(a+b)\,x}\right] \text{ in the interval } [a, b].$$

Solution. Here, we have

$$f(a) = \log\left[\frac{a^2+ab}{(a+b)\,a}\right] = \log 1 = 0$$

and

$$f(b) = \log\left[\frac{b^2+ab}{(a+b)\,b}\right] = \log 1 = 0$$

$\Rightarrow \quad f(a) = f(b) = 0.$

Also, it can be easily seen that $f(x)$ is continuous on $[a, b]$ and differentiable on $]a, b[$.

Thus all the three conditions of Rolle's theorem are satisfied.

Hence $f'(x) = 0$ for at least one value of x in $]a, b[$.

Now $$f'(x) = 0 \Rightarrow \frac{2x}{x^2 + ab} - \frac{1}{x} = 0$$

$$\Rightarrow \quad 2x^2 - (x^2 + ab) = 0$$

$$\Rightarrow \quad x^2 = ab \text{ or } x = \sqrt{ab}.$$

Obviously $\sqrt{ab} \in]a, b[$ [being the geometric mean of a and b]

Hence, the Rolle's theorem is verified.

Example 3. *Verify Rolle's theorem for*

$$f(x) = x(x+3)\, e^{-x/2} \text{ in } [-3, 0].$$

Solution. Here, we have

$$f(x) = x(x+3)\, e^{-x/2}$$

$$\therefore \quad f'(x) = (2x+3)\, e^{-x/2} + (x^2+3x)\, e^{-x/2} \cdot \left(-\frac{1}{2}\right)$$

$$= e^{-x/2}\left[2x + 3 - \frac{1}{2}[x^2 + 3x]\right]$$

$$= -\frac{1}{2}[x^2 - x - 6]\, e^{-x/2}.$$

$\Rightarrow$ $f'(x)$ exist for every value of x in the interval $[-3, 0]$. Hence, $f(x)$ is differentiable and hence, continuous in the interval $[-3, 0]$.

Also, we have

$$f(-3) = f(0) = 0$$

$\Rightarrow$ All the three conditions of Rolle's theorem are satisfied. So

$$f'(x) = 0 \Rightarrow \frac{1}{2}(x^2 - x - 6)\, e^{-x/2} = 0$$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow x = 3, -2.$$

Since, the value $x = -2$ lies in the open interval $]-3, 0[$, the Rolle's theorem is verified.

Example 4. *If $a + b + c = 0$, then show that the quadratic equation $3ax^2 + 2bx + c = 0$ has at least one root in* $]0, 1[$.

Solution. Let us define a funtion $f(x)$ such that

$$f(x) = ax^3 + bx^2 + cx + d.$$

Here we have $f(0) = d$ and $f(1) = a + b + c + d = d$ $\quad (\because \; a + b + c = 0)$

Obviously, $f(x)$ is continuous and differentiable in $]0, 1[$ (being a polynomial).

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem in $[0, 1]$. Hence, there is at least one value of x in the open interval $]0, 1[$ where $f'(x) = 0$

i.e., $\quad 3ax^2 + 2bx + c = 0$ has at least one root in $]0, 1[$.

Examply 5. *Find 'c' of the mean value theorem, if*

$f(x) = x(x-1)(x-2)$; $a = 0$, $b = 1/2$.

Solution. Here, we have $f(a) = f(0) = 0$

$$f(b) = f\left(\frac{1}{2}\right) = \frac{3}{8}$$

$$\therefore \quad \frac{f(b) - f(a)}{b - a} = \frac{\frac{3}{8} - 0}{\frac{1}{2} - 0} = \frac{3}{4}.$$

Now $$f(x) = x^3 - 3x^2 + 2x$$

$$\therefore \quad f'(x) = 3x^2 - 6x + 2$$

$$\Rightarrow \quad f'(c) = 3c^2 - 6c + 2.$$

Putting all these values in the Lagrange's mean value theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c), \; (a < c < b)$$

we get $$\frac{3}{4} = 3c^2 - 6c + 2 \text{ or } c = 1 \pm \frac{\sqrt{21}}{6}.$$

Hence $c = \frac{1-\sqrt{21}}{6}$ lies in the open interval $]0, \frac{1}{2}[$ therefore, it is the required value.

Example 6. *Show that*

$$\frac{x}{1+x} < \log(1+x) < x, \text{ for } x > 0.$$

Solution. Let, $$f(x) = \log(1+x) - \frac{x}{1+x}.$$

Obviously $$f(0) = 0.$$

Then $$f'(x) = \frac{1}{1+x} - \frac{1.(1+x) - x.1}{(1+x)^2}$$

$$= \frac{1}{1+x} - \frac{1}{(1+x)^2}$$

$$= \frac{x}{(1+x)^2}.$$

Here, we observe that $f'(x) > 0$, for $x > 0$.

$\Rightarrow$ $f(x)$ is monotonically increasing in the interval $[0, \infty[$. Therefore

$$f(x) > f(0), \quad \text{for} \quad x > 0$$

$$\Rightarrow \quad \left[\log(1+x) - \frac{x}{1+x}\right] > 0, \quad \text{for} \quad x > 0$$

$$\Rightarrow \quad \log(1+x) > \frac{x}{1+x}, \quad \text{for} \quad x > 0. \qquad ...(1)$$

Now, let $$F(x) = x - \log(1+x)$$

Obviously $$F(0) = 0.$$

Then $$F'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}.$$

Here, we observe that $F'(x) > 0$, for $x > 0$. Hence $F(x)$ is mono- tonically increasing in the interval $[0, \infty[$.

$$\therefore \quad F(x) > F(0), \quad \text{for } x > 0$$

$$\Rightarrow \quad [x - \log(1+x)] > 0, \quad \text{for } x > 0$$

$$\Rightarrow \quad x > \log(1+x), \quad \text{for } x > 0. \qquad ...(2)$$

Now, from (1) and (2), we get

$$\frac{x}{1+x} < \log(1+x) < x, \text{ for } x > 0.$$

Example 7. *Verify Lagrange's mean value theorem for the function*

$$f(x) = \sin x \text{ in } \left[0, \frac{\pi}{2}\right].$$

Solution. The function $f(x) = \sin x$ is continuous and differentiable on **R**. Hence it is continuous as well as differentiable in $[0, \pi/2]$. Then, by Lagrange's mean value theorem, there must exists at least one c in $]0, \pi/2[$ such that

$$\frac{f(\pi/2) - f(0)}{\pi/2 - 0} = f'(c). \qquad ...(1)$$

Here $f(0) = 0$, $f(\pi/2) = 1$

$$f'(x) = \cos x \quad \Rightarrow \quad f'(c) = \cos c.$$

Put all these values in (1), we have

$$\frac{1-0}{\pi/2} = \cos c \Rightarrow \cos c = \frac{2}{\pi} \Rightarrow c = \cos^{-1}\left(\frac{2}{\pi}\right).$$

Since, $0 < 2/\pi < 1$, therefore the value of $c = \cos^{-1}\left(\frac{2}{\pi}\right)$ lies in $\left]0, \frac{\pi}{2}\right[$, so the required value of c. Hence, Lagrange's mean value theorem is satisfied.

Example 8. *Verify Cauchy's mean value theorem for the function x^2 and x^3 in the interval* $[1, 2]$.

Soltuion. Let us suppose $f(x) = x^2$ and $g(x) = x^3$.

Then, obviously $f(x)$ and $g(x)$ are continuous in $[1, 2]$ and differentiable in $]1, 2[$.

Also $g(x) = 3x^2 \neq 0$ for any point in $]1, 2[$.

Then, by Cauchy's mean value theorem there exist at least one real number $c \in]1, 2[$, such that

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}. \qquad ...(1)$$

After solving, we get $c = \frac{14}{9}$, which lies in the open interval $]1, 2[$. Hence, Cauchy's mean value theorem is verified.

Example 9. *If $f(x)$, $g(x)$ and $h(x)$ are functions such that*

(i) $f(x)$, $g(x)$ and $h(x)$ are continuous on $[a, b]$

(ii) $f(x)$, $g(x)$ and $h(x)$ are differentiable on $]a, b[$,

then
$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} = 0 \quad \text{where } c \in]a, b[.$$

Solution. Consider the function $f(x)$ such that

$$F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} = 0. \qquad ...(1)$$

Obviously, $F(x)$ is of the form $A f(x) + B g(x) + C h(x)$, where A, B, C are some real numbers.

From the condition (i) and (ii), $F(x)$ is continuous on $[a, b]$ and differentiable on $]a, b[$.

Also $F(a) = F(b) = 0$.

$\Rightarrow$ $F(x)$ satisfies all the conditions of Rolle's theorem. Hence, there exists a $c \in]a, b[$ such that $F'(c) = 0$

i.e.,
$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} = 0.$$

Example 10. *Show that*

$$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$$

where
$$0 < \alpha < \theta < \beta < \frac{\pi}{2}.$$

Solution. Let $f(x) = \sin x$ and $g(x) = \cos x$, for $x \in [\alpha, \beta]$, where $0 < \alpha < \beta < \pi/2$.

$\therefore \qquad f'(x) = \cos x$ and $g'(x) = -\sin x$.

It can be easily seen that both the functions $f(x)$ and $g(x)$ are continuous in the closed internal $[\alpha, \beta]$ and differentiable in the open interval $]\alpha, \beta[$.

Hence, by Cauchy's mean value theorem there exist at least one $\theta \in R$, $\theta \in]\alpha, \beta[$ such that

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}$$

$$\Rightarrow \quad \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$

$$\Rightarrow \quad \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \text{ where } 0 < \alpha < \theta < \beta < \pi/2$$

• SUMMARY

- Left hand derivative $= Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$
- Right hand derivative $= Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$
- f is differentiable at $x = a$ if $Lf'(a) = Rf'(a)$.

- Continuity is the necessary condition for the existence of a finite derivative of a function f.
- $(f \pm g)'(a) = f'(a) \pm g'(a)$
- $(cf)'(a) = cf'(a)$
- $(fg)'(a) = f(a)\, g'(a) + f'(a)\, g(a)$
- $\left(\frac{1}{f}\right)'(a) = -\frac{f'(a)}{[f(a)]^2}$
- $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\, g(a) - f(a)\, g'(a)}{[g(a)]^2}$
- **Chain Rule :** $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.
- **Rolle's theorem :** If a function f defined on $[a, b]$ is such that it is
  (i) continuous on $[a, b]$,
  (ii) differentiable on (a, b), and
  (iii) $f(a) = f(b)$
  then there exists atleast one value of x say c in (a, b) such that $f'(c) = 0$.
- **Lagrange's Mean Value Theorem :** If a function f defined on $[a, b]$ is such that it is
  (i) continuous on $[a, b]$, and
  (ii) differentiable on (a, b),
  then there exists atleast one value of x say c in (a, b) such that
  $$\frac{f(b) - f(a)}{b - a} = f'(c).$$
- **Cauchy's Mean Value Theorem :** If functions f and g defined on $[a, b]$ are such that
  (i) f and g are continuous on $[a, b]$,
  (ii) f and g are differentiable on (a, b), and
  (iii) $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists atleast one value of x say $c \in (a, b)$ such that
  $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

• STUDENT ACTIVITY

1. State and prove Lagrange's mean value theorem.

2. Show that $\frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \theta,\ 0 < \alpha < \theta < \beta < \frac{\pi}{2}$.

• TEST YOURSELF-2

1. Discuss the applicability of Rolle's theorem of the following functions :
   (a) $f(x) = 2 + (x-1)^{2/3}$ in the interval [0, 2]
   (b) $f(x) = x^2$ in $2 \le x \le 3$
   (c) $f(x) = \tan x$ in $0 \le x \le \pi$
   (d) $f(x) = x^4 - 3x^2 + 4$ in the interval [– 4, 4]
2. Show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x - 1 = 0$.
3. Let $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + \frac{a_{n-1}}{2} + a_n = 0$. Show that there exists at least one real x between 0 and 1 such that
$$a_0x^n + a_1x^{n-1} + \ldots + a_n = 0.$$
4. Verify the Rolle's theorem for the following functions :
   (a) $f(x) = x^3 - 3x + 2$ on the interval [1, 2]
   (b) $f(x) = x^2$ on the interval [– 1, 1]
   (c) $f(x) = x^4 - 1$ on the interval [– 1, 1]
5. Verify the Lagrange's theorem for the following functions :
   (a) $f(x) = x^3$ in [– 1, 1]   (b) $f(x) = \sin x$ in $[0, \pi/2]$
   (c) $f(x) = 2x^2 - 7x + 10$, $x \in [2, 5]$.
6. Find the value of c, of mean value theorem, when
   (a) $f(x) = \sqrt{x^2 - 4}$ in the interval [2, 4]
   (b) $f(x) = 2x^2 + 3x + 4$ in the interval [1, 2]
   (c) $f(x) = x(x-1)$ in the interval [1, 2].
7. (a) If $f(x) = \sqrt{x}$ and $g(x) = 1/\sqrt{x}$, then show by Cauchy's mean value theorem c is the geometric mean between a and b.
   (b) If $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$, then c is the harmonic mean between a and b.

ANSWERS

1.	(a) Not applicable	(b) Not applicable	
	(c) Not applicable	(d) Verified	
4.	(a) Verified	(b) Verified	
	(c) Verified	(d) Verified	
5.	(a) Verified	(b) Verified	(c) Verified.
6.	(a) $c = \pm\sqrt{6}$	(b) $c = 3/2$	(c) $c = 3/2$.

OBJECTIVE EVALUATION

Fill in the Blanks :

1. Every differentiable function is ....
2. Every continuous function is ....
3. Sum and difference of two differentiable functions is again ....
4. The first mean value theorem is also known as ....
5. If $f'(x) > 0$ then $f(x)$ is known as ....

True or False :

Write T for true and F for False :

1. Every continuous function is differentiable. *(T/F)*
2. Every differentiable function is continuous. *(T/F)*
3. Every differentiable function is bounded. *(T/F)*
4. A function is said to be differentiable if $Lf'(x) = Rf'(x)$. *(T/F)*
5. If $f'(x) > 0$. Then $f(x)$ is an increasing function. *(T/F)*
6. The function $f(x) = |x|$ is differentiable everywhere. *(T/F)*

Multiple Choice Questions :

Choose the most appropriate one :

1. A function $f : [a, b] \to R$ is said to be differentiable if f is :
   (a) differentiable at each point of $[a, b]$
   (b) differentiable at the ends points only
   (c) differentiable at each point of $[a, b]$ except the end points
   (d) none of these.
2. A function $f(x)$ is said to be differentiable at $x = a$, if :
   (a) right hand and left hand derivatives at a exist and equal
   (b) only right hand derivative must exist
   (c) only left hand derivative must exist
   (d) none of these.
3. Every differentiable function is :
   (a) necessarily continuous (b) never continuous
   (c) may or may not be continuous (d) none of these.
4. If f is finitely differentiable in a closed interval $[a, b]$ and $f'(a), f'(b)$ are of opposite sign, then :
   (a) $f'(c) = 0 \ \forall c \in [a, b]$ (b) $f'(c) = 0$ for at least one $c \in]a, b[$
   (c) $f'(c) = 0 \ \forall c \in]a, b[$ (d) None of these.
5. Every continuous function is :
   (a) necessarily differentiable (b) never differentiable
   (c) may or may not be differentiable (d) none of these.

ANSWERS

Fill in the Blanks :

1. Continuous 2. not necessarily differentiable
3. Differentiable 4. Lagrange's mean value theorem
5. Increasing function

True or False :

1. F 2. T 3. T 4. T 5. T 6. F

Multiple Choice Questions :

1. (a) 2. (a) 3. (a) 4. (b) 5. (c)

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UNIT 3

LIMIT AND CONTINUITY OF FUNCTIONS OF TWO VARIABLES

STRUCTURE

- Function of Two Variables
- Limit
- Neighbourhood
- Algebra of Limits
- Continuity of a Function of Two Variables
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What are functions of two variables ?
- What are simultaneous and iterated limits ?
- How to check the continuity of functions of two variables ?

• 3.1. FUNCTION OF TWO VARIABLES

Let f be a function from a set of ordered pair of real numbers to a set of real numbers; then f is said to be a real valued function of two real variables or, briefly, a real function of two variables. The value that f assumes at the arguments (x, y) is naturally written $f(x, y)$. Let us suppose this value is called z. Then we write $z = f(x, y)$, where x and y are the independent variables and z is the dependent variable.

We shall write $z = z(x, y)$, which means that we are considering some function of two variables, where the independent variables are x and y and the dependent variable is z.

If to each pair of values of x and y there exists only one value of z, then the function is said to be single valued function. On the other hand, if there are two or more values of z correspond to some x and y or all of the values assigned to x and y, the function is called multiple valued.

Let $f(x, y)$ is a function of two variables x and y, then we say $\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y)$ exists and is equal to l, if for every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$|f(x, y) - l| < \varepsilon$$

for all values of x and y in the neighbourhood of (x_0, y_0) defined by

$$|x - x_0| < \delta, \quad |y - y_0| < \delta.$$

• 3.2. LIMIT

Let $f(x, y)$ is a function of two variables x and y, we define several kind of limits.

If (x_0, y_0) is the limiting point of a set of values on two dimensional space, then we have

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y), \quad \lim_{x \to x_0} \lim_{y \to y_0} f(x, y), \quad \lim_{y \to y_0} \lim_{x \to x_0} f(x, y).$$

Then limit of the first kind is known as simultaneous limit and the last two types are known as iterated limits.

Non-existence of a limit. To determine whether a simultaneous limit exist or not, it is a difficult matter but a simple consideration, which we describe, says us to decide about the non-existence of a limit.

If $$\lim_{(x, y) \to (a, b)} f(x, y) = l$$

and if ϕ is any function of a single real variable such that

$$\lim_{x \to a} \phi(x) = b.$$

Then $$\lim_{x \to a} f[x, \phi(x)] = l.$$

Thus, we can determine two functions ϕ_1 and ϕ_2 such that

$$\lim_{x \to a} f[x, \phi_1(x)] \neq \lim_{x \to a} f[x, \phi_2(x)].$$

Then, we can say that the simultaneous limit

$$\lim_{(x, y) \to (a, b)} f(x, y)$$

does not exist.

SOLVED EXAMPLES

Example 1. *Show that the limit,* $\lim_{(x, y) \to (0, 0)} f(x, y)$, *where* $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ *does not exists.*

Solution. Here $$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

Now taking $y = m \cdot x$, then, we have $\lim_{x \to 0} f(x, mx) = \dfrac{1 - m^2}{1 + m^2}$, which depend upon m.

Since, $\lim_{x \to 0} f(x, mx)$ is not unique. Hence $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exists.

Example 2. *Show that the simultaneous limit,* $\lim_{\substack{x \to 0 \\ y \to 0}} \dfrac{xy^3}{x^2 + y^6}$ *does not exists.*

Solution. Let (x, y) tends to $(0, 0)$ through the line $y = x$, which is a line through the origin. Put $y = x$, in the function, we get

$$\lim_{x \to 0} \frac{x^4}{x^2 + x^6} = \lim_{x \to 0} \frac{x^2}{1 + x^2} = 0.$$

Again, let $(x, y) \to (0, 0)$ through the curve $x = y^3$.

Put $y = x^3$, in the given function, we obtain

$$\lim_{y \to 0} \frac{y^6}{y^6 + y^6} = \frac{1}{2}$$

$\Rightarrow$The limit obtained by two different methods are different.

Hence, the simultaneous limit does not exist.

3.3. NEIGHBOURHOOD

Rectangular neighbourhood of a point (a, b).

Let neighbourhood of a point (a, b) in the xy-plane be determined by a positive number δ is the square bounded by the lines

$$x = a - \delta, \quad x = a + \delta,$$
$$y = b - \delta, \quad y = b + \delta.$$

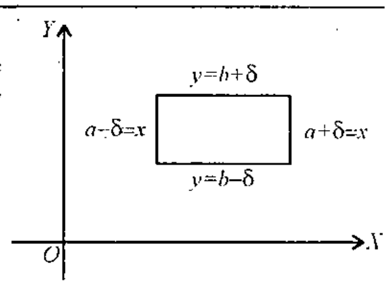

Fig. 1

If a point (x, y) lies in the neighbourhood, we have

$$a - \delta < x < a + \delta \Rightarrow |x - a| < \delta$$
$$b - \delta < y < b + \delta \Rightarrow |y - b| < \delta.$$

The centre of the square is at the point (a, b). This square is called the neighbourhood of the point (a, b). For every value of δ, we will get a neighbourhood.

Circular Neighbourhood of a Point (*a, b*).

A circular neighbourhood of a point (a, b) in R^2 is the set of all points (x, y) whose distance from the point (a, b) is less than some given $\delta > 0$ *i.e.* the set of all points (x, y) such that

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta$$

i.e. $$|(x, y) - (a, b)| < \delta$$

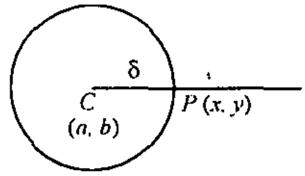

Fig. 2

Here, $|(x, y) - (a, b)|$ stands for distance between the points (x, y) and (a, b) *i.e.*, $\sqrt{(x-a)^2 + (y-b)^2}$.

• 3.4. ALGEBRA OF LIMITS

If $\lim\limits_{(x, y) \to (a, b)} f(x, y) = l_1$ and $\lim\limits_{(x, y) \to (a, b)} g(x, y) = l_2$

then

(i) $\lim\limits_{(x, y) \to (a, b)} [f(x, y) + g(x, y)] = l_1 + l_2$

(ii) $\lim\limits_{(x, y) \to (a, b)} [f(x, y) - g(x, y)] = l_1 - l_2$

(iii) $\lim\limits_{(x, y) \to (a, b)} [f(x, y) \cdot g(x, y)] = l_1 \cdot l_2$

(iv) $\lim\limits_{(x, y) \to (a, b)} \left[\dfrac{f(x, y)}{g(x, y)}\right] = \dfrac{l_1}{l_2}$ (provided $l_2 \neq 0$).

Theorem 1. *Let $z = f(x, y)$ be a function, then $\lim\limits_{(x, y) \to (a, b)} f(x, y)$, if exists is unique.*

Proof. Let $z = f(x, y)$ be a function.

Let if possible

$$\lim_{(x, y) \to (a, b)} f(x, y) = l_1$$

and $$\lim_{(x, y) \to (a, b)} f(x, y) = l_2.$$

Now, to prove $l_1 = l_2$.

Let us first suppose $\lim\limits_{(x, y) \to (a, b)} f(x, y) = l_1$, then by the definition of limit, we have

"given $\varepsilon > 0$, $\exists$ $\delta_1 > 0$ such that"

$$|f(x, y) - l_1| < \varepsilon/2 \quad \text{whenever} \quad |(x, y) - (a, b)| < \delta_1 \qquad \text{...(1)}$$

Now suppose $\lim\limits_{(x, y) \to (a, b)} f(x, y) = l_2$

"given $\varepsilon > 0$, $\exists$ $\delta_2 > 0$ such that"

$$|f(x, y) - l_2| < \varepsilon/2 \quad \text{whenever} \quad |(x, y) - (a, b)| < \delta_2 \qquad \text{...(2)}$$

Let $\delta = \min\{\delta_1, \delta_2\}$.

Hence, we have

$$|f(x, y) - l_1| < \varepsilon/2 \text{ and } |f(x, y) - l_2| < \varepsilon/2 \text{ whenever } |(x, y) - (a, b)| < \delta$$

Now, consider

$$\begin{aligned} |l_1 - l_2| &= |l_1 - f(x, y) + f(x, y) - l_2| \\ &\leq |l_1 - f(x, y)| + |f(x, y) - l_2| \qquad \text{(By triangular inequality)} \\ &\leq |f(x, y) - l_1| + |f(x, y) - l_2| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

Since ε is arbitrary and small, hence

$$l_1 - l_2 = 0 \Rightarrow l_2 = l_2$$

$\Rightarrow$ limit of a function is unique.

SOLVED EXAMPLES

Example 1. *Let $f : R^2 \to R$ be defined by*

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

Prove that, $\lim_{(x, y) \to (0, 0)} f(x, y)$ *does not exists.*

Solution. Since, if $\lim_{(x, y) \to (a, b)} f(x, y)$ exists, then this limit is independent of the path along which we approach the point (a, b).

Let $(x, y) \to (0, 0)$ along the path $y = mx$, where $m \in \mathbf{R}$

As $x \to 0$, from $y = mx$, we have $y \to 0$.

Consider $\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \dfrac{xy}{x^2 + y^2}$

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \frac{xy}{x^2 + y^2}$$

Putting $y = mx$

$$= \lim_{x \to 0} \frac{x\, mx}{x^2 + m^2x^2}$$

$$= \lim_{x \to 0} \frac{mx^2}{x^2(1 + m^2)}$$

$$= \lim_{x \to 0} \frac{m}{1 + m^2} = \frac{m}{1 + m^2}$$

which will be different for different values of m.

Therefore, $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exists.

Example 2. *If* $f(x, y) = y \sin \dfrac{1}{x} + x \sin \dfrac{1}{y}$ *where* $x \neq 0,\ y \neq 0$.

Then prove that

$$f(x, y) \to 0 \ \text{as}\ (x, y) \to (0, 0).$$

Solution. Let ε be any given arbitrary small positive number since, $\varepsilon > 0$, let us take $\delta = \varepsilon$.

Also, let $|x - 0| < \varepsilon/2, \quad |y - 0| < \varepsilon/2$

$$\therefore \quad |(x, y) - (0, 0)| = \sqrt{(x - 0)^2 + (y - 0)^2}$$

$$\leq |x - 0| + |y - 0|$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon$$

$$\Rightarrow \quad |f(x, y) - 0| = \left| y \sin \frac{1}{x} + x \sin \frac{1}{y} \right|$$

$$\leq \left| y \sin \frac{1}{x} \right| + \left| x \sin \frac{1}{y} \right|$$

$$\leq |y| \left| \sin \frac{1}{x} \right| + |x| \left| \sin \frac{1}{y} \right|$$

$$\leq |y| + |x| \qquad \left[\because \left| \sin \frac{1}{x} \right| \leq 1 \text{ and } \left| \sin \frac{1}{y} \right| \leq 1 \right]$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon$$

$$\Rightarrow \quad |f(x, y) - 0| < \varepsilon$$

Hence $\lim_{(x, y) \to (0, 0)} f(x, y) = 0$

Example 3. *Show that the simultaneous limit* $\lim_{(x, y) \to (0, 0)} \dfrac{2xy^2}{x^2 + y^4}$ *does not exit.*

Solution. Let $y = mx$,

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{2x\,(mx)^2}{x^2+m^4x^4}$$

$$= \lim_{x\to 0} \frac{2xm^2}{1+m^4x^2} = 0$$

when $y^2 = x$, then

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{2x\,(x)}{x+x^2}$$

$$= \lim_{x\to 0} \frac{2x^2}{2x^2} = 1.$$

Thus along the line $y = mx$, and along the curve $y^2 = x$, imultaneous limit are different, hence the limit does not exit.

• 3.5. CONTINUITY OF A FUNCTION OF TWO VARIABLES

(i) A function $f(x, y)$ is said to be continuous at the point (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y)$ exists and equal to $f(a, b)$.

(ii) A function $f(x, y)$ is said to be continuous at (a, b), if for every $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(x,y) - f(a,b)| < \varepsilon, \text{ whenever } |x-a| < \delta,\ |y-b| < \delta.$$

SOLVED EXAMPLES

Example 1. *Show that the function $f : R^2 \to \mathbf{R}$ defined by*

$$f(x,y) = \begin{cases} \dfrac{xy\,(x^2-y^2)}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , \textit{otherwise} \end{cases}$$

is continuous at $(0, 0)$.

Solution. Let $\varepsilon > 0$ be given.

Now, let us suppose $|x - 0| < \sqrt{\varepsilon}$ and $|y - 0| < \sqrt{\varepsilon}$

Consider,

$$|f(x,y) - f(0,0)| = \left| \frac{xy\,(x^2-y^2)}{x^2+y^2} - 0 \right|$$

$$= |xy| \left| \frac{x^2-y^2}{x^2+y^2} \right|$$

$$\leq |xy| \qquad \left[\because\ |x^2-y^2| \leq |x^2+y^2|\ \left|\frac{x^2-y^2}{x^2+y^2}\right| \leq 1 \right]$$

$\Rightarrow \qquad |f(x,y) - f(0,0)| \leq |x|\,|y|$

$\Rightarrow \qquad |f(x,y) - f(0,0)| < \sqrt{\varepsilon}\,.\,\sqrt{\varepsilon}$

$\Rightarrow \qquad |f(x,y) - f(0,0)| < \varepsilon.$

Hence, we have $\lim_{(x,y)\to(0,0)} f(x, y)$ exists and equal to $f(0, 0)$.

Example 2. *Show that the function*

$$f(x,y) = \frac{xy^3}{x^2+y^6},\quad x \neq 0, y \neq 0 \textit{ and } f(0,0) = 0$$

is not continuous at $(0, 0)$ *in* (x, y).

Solution. Here $f(0, 0) = 0$ (given)

Let us suppose $(x, y) \to (0, 0)$ through the curve $x = y^3$.

Then $\qquad \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} \dfrac{y^6}{y^6+y^6} = \dfrac{1}{2}.$

Again, let $(x, y) \to (0, 0)$ through the line $y = x$, then

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{x \cdot x^3}{x^2 + x^6}$$

$$= \lim_{x \to 0} \frac{x^2}{1 + x^4} = 0$$

Since, the limit obtained by two different approaches are different, Therefore, $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exists. Hence, the given function is not continuous.

Example 3. *Show that the function*

$$f(x, y) = \frac{xy}{\sqrt{(x^2 + y^2)}}, \quad x \neq 0, y \neq 0 \text{ and } f(0, 0) = 0$$

is continuous at the origin in (x, y) *together.*

Solution. Let $\varepsilon > 0$ be given.

Let us suppose $\quad x = r \cos \theta, \; y = r \sin \theta$

Then $\quad f(r \cos \theta, r \sin \theta) = \dfrac{r^2 \cos \theta \sin \theta}{r \sqrt{\sin^2 \theta + \cos^2 \theta}} = r \cos \theta \sin \theta$

$$= \frac{1}{2} r \sin 2\theta$$

Now, consider

$$|f(r \cos \theta, r \sin \theta) - f(0, 0)| = |f(r \cos \theta, r \sin \theta)|$$

$$= \left| \frac{1}{2} r \sin 2\theta \right|$$

$$= \frac{1}{2} r \, |\sin 2\theta|$$

$$\leq \frac{1}{2} r \qquad [\because \; |\sin 2\theta| \leq 1]$$

Now, if we choose $r = 2\varepsilon$.

Therefore we have $\varepsilon > 0$ such that

$$|f(r \cos \theta, r \sin \theta)| < \varepsilon \quad \text{for all values of } \theta \qquad \ldots(1)$$

Equation (1) is true for all points within a circle about the origin and radius $r = 2\varepsilon$. Therefore $f(r \cos \theta, r \sin \theta)$ is uniformly continuous in r for all values of θ. Hence $f(x, y)$ is continuous in (x, y) at the origin.

Example 4. *Let* $f : R^2 \to \mathbf{R}$ *be a function, defined by*

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{when } (x, y) \neq (0, 0) \\ 0 & \text{when } (x, y) = (0, 0) \end{cases}$$

Show that f is not continuous at $(0, 0)$ *but is continuous in each variable separately.*

Solution. For point (x, y) on the x-axis, we have $y = 0$ and $f(x, y) = f(x, 0) = 0$, so the function has the constant value, 0, everywhere on the x-axis, which gives that $f(x, y)$ is continuous at $x = 0$.

Similarly $f(x, y)$ has the constant value, 0, at all points on the y-axis, so if we put $x = 0$, the function $f(x, y)$ is continuous at $y = 0$. Now, we shall show that $f(x, y)$ is not continuous at origin.

Let $\quad y = x.$

Then $\quad f(x, y) = f(x, x) = \dfrac{x^2}{2x^2} = \dfrac{1}{2}.$

Also $\quad f(0, 0) = 0$ (given)

Since there are points on the line arbitrarily close to the origin and since $f(0, 0) \neq \frac{1}{2}$, the function of two variable $f(x, y)$ is not continuous at the origin.

• SUMMARY

- **Simultaneous limit :**

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y).$$

- **Interated limits :**

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x, y) \text{ and } \lim_{y \to y_0} \cdot \lim_{x \to x_0} f(x, y)$$

- **Rectangular *nbd* of a point (a, b)**

$$|x - a| < \delta, \quad |y - b| < \delta$$

- **Circular *nbd* of a point (a, b)**

$$|(x, y) - (a, b)| < \delta$$

or $$\sqrt{(x - a)^2 + (y - b)^2} < \delta$$

- **Continuity of $f(x, y)$ at (a, b)**

$f(x, y)$ is continuous at (a, b) if

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(a, b).$$

• STUDENT ACTIVITY

1. Show that $\lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{xy^3}{x^2 + y^6}$ does not exist.

2. Show that the function $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$, $x \neq 0, y \neq 0$ and $f(0, 0) = 0$ is continuous at $(0, 0)$.

• TEST YOURSELF

1. Let $f: R^2 \to \mathrm{R}$ be defined by $f(x, y) = x^2 + y^2$, show that

$$\lim_{(x, y) \to (0, 0)} f(x, y) = 0.$$

2. Show that $\lim_{(x, y) \to (0, 0)} \frac{2x^3 - y^3}{x^2 + y^2} = 0.$

3. Show that $\lim_{(x, y) \to (0, 0)} \frac{2x - y}{x^2 + y^2}$ does not exists.

4. Prove that $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exists, where

$$f(x, y) = \frac{xy^2}{x^2 + y^4}, \quad (x, y) \neq (0, 0).$$

5. Let
$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{when } (x, y) \neq (0, 0) \\ 0, & \text{when } (x, y) = (0, 0) \end{cases}$$

Show that $f(x, y)$ is continuous at $(0, 0)$.

6. Let $f(x, y) = \begin{cases} x \sin \dfrac{1}{y}, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$

is continuous at $(0, 0)$.

7. Show that the function $f(x, y)$ be defined as

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } x \neq 0, y \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$$

is discontinuous at the point $(0, 0)$.

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UNIT 4

PARTIAL DIFFERENTIATIONS

STRUCTURE

- Introduction
- Rules of Partial Differentiation
- Partial derivatives of the higher order
- Homogeneous functions
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :
- How to find the partial derivatives of the functions ?
- How to apply Euler's theorem ?

• 4.1. INTRODUCTION

We know that the differential coefficient of $f(x)$ with respect to x is $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$, provided this limit exists, and it is denoted by

$$f'(x) \quad \text{or} \quad \frac{d}{dx}[f(x)].$$

If $u = f(x, y)$ be a continuous function of two independent variables x and y, then the differential coefficient of u w.r.t. x (regarding y as constant) is called the partial derivative or partial differential co-efficient of u w.r.t. x and is denoted by various symbols such as

$$\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), f_x.$$

Symbolically, if $u = f(x, y)$, then

$$\lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

if it exists, it is called the partial derivative or partial differential co-efficient of u w.r.t. x and is denoted by

$$\frac{\partial u}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x \quad \text{or} \quad u_x.$$

Similarly, by keeping x constant and allowing y alone to vary, we can define the partial derivative or partial differential coefficient of u w.r.t. y. It is denoted by any one of the symbols $\frac{\partial u}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), f_y$.

Symbolically $$\frac{\partial u}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}.$$

provided this limit exists.

If $$u = ax^2 + 2hxy + by^2$$

then $$\frac{\partial u}{\partial x} = 2ax + 2hy$$

and $$\frac{\partial u}{\partial y} = 2hx + 2by.$$

• 4.2. RULES OF PARTIAL DIFFERENTIATION

Rule (1) :

(a) If u is a function of x, y and we are to differentiate partially w.r.t. x, then y is treated as constant.

(b) Similarly, if we are to differentiate u partially w.r.t. y, then x is treated as constant.

(c) If u is a function of x, y, z and we are to differentiate partially w.r.t. x, then y and z are treated as constant.

Rule (2) : If $z = u \pm v$, where u and v are functions of x and y, then

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} \pm \frac{\partial v}{\partial y}.$$

Rule (3) : If $z = uv$, where u and v are functions of x and y, then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

and
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}$$

Rule (4) : If $z = \frac{u}{v}$, where u, v are functions of x and y, then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

and
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}.$$

Rule (5) : If $z = f(u)$, where u is a function of x and y, then

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y}.$$

REMARKS

- Partial means a 'part of'.
- If z is a function of one variable x, then $\frac{\partial z}{\partial x} = \frac{dz}{dx}$.
- If z is a function of two variable x_1 and x_2, we get $\frac{\partial z}{\partial x_1}$ and $\frac{\partial z}{\partial x_2}$.
- If z is a function of n variables $x_1, x_2 \ldots x_n$, we can find $\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots \frac{\partial z}{\partial x_n}$.

Symmetric Funcrtion of x and y. *A function $u = u(x, y)$ is said to be symmetric if, on interchanging x and y, u remains unchanged.*

• 4.3. PARTIAL DERIVATIVES OF THE HIGHER ORDER

We can find partial derivative of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ just as we found those of u for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are itself functions of x and y.

The four derivatives, thus obtained, called the *second order partial derivatives of u or $f(x, y)$ are*

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right); \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right); \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

and are denoted as

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y\,\partial x}, \frac{\partial^2 u}{\partial x\,\partial y}, \frac{\partial^2 u}{\partial y^2} \text{ or } f_{xx}, f_{yx}, f_{xy}, f_{yy}.$$

REMARKS

- $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$ and $\dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$
- $\dfrac{\partial^2 u}{\partial x\,\partial y} \neq \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial u}{\partial y}$.
- The partial derivatives $\dfrac{\partial^2 u}{\partial x\,\partial y}$ and $\dfrac{\partial^2 u}{\partial y\,\partial x}$ are distinguished by the order in which u is successively differentiated w.r.t. x and y, but it will be seen that, in general they are equal.

SOLVED EXAMPLES

Example 1. *Verify that* $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$, *when* $u = x \sin y + y \sin x$.

Solution.
$$u = x \sin y + y \sin x. \qquad \ldots (1)$$

Differentiating partially both sides of (1) w.r.t. x and y respectively, we get

$$\frac{\partial u}{\partial x} = \sin y + y \cos x \qquad \ldots (2)$$

and
$$\frac{\partial u}{\partial y} = x \cos y + \sin x. \qquad \ldots (3)$$

Again differentiating (2) partially w.r.t. y and (3) w.r.t. x, we get

$$\frac{\partial^2 u}{\partial y\,\partial x} = \cos y + \cos x \qquad \ldots (4)$$

and
$$\frac{\partial^2 u}{\partial x\,\partial y} = \cos y + \cos x \qquad \ldots (5)$$

From (4) and (5), we obtain

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

Example 2. *If* $u = x^2y + y^2z + z^2x$, *then show that* $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = (x + y + z)^2$.

Solution. Given that

$$u = x^2y + y^2z + z^2x. \qquad \ldots (1)$$

Differentiating partially both sides of (1) w.r.t. x, y and z respectively, we get

$$\frac{\partial u}{\partial x} = 2xy + z^2 \qquad \ldots (2)$$

$$\frac{\partial u}{\partial y} = x^2 + 2yz \qquad \ldots (3)$$

and
$$\frac{\partial u}{\partial z} = y^2 + 2zx. \qquad \ldots (4)$$

Adding (2), (3) and (4), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 2xy + z^2 + x^2 + 2yz + y^2 + 2zx$$
$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$
$$= (x + y + z)^2.$$

Hence proved.

Example 3. *If* $z = f(x + ay) + \phi\,(x - ay)$, *prove that* $\dfrac{\partial^2 z}{\partial y^2} = a^2\,\dfrac{\partial^2 z}{\partial x^2}$.

Solution. Given that

$$z = f(x + ay) + \phi(x - ay). \qquad \ldots (1)$$

Differentiating partially both sides of (1) w.r.t. x and y respectively, we get

$$\frac{\partial z}{\partial x} = f'\,(x + ay) + \phi'(x - ay) \qquad \ldots (2)$$

and $$\frac{\partial z}{\partial y} = af'(x+ay) - a\phi'(x-ay). \quad \ldots (3)$$

Again differentiating partially both sides of (2), w.r.t. x and (3) w.r.t. y, we get

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \quad \ldots (4)$$

and $$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2\phi''(x-ay). \quad \ldots (5)$$

From (4) and (5), we get,

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

Hence Proved.

Example 4. If $u = log\,(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

Solution. Here, we have

$$u = \log(x^3 + y^3 + z^3 - 3xyz).$$

Differentiating partially with respect to x, we have

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3x^2 - 3yz)$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{3(x^2 - yz)}{x^3 + y^3 + z^3 - 3xyz} \quad \ldots (1)$$

Similarly $$\frac{\partial u}{\partial y} = \frac{3(y^2 - zx)}{x^3 + y^3 + z^3 - 3xyz} \quad \ldots (2)$$

and $$\frac{\partial u}{\partial z} = \frac{3(z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz} \quad \ldots (3)$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - yz - zx - xy)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - yz - zx - xy)}{(x+y+z)(x^2 + y^2 + z^2 - yz - zx - xy)} = \frac{3}{x+y+z}$$

Also, $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$= 3\left[-\frac{1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2} - \frac{1}{x+y+z)^2}\right] = -\frac{9}{(x+y+z)^2}$$

Example 5. If $u = f(x)$, where $r^2 = x^2 + y^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

Solution. Here, we have

$$r^2 = x^2 + y^2$$

$$\Rightarrow \quad 2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

and $$2r\frac{\partial r}{\partial y} = 2y \quad \text{or} \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \ldots (1)$$

Since $$u = f(r)$$

$\Rightarrow$ $$\frac{\partial u}{\partial r} = [f'(r)] \cdot \frac{\partial r}{\partial x} = \frac{x}{r} f'(r)$$

and $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left[x \cdot \frac{1}{r} \cdot f'(r)\right]$$

$$= 1 \cdot \frac{1}{r} f'(r) + [xf'(r)]\left[-\frac{1}{r^2} \cdot \frac{\partial r}{\partial x}\right] + \frac{x}{r}[f''(r)] \frac{\partial r}{\partial x}$$

$$= \frac{1}{r} f'(r) - \frac{x}{r^2} \cdot \frac{x}{r} f'(r) + \frac{x^2}{r^2} f''(r)$$

$$= \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r). \qquad \ldots (2)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r) \qquad \ldots (3)$$

Adding (2) and (3), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2}{r} f'(r) - \frac{x^2 + y^2}{r^3} f'(r) + \frac{x^2 + y^2}{r^2} f''(r)$$

$$= \frac{2}{r} f'(r) - \frac{r^2}{r^3} f'(r) + \frac{r^2}{r^2} \cdot f''(r)$$

$$= \frac{2}{r} f'(r) - \frac{1}{r} f'(r) + f''(r) = f''(r) + \frac{1}{r} f'(r).$$

Example 6. *If* $x^x y^y z^z = c$. *Show that at* $x = y = z$.

$$\frac{\partial^2 z}{\partial x \, \partial y} = -[x \log ex]^{-1}.$$

Solution. Here, we have

$$x^x y^y z^z = c. \qquad \ldots (1)$$

Here, we observe that z can be regarded as a function of two independent variable x and y. Taking logs of both the sides of (1), we have

$$x \log x + y \log y + z \log z = \log c. \qquad \ldots (2)$$

Diff. (2) partially w.r.t. x, we get

$$x \cdot \frac{1}{x} + 1 \cdot \log x + \left[z \cdot \frac{1}{z} + 1 \cdot \log z\right] \frac{\partial z}{\partial x} = 0.$$

$\Rightarrow$ $$\frac{\partial z}{\partial x} = -\frac{(1 + \log x)}{(1 + \log z)} \qquad \ldots (3)$$

Similarly differentiating (2), w.r.t. y, we get

$$\frac{\partial z}{\partial y} = -\frac{(1 + \log y)}{(1 + \log z)} \qquad \ldots (4)$$

Also, $$\frac{\partial^2 z}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\left(\frac{1 + \log y}{1 + \log z}\right)\right]$$

$$= -(1 + \log y) \frac{\partial}{\partial x} [(1 + \log z)^{-1}]$$

$$= -(1 + \log y) \cdot \left[-(1 + \log z)^{-2} \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x}\right]$$

$$= \frac{(1 + \log y)}{z(1 + \log z)^2} \cdot \left[-\left(\frac{1 + \log x}{1 + \log z}\right)\right]$$

At $x = y = z$, we have

$$\frac{\partial^2 z}{\partial x \, \partial y} = -\frac{(1 + \log x)^2}{x(1 + \log x)^3} = -\frac{1}{x(1 + \log x)} = \frac{1}{x[\log e + \log x]} \quad [\because \log e = 1]$$

$$= -\frac{1}{x \log(ex)} = [-x \log ex]^{-1}.$$

• TEST YOURSELF

1. Find the first partial derivatives of

(i) $\log(x^2+y^2)$ (ii) $\cos^{-1}\left(\frac{x}{y}\right)$

2. Find the second order partial derivatives of $\log(e^x+e^y)$.

3. Verify that $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y\,\partial x}$.

(i) $z=\log(y\sin x+x\sin y)$ (ii) $z=\log\left(\frac{x^2+y^2}{xy}\right)$

(iii) $z=\log\left(\frac{x^2+y^2}{x+y}\right)$ (iv) $z=\sin^{-1}\frac{x}{y}$ (v) $z=x^y$.

4. If $x=r\cos\theta$, $y=r\sin\theta$, show that $\frac{\partial r}{\partial x}=\frac{\partial x}{\partial r}$, $\frac{\partial x}{r\,\partial\theta}=r\frac{\partial\theta}{\partial x}$

5. If $u=\log(\tan x+\tan y)$, prove that $\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}=2$.

6. If $u=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$, prove that $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{x^2-y^2}{x^2+y^2}$.

7. If $u=2(ax+by)^2-(x^2+y^2)$ and $a^2+b^2=1$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

8. If $u=\log(x^3+y^3-x^2y-xy^2)$, prove that

(i) $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=2(x+y)^{-1}$ (ii) $\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 u}{\partial y^2}=-4(x+y)^{-2}$

9. If $u=f(x+2y)+g(x-2y)$, show that $4\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$.

10. If $u=e^{xyz}$, show that $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=(1+3xyz+x^2y^2z^2)\,e^{xyz}$.

ANSWERS

1. (i) $\frac{2x}{x^2+y^2}$, $\frac{2y}{x^2+y^2}$ (ii) $-\frac{1}{\sqrt{y^2-x^2}}$, $\frac{x}{y\sqrt{y^2-x^2}}$

2. $\frac{e^{x+y}}{(e^x+e^y)^2}$, $\frac{-e^{x+y}}{(e^x+e^y)^2}$, $\frac{e^{x+y}}{(e^x+e^y)^2}$

• 4.4. HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ is said to be homogeneous functon of order n, if the degree of each of its terms in x and y is equal to n. Thus

$$a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\ldots\ldots+a_{n-1}xy^{n-1}+a_ny^n \quad \ldots(1)$$

is homogeneous function in x and y of order n.

REMARKS

➤ This definition of homogeneity applies to polynomial functions only. To widen the concept of homogeneity so as to bring even transcendental functions within its scope, we define u as a homogeneous function in x and y of order or degree n, if it can be expressed in the form of $x^nf\left(\frac{y}{x}\right)$.

This definition also covers the polynomial function (1), which can be written as

$$x^n\left[a_0+a_1\frac{y}{x}+a_2\left(\frac{y}{x}\right)^2+\ldots+a_n\left(\frac{y}{n}\right)^n\right]=x^nf\left(\frac{y}{x}\right).$$

∴ It is a homogeneous function of order n.

➤ A homogeneous function in x and y of degree n can also be written as $y^nf\left(\frac{x}{y}\right)$.

Some Important Theorem

Theorem 1. *If u is a homogeneous function of x and y of degree n, then each of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are homogeneous function of degree $(n-1)$.*

Proof. Since, u is a homogeneous function of x and y of degree n therefore, u can be expressed as

$$u = x^n f\left(\frac{y}{x}\right). \quad \ldots (1)$$

Now from (1)

$$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= x^{n-1}\left[nf\left(\frac{y}{x}\right) + f'\left(\frac{y}{x}\right)\left(-\frac{y}{x}\right)\right]$$

$$= x^{n-1} \times \text{a function of } \frac{y}{x}$$

$$= x^{n-1} g\left(\frac{y}{x}\right) \text{(say)}.$$

which is a homogeneous function of degree $(n - 1)$.

Also, $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right).\frac{1}{x}\ x^{n-1} f'\left(\frac{y}{x}\right) = x^{n-1} \times \text{a function of} \left(\frac{y}{x}\right)$$

$$= x^{n-1}\,\alpha\left(\frac{y}{x}\right) \text{say}$$

which is a homogeneous function of degree $(n - 1)$.

Theorem 2. [Euler's Theorem on Homogeneous functions].

If u is a homogeneous function of x and y of order n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Proof. Since, u is a homogeneous function of x and y of degree n, then it can be expressed as

$$u = x^n f\left(\frac{y}{x}\right).$$

$\therefore$ $$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = nx^{n-1} f\left(\frac{y}{x}\right) - yx^{n-2} f'\left(\frac{y}{x}\right).$$

Also, $$\frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right).\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right).$$

Now, $$\text{L.H.S.} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x\left[nx^{n-1} f\left(\frac{y}{x}\right) - yx^{n-2} f'\left(\frac{y}{x}\right)\right] + yx^{n-1} f'\left(\frac{y}{x}\right)$$

$$= nx^n f\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right) + yx^{n-1} f'\left(\frac{y}{x}\right) = nx^n f\left(\frac{y}{x}\right) = nu$$

$$= \text{R.H.S.}$$

Theorem 3. *If u is a homogeneous function of x and y of degree n, then prove that*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)\,u.$$

Proof. Since, u is a homogeneous function of x, y of degree n therefore, by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu. \quad \ldots (1)$$

Differentiating (1) partially w.r.t. x, get

$$\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(nu)$$

($\because$ Each of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ is a function of both x and y)

$\Rightarrow$ $$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}.1 + y\frac{\partial^2 u}{\partial x\,\partial y} = n\frac{\partial u}{\partial x}$$

$\Rightarrow$ $$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (n-1)\frac{\partial u}{\partial x} \quad \ldots (2)$$

Again differentiating (2) partially w.r.t., y, we get

$$y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial x\,\partial y}=(n-1)\frac{\partial u}{\partial y}. \qquad \because \frac{\partial^2 u}{\partial y\,\partial x}=\frac{\partial^2 u}{\partial x\,\partial y} \qquad \ldots (3)$$

Now, multiply (2) by x; (3) by y and then adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$=(n-1)\,nu=n(n-1)\,u.$$

SOLVED EXAMPLES

Example 1. *Verify the Euler's theorem for the function $u = axy + byz + czx$.*

Solution. Here, we have

$$u = axy + byz + czx$$

which is a homogeneous function of x, y and z of degree 2.

To verify the Euler's theorem, we must show

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2u.$$

Now $\qquad \dfrac{\partial u}{\partial x}=ay+cz,\quad \dfrac{\partial u}{\partial y}=ax+bz,\quad \dfrac{\partial u}{\partial z}=by+cx.$

$$\therefore \qquad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=x\,(ay+cz)+y\,(ax+bz)+z\,(by+cx)$$

$$=2\,(axy+byz+czx)=2u.$$

Hence, Euler's theorem is verified.

Example 2. *If $u=\sin^{-1}\left[\dfrac{x^2+y^2}{x+y}\right]$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\tan u$.*

Solution. Here, we have

$$\sin u=\frac{x^2+y^2}{x+y}$$

Let $\qquad v=\dfrac{x^2+y^2}{x+y}$

$\Rightarrow$ v is a homogeneous of x and y of degree 1.

Then, by Euler's theorem, we have

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=v \qquad \ldots (1)$$

$$v=\sin u \;\Rightarrow\; \frac{\partial v}{\partial x}=\cos u\frac{\partial u}{\partial x} \text{ and } \frac{\partial v}{\partial y}=\cos u\frac{\partial u}{\partial y}.$$

Putting now these values in (1), we get

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=v$$

$$\Rightarrow \qquad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{v}{\cos u}=\frac{\sin u}{\cos u}=\tan u.$$

Example 3. *If $u=\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$.*

Solution. $\qquad u=\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$

$$=x^0\left[\sin^{-1}\left(\frac{1}{y/x}\right)+\tan^{-1}\left(\frac{y}{x}\right)\right]=x^0\left(\text{A function of }\frac{y}{x}\right)$$

$\Rightarrow$ u is a homogeneous function of order 0.

Then, by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0\times u=0.$$

• SUMMARY

- If $f = f(x, y)$, then $\dfrac{\partial f}{\partial x} = \lim\limits_{\delta x \to 0} \dfrac{f(x + \delta x, y) - f(x, y)}{\delta x}$

  $\dfrac{\partial f}{\partial y} = \lim\limits_{\delta y \to 0} \dfrac{f(x, y + \delta y) - f(x, y)}{\delta y}$

  Similarly we can find $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y^2}$ etc.

- **Homogeneous functions :** A function of the form $u = x^n f\left(\dfrac{y}{x}\right)$ is said to be a homogeneous function of degree n.
- **Euler's theorem on homogeneous function :** If u is a homogeneous function of degree n in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

• STUDENT ACTIVITY

1. If $z = f(x + ay) + \phi(x - ay)$, prove that

$$\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$$

2. If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

• TEST YOURSELF

1. Verify the Euler's theorem for the following functions

   (i) $u = \dfrac{x(x^3 - y^3)}{x^3 + y^3}$ (ii) $u = x^n \sin\left(\dfrac{y}{x}\right)$ (iii) $u = x^n \sin\left(\dfrac{y}{x}\right)$

   (iv) $u = \dfrac{1}{\sqrt{x^2 + y^2}}$ (v) $u = x^4 \log \dfrac{y}{x}$

2. (i) If $u = xf\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$.

   (ii) If $u = f\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

(iii) If $u = xy\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$.

3. If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

4. If $u = \tan^{-1}\frac{y}{x}$, show that (using Euler's Theorem)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

5. If $u = \sin^{-1}\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

6. (i) If $u = \log\frac{x^4 + y^4}{x + y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

(ii) If $u = \log\frac{x^3 + y^3}{x + y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$.

OBJECTIVE EVALUATION

Fill in the Blanks :

1. $\cos^{-1}\frac{y}{x}$ is a homogeneous function of degree ....
2. If $\phi = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$, then $x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y}$ is ....
3. If $u = e^{my}\cos mx$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =$ ....

True or False :

Write 'T' for True and 'F' for False :

1. An exporession in which every term is of same degree is called homogeneous function. *(T/F)*
2. In homogeneous function every term is not necessarily of same degree. *(T/F)*
3. If u is a homogeneous function of x and y of degree n, then $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are also homogeneous function f degree n. *(T/F)*

Multiple Choice Questions :

Choose the most appropriate one :

1. $\sin^{-1}(y/x)$ is a homogeneous function of degree :
   (a) 1 (b) 2 (c) 3 (d) 0
2. If $z = xyf\left(\frac{y}{x}\right)$ then $x\frac{2z}{2x} + y\frac{2z}{2y}$ is equal to :
   (a) z (b) $2z$ (c) xy (d) yz
3. If $f = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$ then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is :
   (a) f (b) $2f$ (c) $\tan f$ (d) $\sin f$

ANSWERS

Fill in the Blanks :

1. 0   2. $\tan \phi$ 3. 0

True or False :

1. T   2. F   3. F

Multiple Choice Questions :

1. (d)   2. (b)   3. (c).

UNIT 5

JACOBIANS

STRUCTURE

- Some definitions of Jacobians
- Theorems of Jacobians
- Jacobian of Implicit Functions
- Necessary and Sufficient Condition for a Jacobian to be Vanished
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- What is Jacobian ?
- How to find Jacobian of a function ?

5.1. SOME DEFINITIONS OF JACOBIANS

(i) If u and v are the functions of two independent variables x and y, then the determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u and v with respect to x and y.

It is denoted by $\dfrac{\partial(u, v)}{\partial(x, y)}$ or $J(u, v)$.

(ii) If u, v and w are the functions of three independent variables x, y and z, then the determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

is called the Jacobian of u, v and w with respect to x, y and z.

It is denoted by $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ or $J(u, v, w)$.

(iii) If $u_1, u_2, \ldots, u_n$ are the n functions of independent variables $x_1, x_2, \ldots, x_n$, then the determinant

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \dfrac{\partial u_n}{\partial x_3} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \ldots, u_n$ with respect to variables $x_1, x_2, \ldots, x_n$.

It is denoted by $\dfrac{\partial (u_1, u_2, \ldots, u_n)}{\partial (x_1, x_2, \ldots, x_n)}$ **or** $J(u_1, u_2, \ldots, u_n)$.

• 5.2. IMPORTANT THEOREMS ON JACOBIANS

Theorem 1. *If u_1, u_2 are functions of y_1, y_2 and y_1, y_2 are functions of x_1, x_2 then*

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}.$$

Proof. Since u_1, u_2 are function of y_1, y_2. Also y_1, y_2 are function of x_1, x_2; therefore, we get

$$\left.\begin{aligned} \frac{\partial u_1}{\partial x_1} &= \frac{\partial u_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} &= \frac{\partial u_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} &= \frac{\partial u_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_1} \\ \text{and} \quad \frac{\partial u_2}{\partial x_2} &= \frac{\partial u_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_2} \end{aligned}\right] \quad \ldots(1)$$

Now, consider

$$\frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} \end{vmatrix} \times \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_1} & \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_2} \\ \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_1} & \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

Now, using relation (1), we get

$$\frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix} = \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}.$$

Theorem 2. *If u_1, u_2, u_3 are functions of y_1, y_2, y_3 and y_1, y_2, y_3 are functions of x_1, x_2, x_3, then*

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \frac{\partial(u_1, u_2, u_3)}{\partial(y_1, y_2, y_3)} \cdot \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}.$$

Proof. Since u_1, u_2 and u_3 are functions of y_1, y_2 and y_3.
Also y_1, y_2 and y_3 are function of x_1, x_2 and x_3 therefore, we get

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_1} + \frac{\partial u_1}{\partial y_3} \cdot \frac{\partial y_3}{\partial x_1} = \sum_{i=1}^{3} \frac{\partial u_1}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_1}$$

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_2} + \frac{\partial u_1}{\partial y_3} \cdot \frac{\partial y_3}{\partial x_2} = \sum_{i=1}^{3} \frac{\partial u_1}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_2}$$

Similarly $\quad \dfrac{\partial u_1}{\partial x_3} = \displaystyle\sum_{i=1}^{3} \frac{\partial u_1}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_3},$

$$\frac{\partial u_2}{\partial x_1} = \sum_{i=1}^{3} \frac{\partial u_2}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_1},$$

$$\frac{\partial u_2}{\partial x_2} = \sum_{i=1}^{3} \frac{\partial u_2}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_2},$$

$$\frac{\partial u_2}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial u_2}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_3},$$

$$\frac{\partial u_3}{\partial x_1} = \sum_{i=1}^{3} \frac{\partial u_3}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_1},$$

$$\frac{\partial u_3}{\partial x_2} = \sum_{i=1}^{3} \frac{\partial u_3}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_2},$$

and
$$\frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial u_3}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_3}.$$

Now, consider

$$\frac{\partial(u_1, u_2, u_3)}{\partial(y_1, y_2, y_3)} \cdot \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} & \frac{\partial u_1}{\partial y_3} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} & \frac{\partial u_2}{\partial y_3} \\ \frac{\partial u_3}{\partial y_1} & \frac{\partial u_3}{\partial y_2} & \frac{\partial u_3}{\partial y_3} \end{vmatrix} \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma \frac{\partial u_1}{\partial y_i} \frac{\partial y_i}{\partial x_1} & \Sigma \frac{\partial u_1}{\partial y_i} \frac{\partial y_i}{\partial x_2} & \Sigma \frac{\partial u_1}{\partial y_i} \frac{\partial y_i}{\partial x_3} \\ \Sigma \frac{\partial u_2}{\partial y_i} \frac{\partial y_i}{\partial x_1} & \Sigma \frac{\partial u_2}{\partial y_i} \frac{\partial y_i}{\partial x_2} & \Sigma \frac{\partial u_2}{\partial y_i} \frac{\partial y_i}{\partial x_3} \\ \Sigma \frac{\partial u_3}{\partial y_i} \frac{\partial y_i}{\partial x_1} & \Sigma \frac{\partial u_3}{\partial y_i} \frac{\partial y_i}{\partial x_2} & \Sigma \frac{\partial u_3}{\partial y_i} \frac{\partial y_i}{\partial x_3} \end{vmatrix}$$

Putting the values of each element of the determinant from the above relation, we get

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

$$= \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}.$$

Theorem 3. *If functions u, v, w of three independent variables x, y and z are not independent, then the Jacobian of u, v, w with respect to x, y, z vanishes.*

Proof. Here, we have, the functions u, v and w (of three independent variables x, y and z) are not independent.

Then there will be a relation

$$F(u, v, w) = 0 \quad \text{...(A)}$$

which will connect these independent variables.

Differentiating (A), with respect to x, y and z, we get

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial x} = 0 \quad \text{...(1)}$$

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial y} = 0 \quad \text{...(2)}$$

and
$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial z} = 0 \quad \text{...(3)}$$

Eliminating $\frac{\partial F}{\partial u}$, $\frac{\partial F}{\partial v}$ and $\frac{\partial F}{\partial w}$ from (1), (2) and (3), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

$$= \frac{\partial(u, v, w)}{\partial(x, y, z)} = 0.$$

• 5.3. JACOBIAN OF IMPLICIT FUNCTIONS

Theorem 1. *If u_1, u_2 be the implicit functions of x_1, x_2 that is*

$$F_1(u_1, u_2, x_1, x_2) = 0$$
$$F_2(u_1, u_2, x_1, x_2) = 0.$$

Then
$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = (-1)^2 \frac{\partial(F_1, F_2)}{\partial(x_1, x_2)} \cdot \frac{\partial(u_1, u_2)}{\partial(F_1, F_2)}.$$

Or
$$\frac{\partial(F_1, F_2)}{\partial(u_1, u_2)} \cdot \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = (-1)^2 \frac{\partial(F_1, F_2)}{\partial(x_1, x_2)}.$$

Proof. Here, we have

$$\left.\begin{array}{l} F_1(u_1, u_2, x_1, x_2) = 0 \\ F_2(u_1, u_2, x_1, x_2) = 0 \end{array}\right] \quad \ldots(1)$$

Differentiating relation (1), partially w.r.t. x_1 and x_2, we get

$$\left.\begin{array}{l} \frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1} = 0 \\ \frac{\partial F_1}{\partial x_2} + \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_2} + \frac{\partial F_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_2} = 0 \\ \frac{\partial F_2}{\partial x_1} + \frac{\partial F_2}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F_2}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1} = 0 \\ \frac{\partial F_2}{\partial x_2} + \frac{\partial F_2}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_2} + \frac{\partial F_2}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_2} = 0 \end{array}\right] \quad \ldots(2)$$

Now, consider

$$\frac{\partial(F_1, F_2)}{\partial(u_1, u_2)} \cdot \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1}, & \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_2} + \frac{\partial F_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_2} \\ \frac{\partial F_2}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F_2}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1}, & \frac{\partial F_2}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_2} + \frac{\partial F_2}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_2} \end{vmatrix}$$

Using, relation (2), we get

$$= \begin{vmatrix} -\frac{\partial F_1}{\partial x_1} & -\frac{\partial F_1}{\partial x_2} \\ -\frac{\partial F_2}{\partial x_1} & -\frac{\partial F_2}{\partial x_2} \end{vmatrix}$$

$$= (-1)^2 \frac{\partial(F_1, F_2)}{\partial(x_1, x_2)}$$

$$\Rightarrow \qquad \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = (-1)^2 \left[\frac{\partial(F_1, F_2)/\partial(x_1, x_2)}{\partial(F_1, F_2)/\partial(u_1, u_2)}\right].$$

Theorem 2. *If u_1, u_2 and u_3 be the implicit functions of x_1, x_2, x_3 that is*

$$F_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$

then,

$$\frac{\partial(F_1, F_2, F_3)}{\partial(u_1, u_2, u_3)} \times \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)}.$$

Proof. Here, we have

$$\left.\begin{array}{l} F_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0 \\ F_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0 \\ F_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0 \end{array}\right] . \qquad ...(1)$$

Differentiating (1), partially with respect to x_1, x_2 and x_3, we get

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1} + \frac{\partial F_1}{\partial u_3} \cdot \frac{\partial u_3}{\partial x_1} = 0.$$

$$\Rightarrow \qquad \left.\begin{array}{l} \displaystyle\sum_{r=1}^{3} \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} = -\frac{\partial F_1}{\partial x_1}. \\ \text{Similary} \\ \displaystyle\sum_{r=1}^{3} \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} = -\frac{\partial F_1}{\partial x_2}, \\ \displaystyle\sum_{r=1}^{3} \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} = -\frac{\partial F_1}{\partial x_3}, \\ \displaystyle\sum_{r=1}^{3} \frac{\partial F_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} = -\frac{\partial F_2}{\partial x_1}, \\ \displaystyle\sum_{r=1}^{3} \frac{\partial F_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} = -\frac{\partial F_2}{\partial x_2}, \\ \displaystyle\sum_{r=1}^{3} \frac{\partial F_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} = -\frac{\partial F_2}{\partial x_3}, \\ \displaystyle\sum_{r=1}^{3} \frac{\partial F_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} = -\frac{\partial F_3}{\partial x_1}, \\ \displaystyle\sum_{r=1}^{3} \frac{\partial F_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} = -\frac{\partial F_3}{\partial x_2}, \\ \text{and} \quad \displaystyle\sum_{r=1}^{3} \frac{\partial F_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} = -\frac{\partial F_3}{\partial x_3}. \end{array}\right] \qquad ...(2)$$

Now, consider

$$\frac{\partial(F_1, F_2, F_3)}{\partial(u_1, u_2, u_3)} \times \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \frac{\partial F_1}{\partial u_3} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial u_3} \\ \frac{\partial F_3}{\partial u_1} & \frac{\partial F_3}{\partial u_2} & \frac{\partial F_3}{\partial u_3} \end{vmatrix} \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} & \Sigma \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} & \Sigma \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} \\ \Sigma \frac{\partial F_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} & \Sigma \frac{\partial F_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} & \Sigma \frac{\partial F_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} \\ \Sigma \frac{\partial F_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} & \Sigma \frac{\partial F_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} & \Sigma \frac{\partial F_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} \end{vmatrix}$$

Now, using (2), we get

$$= \begin{vmatrix} -\frac{\partial F_1}{\partial x_1} & -\frac{\partial F_1}{\partial x_2} & -\frac{\partial F_1}{\partial x_3} \\ -\frac{\partial F_2}{\partial x_1} & -\frac{\partial F_2}{\partial x_2} & -\frac{\partial F_2}{\partial x_3} \\ -\frac{\partial F_3}{\partial x_1} & -\frac{\partial F_3}{\partial x_2} & -\frac{\partial F_3}{\partial x_3} \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} \end{vmatrix}$$

$$= (-1)^3 \frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)}.$$

• 5.4. NECESSARY AND SUFFICIENT CONDITION FOR A JACOBIAN TO BE VANISHED

Theorem 1. *If $v_1, v_2, \ldots, v_n$ be the functions of n independent variables $x_1, x_2, \ldots, x_n$ such that*

$$F(v_1, v_2, \ldots, v_n) = 0$$

it is necessary and sufficient that the Jacobian

$$\frac{\partial(v_1, v_2, \ldots, v_n)}{\partial(x_1, x_2, \ldots, x_n)} \text{ should vanish identically.}$$

Proof. Necessary Condition. Here, we have, if there exists a relation of $v_1, v_2, \ldots, v_n$ such that

$$F(v_1, v_2, \ldots, v_n) = 0. \qquad \ldots(1)$$

The, Jacobian is necessarily zero.

Differentiating (1) partially with respect to $x_1, x_2, \ldots, x_n$, we get

$$\frac{\partial F}{\partial v_1} \cdot \frac{\partial v_1}{\partial x_1} + \frac{\partial F}{\partial v_2} \cdot \frac{\partial v_2}{\partial x_1} + \ldots + \frac{\partial F}{\partial v_n} \cdot \frac{\partial v_n}{\partial x_1} = 0,$$

$$\frac{\partial F}{\partial v_1} \cdot \frac{\partial v_1}{\partial x_2} + \frac{\partial F}{\partial v_2} \cdot \frac{\partial v_2}{\partial x_2} + \ldots + \frac{\partial F}{\partial v_n} \cdot \frac{\partial v_n}{\partial x_2} = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{\partial F}{\partial v_1} \cdot \frac{\partial v_1}{\partial x_n} + \frac{\partial F}{\partial v_2} \cdot \frac{\partial v_2}{\partial x_n} + \ldots + \frac{\partial F}{\partial v_n} \cdot \frac{\partial v_n}{\partial x_n} = 0.$$

Now, eliminating $\frac{\partial F}{\partial v_1}, \frac{\partial F}{\partial v_2}, \ldots, \frac{\partial F}{\partial v_n}$ from these equations, we get

$$\begin{vmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \cdots & \frac{\partial v_n}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \cdots & \frac{\partial v_n}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial v_1}{\partial x_n} & \frac{\partial v_2}{\partial x_n} & \cdots & \frac{\partial v_n}{\partial x_n} \end{vmatrix} = 0$$

$$\Rightarrow \qquad \frac{\partial(v_1, v_2, \ldots, v_n)}{\partial(x_1, x_2, \ldots, x_n)} = 0.$$

Sufficient Condition. If the Jacobian $J(v_1, v_2, \ldots, v_n)$ is zero, then to show that there must exists a relation between $v_1, v_2, \ldots, v_n$.

The equation connecting the functions $v_1, v_2, \ldots, v_n$ and the variables $x_1, x_2, \ldots, x_n$ can be written as

$$g_1(x_1, x_2, \ldots, x_n, v_1) = 0$$
$$g_2(x_2, x_3, \ldots, x_n, v_1, v_2) = 0$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$g_k(x_k, x_{k+1}, \ldots, x_n, v_1, v_2, \ldots, v_k) = 0$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$g_n(x_n, v_1, v_2, \ldots, v_n) = 0.$$

Then, we have

$$J = \frac{\partial(v_1, v_2, \ldots, v_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\left[\dfrac{\partial(g_1, g_2, \ldots, g_n)}{\partial(x_1, x_2, \ldots, x_n)}\right]}{\left[\dfrac{\partial(g_1, g_2, \ldots, g_n)}{\partial(v_1, v_2, \ldots, v_n)}\right]}$$

$$= (-1)^n \frac{\left(\dfrac{\partial g_1}{\partial x_1} \cdot \dfrac{\partial g_2}{\partial x_2} \cdots \dfrac{\partial g_n}{\partial x_n}\right)}{\left(\dfrac{\partial g_1}{\partial v_1} \cdot \dfrac{\partial g_2}{\partial v_2} \cdots \dfrac{\partial g_n}{\partial v_n}\right)}.$$

If $J = 0$, then

$$\frac{\partial g_1}{\partial x_1} \cdot \frac{\partial g_2}{\partial x_2} \cdots \frac{\partial g_r}{\partial x_r} \cdots \frac{\partial g_n}{\partial x_n} = 0$$

$\Rightarrow$ At least one of $\dfrac{\partial g_1}{\partial x_1} \cdot \dfrac{\partial g_2}{\partial x_2} \ldots \dfrac{\partial g_n}{\partial x_n}$ is zero.

$\Rightarrow$ $\dfrac{\partial g_k}{\partial x_k} = 0$ for some value of k between 1 and n.

$\Rightarrow$ For that particular value of k, the function g_k must not contain x_k and hence

$$g_k(x_{k+1}, \ldots, x_n, v_1, v_2, \ldots, v_k) = 0. \qquad \ldots(2)$$

Now we may easily eliminate the variables $x_{k+1}, x_{k+2}, \ldots, x_n$ between (2) and $g_{r+1} = 0, g_{r+2} = 0, \ldots, g_n = 0$ and an equation between $v_1, v_2, \ldots, v_n$ alone, can be obtained.

SOLVED EXAMPLES

Example 1. *If $x = r\cos\theta$, $y = r\sin\theta$, show that*

(*a*) $\dfrac{\partial(x, y)}{\partial(r, \theta)} = r$ (*b*) $\dfrac{\partial(r, \theta)}{\partial(x, y)} = \dfrac{1}{r}$.

Solution. (a) Here, we have

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r\cos^2\theta + r\sin^2\theta = r$$

(b) From the given relation, we get

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = y/x.$$

Now differentiating partially w.r.t. x and y, we obtain

$$2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$
$$2r\frac{\partial r}{\partial y} = 2y \quad \text{or} \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

and $$\tan\theta = y/x \quad \Rightarrow \sec^2\theta\frac{\partial\theta}{\partial x} = -\frac{y}{x^2}$$

or $$\frac{\partial\theta}{\partial x} = -\frac{y}{x^2\sec^2\theta} = -\frac{y}{r^2\cos^2\theta\sec^2\theta} = -\frac{y}{r^2}$$

and $$\sec^2\theta \frac{\partial\theta}{\partial y} = \frac{1}{x} \quad \text{or} \quad \frac{\partial\theta}{\partial y} = \frac{1}{x\sec^2\theta} = \frac{\cos^2\theta}{x} = \frac{x^2}{r^2}\cdot\frac{1}{x} = \frac{x}{r^2}$$

$$\therefore \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{vmatrix} = \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix}$$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}.$$

Example 2. *If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$, show that*

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta.$$

Solution. Here, we have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\,(r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi)$$
$$+ r\sin\theta\,(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi)$$

[expanding the determinant along the third row]

$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta = r^2\sin\theta\,(\cos^2\theta + \sin^2\theta)$$
$$= r^2\sin\theta.$$

Example 3. *If $x = c\cos u \cosh v$ and $y = \sin u \sinh v$ prove that*

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{c^2}(\cos 2u - \cosh 2v).$$

Solution. We have

$$x = c\cos u\cosh v \quad \text{and} \quad y = c\sin u\sinh v$$

$$\therefore \quad \frac{\partial x}{\partial u} = -c\sin u\cosh v, \quad \frac{\partial x}{\partial v} = \cos u\sinh v$$

and $$\frac{\partial y}{\partial u} = c\cos u\sinh v, \quad \frac{\partial y}{\partial v} = c\sin u\cosh v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -c\cos u\cosh y & c\cos u\sinh v \\ c\cos u\sinh v & c\sin u\cosh v \end{vmatrix}$$

$$= -c^2\sin^2 u\cosh^2 v - c^2\cos^2 u\sinh^2 v$$

$$= -\frac{c^2}{2}[2\sin^2 u\cosh^2 v + 2\cos^2 u\sinh^2 v]$$

$$= -\frac{c^2}{2}[(1-\cos 2u)\cosh^2 v + (1+\cos 2u)\sinh^2 v]$$

$$= -\frac{c^2}{2}[\cos 2u\,(\sinh^2 v - \cosh^2 v) + \cosh^2 v + \sinh^2 v]$$

$$= -\frac{c^2}{2}[-\cos 2u + \cosh 2v]$$

$$= \frac{c^2}{2}[\cos 2u - \cosh 2v].$$

Example 4. *If* $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$, *then prove that*

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{2}\frac{y^2 - x^2}{uv(u - v)}.$$

Solution. Here we can write above relations, as

$$F_1 \equiv u^3 + v^3 - x - y = 0$$

$$F_2 \equiv u^2 + v^2 - x^3 - y^3 = 0.$$

Now $$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\partial(F_1, F_2)}{\partial(x, y)} \Big/ \frac{\partial(F_1, F_2)}{\partial(u, v)}. \quad \text{...(a)}$$

We have $$\frac{\partial(F_1, F_2)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$$

$$= 3y^2 - 3x^2 = 3(y^2 - x^2)$$

and $$\frac{\partial(F_1, F_2)}{\partial(u, v)} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix} = 6u^2v - 6uv^2 = 6uv(u - v)$$

$\therefore$ From (a)

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{3(y^2 - x^2)}{6uv(u - v)} = \frac{1}{2}\frac{y^2 - x^2}{uv(u - v)}.$$

Example 5. *If*

$$u = x + 2y + z, \; v = x - 2y + 3z \;\; \textit{and} \;\; w = 2xy - xz + 4yz - 2z^2,$$

then prove that they are not independent.

Find the relation between u, v and w.

Solution. We have

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y - z & 2x - 4y + 6z & -x + 2y - 3z \end{vmatrix} \text{ by } c_2 - 2c_1 \text{ and } c_3 - c_1$$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 2y - z & -x + 2y - 3z & -x + 2y - 3z \end{vmatrix} = 0.$$

Here last two columns be identical. So the Jacobian of the functions u, v, w is zero, therefore these functions are not independent so there must be exists a relation between them.

We have $$u^2 - v^2 = (x + 2y + z)^2 - (x - 2y + 3z)^2$$

$$= (2x + 4z)(4y - 2z)$$

$$= 4(x + 2z)(2y - z)$$

By simplification

$$= 4(2xy - xz + 4yz - 2z^2) = 4w.$$

Therefore $u^2 - v^2 = 4w$, required relation between u, v and w.

Example 6. *Show that the functions*

$$u = x + y + z, \; v = xy + yz + zx, \; w = x^3 + y^3 + z^3 - 3xyz$$

are not independent, also find the relation between u, v and w.

Solution. We have

6. If $u = x^2 + y^2 + z^2$, $v = x + y + z$, $w = xy + yz + zx$. Show that the Jacobian $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ vanish identically. Also find the relation between u, v and w.

ANSWERS

1. $-\frac{y}{2x}$ 5. $v^2 = u + 2w$ 6. $\tan v = u$

OBJECTIVE EVALUATION

Fill in the Blanks :

1. If u and v be the functions of two independent variables x and y, then Jacobian of u and v with respect to x and y is denoted by ....
2. The function u, v and w of three independent variables x, y and z will not .... if $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$.
3. If $x = r\cos\theta$, $y = r\sin\theta$, then the value of $\frac{\partial(x, y)}{\partial(r, \theta)}$ = ...
4. The value of $\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)}$ = ...

True or False :

Write 'T' for true and 'F' for false.

1. If u_1, u_2 are functions of y_1, y_2 and y_1, y_2 are functions of x_1, x_2 then $\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}$ (T/F)
2. $\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} \cdot \frac{\partial(x_1, x_2)}{\partial(u_1, u_2)} = 0$.
3. If the functions u, v, w, of three independent variables x, y, z are not independent then the Jacobian of u, v, w with respect to x, y, z vanishes. (T/F)

Multiple Choice Questions :

Choose the most appropriate one.

1. If the functions u, v, w of three independent variable x, y and z and $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$ then the functions are :
   (a) Independent (b) Not independent
   (c) May be independent (d) None of these.
2. The value of $\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)}$ is :
   (a) 1 (b) 0 (c) ∞ (d) None of these.
3. The necessary and sufficient condition for the existence of a relation $F(u_1, u_2, \ldots, u_n) = 0$ is that the Jacobian must be :
   (a) equal to 1 (b) equal to 2
   (c) vanish identically (d) none of these.

ANSWERS

Fill in the Blanks :

1. $\frac{\partial(u, v)}{\partial(x, y)}$ or $J(u, v)$ 2. Independent 3. r

True or False :

1. T 2. F 3. T

Multiple Choice Questions :

1. (b) 2. (a) 3. (c)

UNIT 6

ENVELOPS AND EVOLUTES

STRUCTURE

- Family of Curves
- Envelope of a family of curves with one parameter
- Working procedure for finding the envelope.
- Envelope of the family of curves of the form $A\lambda^2 + B\lambda + C = 0$.
- Envelope of the family of curves with two parameters connected by a relation
- Geometrical interpretation of the envelope
- Evolute
- Evolute of pedal form of curves
  - Summary
  - Student Activity
  - Test Yourself

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to define a family of curves with one and two parameters.
- How to define envelope and evolute of a given curve.

6.1. FAMILY OF CURVES

(i) Family of curves with one parameter. An equation in two variables x and y of the form

$$F(x, y, \lambda) = 0$$

where λ is any constant, is known as a curve.

If λ takes all real values, then the equation $F(x, y, \lambda) = 0$ is known as *family of curves* with one parameter λ.

(ii) Family of curves with two parameters. An equation in two variables x and y of the form

$$F(x, y, \lambda, \mu) = 0$$

is known as a family of curves with two parameters λ and μ if λ and μ take all real values.

For Example (1). The euation $x \cos \lambda + y \sin \lambda = p$ represents a family of straight lines with one parameter λ.

(2) The equation $y = mx + a/m$ represents a family of straight lines which are the tangents to parabola $y^2 = 4ax$ with one parameter m.

(3) The equation $(y - \alpha)^2 + (y - \beta)^2 = a^2$ represents a family of circles with centred at (α, β) and radius a with two parameters α and β.

6.2. ENVELOPE OF A FAMILY OF CURVES WITH ONE PARAMETER

Let $F(x, y, \lambda) = 0$ be a family of curves with parameter λ and let $F(x, y, \lambda) = 0$ and $F(x, y, \lambda + \delta\lambda) = 0$ be two members of a family of curves $F(x, y, \lambda) = 0$ corresponding to the parameter λ and $\lambda + \delta\lambda$, suppose P is a point of intersection of two members $F(x, y, \lambda) = 0$ and $F(x, y, \lambda + \delta\lambda) = 0$. As $\delta\lambda \to 0$, the point P tends to a definite point Q which depends upon λ. Thus the locus of such points Q gives an envelope of the family.

Definition. *The locus of the limiting positions of the points of intersection of any two members of the family of curves $F(x, y, \lambda) = 0$, when one of them tends to coincide with the other fixed point.*

REMARK

➤ The envelope of a family of curves is the locus of the points of intersection of consecutive members of the family.

• 6.3. WORKING PROCEDURE FOR FINDING THE ENVELOPE

Let $F(x, y, \lambda) = 0$ be a family of curve with one parameter λ.

Suppose $F(x, y, \lambda) = 0$ and $F(x, y, \lambda + \delta\lambda) = 0$ are two consecutive members of the family of curves corresponding to λ and $\lambda + \delta\lambda$. Thus the co-ordinates of the point of intersection of these two members are obtained by the equations.

$$F(x, y, \lambda) = 0 \qquad \ldots (1)$$

and
$$F(x, y, \lambda) - F(x, y, \lambda + \delta\lambda) = 0 \qquad \ldots (2)$$

Divide the equation (2) by $\delta\lambda$, we get

$$\frac{F(x, y, \lambda) - F(x, y, \lambda + \delta\lambda)}{\delta\lambda} = 0$$

or
$$\frac{F(x, y, \lambda + \delta\lambda) - F(x, y, \lambda)}{\delta\lambda} = 0$$

Taking limit as $\delta\lambda \to 0$, we get

$$\frac{\partial F(x, y, \lambda)}{\delta\lambda} = 0 \qquad \ldots (3)$$

Now eliminating λ between $F(x, y, \lambda) = 0$ and $\frac{\partial F(x, y, \lambda)}{\partial\lambda} = 0$, we therefore, obtain the envelope of the family of curves $F(x, y, \lambda) = 0$.

Remember. To obtain an envelope of the family of curves $F(x, y, \lambda) = 0$, we use following steps:

Step I. *Differentiate partially* $F(x, y, \lambda) = 0$ *with respect to* λ, *we get*

$$\frac{\partial F}{\partial\lambda} = 0.$$

Step II. *Now eliminating* λ *between* $F(x, y, \lambda)$ *and* $\frac{\partial F}{\partial\lambda} = 0$, *we therefore obtain envelope of the given family of curves.*

SOLVED EXAMPLES

Example 1. *Find the envelope of the family of straight lines* $y = mx + \frac{a}{m}$, *the parameter being* m.

Solution. Here, the family of straight lines is

$$y = mx + \frac{a}{m} \qquad \ldots (1)$$

Differentiating (1) partially with respect to m, we get

$$0 = x - \frac{a}{m^2} \qquad \ldots (2)$$

Eliminating m between (1) and (2), we get

From (2), we have

$$m^2 = \frac{a}{x}$$

From (1), we have

$$ym = m^2x + a$$

$$\Rightarrow \quad y^2m^2 = (m^2x + a)^2$$

$$\Rightarrow \quad y^2\left(\frac{a}{x}\right) = \left(\frac{a}{x}.x + a\right)^2 \qquad \left(\because m^2 = \frac{a}{x}\right)$$

$$\Rightarrow \quad \frac{ay^2}{x} = (2a)^2$$

$\Rightarrow \qquad y^2 = 4ax.$

This is the required envelope.

Example 2. *Find the envelope of the family of straight lines : x cosec θ – y cot θ = c, the parameter being θ.*

Solution. Since the family of straight lines is

$$x \operatorname{cosec} \theta - y \cot \theta = c. \qquad \ldots (1)$$

Differentiating (1) partially with respect to θ, we get

$$-x \operatorname{cosec} \theta \cot \theta + y \operatorname{cosec}^2 \theta = 0$$

or
$$x \cot \theta - y \operatorname{cosec} \theta = 0. \qquad \ldots (2)$$

Eliminating θ between (1) and (2), we get

$$(x \operatorname{cosec} \theta - y \cot \theta)^2 - (x \cot \theta - y \operatorname{cosec} \theta)^2 = c^2$$

or
$$x^2 (\operatorname{cosec}^2 \theta - \cot^2 \theta) - y^2 (\operatorname{cosec}^2 \theta - \cot^2 \theta) - 2xy \operatorname{cosec} \theta \cot \theta + 2xy \operatorname{cosec} \theta \cot \theta = c^2$$

or
$$x^2 - y^2 = c^2. \qquad (\because \operatorname{cosec}^2 \theta - \cot^2 \theta = 1)$$

This is the required envelope.

• 6.4. ENVELOPE OF THE FAMILY OF CURVES OF THE FORM $A\lambda^2 + B\lambda + C = 0$

Since the family of curve is

$$A\lambda^2 + B\lambda + C = 0 \qquad \ldots (1)$$

Differentiating (1) partially w.r.t. to λ, we get

$$2A\lambda + B = 0. \qquad \ldots (2)$$

Eliminating λ between (1) and (2), we get

$$A\left[-\frac{B}{2A}\right]^2 + B\left[-\frac{B}{2A}\right] + C = 0$$

or
$$\frac{B^2}{4A} - \frac{B^2}{2A} + C = 0$$

or
$$B^2 - 4AC = 0.$$

This is the required equation of an envelope.

REMARK

➤ If the quation of the family of curves is a quadratic equation in parameter, then its envelope is obtained by $D = 0$, where D is the discriminant of the quadratic.

• 6.5. ENVELOPE OF THE FAMILY OF CURVES WITH TWO PARAMETERS CONNECTED BY A RELATION

Let $F(x, y, \lambda, \mu) = 0$ be a family of curves with two parameters λ and μ. Let $f(\lambda, \mu) = 0$ be a relation between λ and μ.

To obtain the envelope, we proceed as follows :

Differentiating the equations $F(x, y, \lambda, \mu) = 0$ and $f(\lambda, \mu) = 0$ with respect to λ regarding x and y as constants and μ as a function of λ, we get two equations. Now eliminating λ, μ between the given equations and two obtained equations. We therefore obtain the envelope.

• 6.6. GEOMETRICAL INTERPRETATION OF THE ENVELOPE

Let the equation of the family of curves be

$$F(x, y, \lambda) = 0 \qquad \ldots (1)$$

where λ is a parameter.

Thus the envelope of (1) is obtained by eliminating between (1) and

$$\frac{\partial F}{\partial \lambda} = 0. \qquad \ldots (2)$$

Therefore, we can say that (2) is taken as the equation of the envelope of (1), if λ is a function of x and y but not constant.

$$\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial \lambda}\cdot\frac{\partial \lambda}{\partial x}\right)+\left(\frac{\partial F}{\partial y}+\frac{\partial F}{\partial \lambda}\cdot\frac{\partial \lambda}{\partial y}\right)\frac{dy}{dx}=0.$$

$$\Rightarrow \qquad \frac{dy}{dx}=-\frac{\dfrac{\partial F}{\partial x}+\dfrac{\partial F}{\partial \lambda}\cdot\dfrac{\partial \lambda}{\partial x}}{\dfrac{\partial F}{\partial y}+\dfrac{\partial F}{\partial \lambda}\cdot\dfrac{\partial \lambda}{\partial y}} \qquad \ldots(3)$$

This gives the slope of the tangent to the envelope of (1) at any point (x, y). Where (x, y) is a common point to the member $F(x, y, \lambda) = 0$ of the family of curves and the envelope.

If $\frac{\partial F}{\partial x} \neq 0$ and $\frac{\partial F}{\partial y} \neq 0$ at (x, y), then the slope of the tangent to the member $F(x, y, \lambda) = 0$ is

$$\frac{dy}{dx}=-\frac{\partial F/\partial x}{\partial F/\partial y} \qquad \ldots(4)$$

But $F(x, y, \lambda) = 0$ is also the equation of the envelope if λ is a function of x and y, which is given by $\frac{\partial F}{\partial \lambda} = 0$.

Since at every point of the envelope $\frac{\partial F}{\partial \lambda} = 0$, then the slopes given by (3) and (4) are same.

Hence the curve of the family and its envelope have the same tangent lines at the common point. Consequently the envelope of a family of curves touch each member of the family.

REMARK

- If $\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$ at any points on the curve, then the envelope may not touch a curve at that points.

SOLVED EXAMPLES

Example 1. *Find the envelope of the family of straight lines $y = mx + a\sqrt{1+m^2}$, the parameter being m.*

Solution. Here the given equation of the family can be written as :

$$(y - mx)^2 = a^2 (1 + m^2)$$

or $$(x^2 - a^2) m^2 - 2mxy - a^2 + y^2 = 0 \qquad \ldots(1)$$

This equation is quadratic in m. Then the envelope of (1) is obtained by equating the discriminant of (1) to zero, we get

$$(-2xy)^2 - 4(x^2 - a^2)(y^2 - a^2) = 0 \qquad (\because B^2 - 4AC = 0)$$

or $$4x^2y^2 - 4[x^2y^2 - x^2a^2 - a^2y^2 + a^4] = 0$$

or $$x^2a^2 + a^2y^2 = a^4$$

or $$x^2 + y^2 + a^2.$$

This is the required equation of envelope.

Example 2. *Find the envelope of the family of circles $(x - c)^2 + y^2 = r^2$ where the parameter being c.*

Solution. Here equation of family of circle is

$$(x - c)^2 + y^2 = r^2 \qquad \ldots(1)$$

It can also be written as

$$c^2 - 2xc + x^2 + y^2 - r^2 = 0. \qquad \ldots(2)$$

This is quadratic in c, so that the envelope is

$$(-2x)^2 - 4 \,.\, 1 \,.\, (x^2 + y^2 - r^2) = 0 \qquad (\because B^2 - 4AC = 0)$$

or $$x^2 - x^2 - y^2 + r^2 = 0$$

or $$y^2 = r^2$$

or $$y = r, \quad y = -r.$$

These are the required envelopes.

Example 3. *Find the envelope of the circles drawn on the radii vectors of the parabola $y^2 = 4ax$ as diameter.*

Solution. Let $(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$. Then the equation of circles drawn on the line joining (0, 0) and $(at^2, 2at)$ as diameter is

$$(x-0)(x-at^2)+(y-0)(y-2at)=0$$

or $$x^2+y^2-axt^2-2aty=0 \qquad \ldots (1)$$

where t being the parameter.

Differentiating (1) partially with respect to t, we get

$$-2axt-2ay=0$$

or $$xt+y=0 \qquad \ldots (2)$$

Eliminating t between (1) and (2), we get

$$x^2+y^2-ax\left(-\frac{y}{x}\right)^2-2a\left(-\frac{y}{x}\right)y=0$$

or $$x^2+y^2-\frac{ay^2}{x}+\frac{2ay^2}{x}=0$$

or $$x(x^2+y^2)+ay^2=0.$$

This is the required envelope.

Example 4. *Find the envelope of the family of straight lines* $\frac{x}{a}+\frac{y}{b}=1$ *where a, b are connected by a relation* $a^2+b^2=c^2$, *c is a constant.*

Solution. Since the equation of family of straight lines is

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \ldots (1)$$

and $$a^2+b^2=c^2. \qquad \ldots (2)$$

Differentiating (1) and (2) w.r.t. a treating x and y as constant and 'b' as a function of 'a', we get

$$-\frac{x}{a^2}-\frac{y}{b^2}\cdot\frac{db}{da}=0$$

or $$\frac{db}{da}=-\frac{x/a^2}{y/b^2} \qquad \ldots (3)$$

and $$2a+2b\frac{db}{da}=0$$

or $$\frac{db}{da}=-\frac{a}{b} \qquad \ldots (4)$$

From (3) and (4), we get

$$-\frac{x/a^2}{y/b^2}=-\frac{a}{b}$$

or $$\frac{x/a}{y/b}=\frac{a^2}{b^2}$$

or $$\frac{x/a}{a^2}=\frac{y/b}{b^2} \qquad \ldots (5)$$

Eliminating a and b between (1), (2) and (5), we get

$$\frac{x/a}{a^2}=\frac{y/b}{b^2}=\frac{x/a+y/b}{a^2+b^2}=\frac{1}{c^2} \qquad \text{[using (1) and (2)]}$$

$\therefore$ $$\frac{x/a}{a^2}=\frac{1}{c^2} \Rightarrow xc^2=a^3$$

$$\Rightarrow a=(xc^2)^{1/3}$$

and $$\frac{y/b}{b^2}=\frac{1}{c^2}$$

$\Rightarrow$ $$yc^2=b^3$$

$\Rightarrow$ $$b=(yc^2)^{1/3}.$$

Putting these values of a and b in (2), we get

$$(xc^2)^{2/3} + (yc^2)^{2/3} = c^2$$

or
$$x^{2/3} + y^{2/3} = c^{2/3}.$$

This is the required envelope.

• TEST YOURSELF–1

1. Find the envelope of the family of straight lines.
$$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$$
where θ being the parameter.
2. Find the envelope of the following families of straight lines :
(i) $y = mx + am^3$, the parameter being m.
(ii) $y = mx + am^p$, the parameter being m.
(iii) $x\cos^3\alpha + y\sin^3\alpha = a$, α being the parameter.
3. Find the envelope of the family of straight lines
$$x\cos\alpha + y\sin\alpha = a$$
where α being the parameter, and interpret the result.
4. Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where two parameters a and b are connected by a relation $a + b = c$, c being the constant.
5. Show that the envelope of the family of straight lines $y = mx + \sqrt{a^2m^2 + b^2}$, α being the parameter is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

ANSWERS

1. $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$
2. (i) $4x^3 + 27ay^2 = 0$ (ii) $(p-1)^{p-1} . x^p + p^p ay^{p-1} = 0$ (iii) $a^2(x^2 + y^2) = x^2y^2$
3. $x^2 + y^2 = a^2$
4. $x^{1/2} + y^{1/2} = c^{1/2}$

• 6.7. EVOLUTE

Definition. *The evolute of a curve is the envelope of the normals to that curve.*

In other words, The locus of the centre of curvature of a curve is called **evolute** for the curve.

Since th centre of curvature of a curve for a given point P on it is the limiting position of the intersection of the normal at P and the normal at other point Q as Q tends to P. Thus the envelope of the normals to a given curve is called an evolute of that curve (Remember).

• 6.8. EVOLUTE OF PEDAL FORM OF CURVES

Let the pedal equation of the given curve be

$$p = f(r) \quad \ldots (1)$$

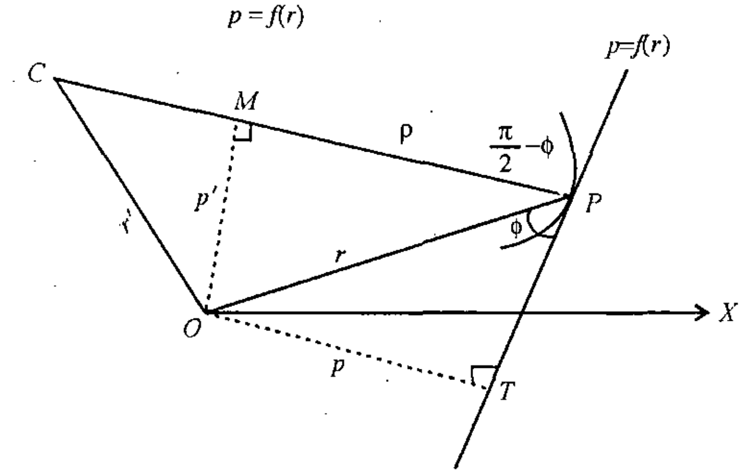

Fig. 2

and let C be the centre of curvature of (1) at the point P. Then $PC = \rho$ (radius of curvature) and the equation joining P and C is the normal to the curve (1) at P. The point C will be on evolute corresponding to the point P on the curve.

Since the evolute of the given curve $p = f(r)$ is the envelope of the normals at P of the curve, so that the normal PC of the given curve is a tangent to the evolute at C.

Here PT is the tangent at P to the given curve $p = f(r)$ and OT is perpendicular to PT such that $OT = p$ and $OP = r$. Now draw a perpendicular OM from O to PC such that $OM = p'$ and $CO = r'$. Then in triangle OPC, we have

$$\cos \angle OPC = \frac{r^2 + \rho^2 - r'^2}{2r\rho}$$

$$\therefore \quad r'^2 = r^2 + \rho^2 - 2r\rho \cos \angle OPC$$

$$= r^2 + \rho^2 - 2r\rho \cos\left(\frac{\pi}{2} - \phi\right)$$

$$= r^2 + \rho^2 - 2r\rho \sin \phi$$

$$r'^2 = r^2 + \rho^2 - 2\rho p \qquad (\because\ p = r \sin \phi)$$

$$r'^2 = r^2 + \rho'^2 - 2\rho. \qquad \ldots (2)$$

Since $OTPM$ is a rectangle, so that $OM = TP = p'$, then in ΔPTO,

$$r^2 = p^2 + p'^2$$

$$\Rightarrow \quad p'^2 = r^2 - p^2. \qquad \ldots (3)$$

Also, we have

$$\rho = r\frac{dr}{dp} \qquad \ldots (4)$$

Now eliminating r, p and ρ between (1), (2), (3) and (4), we get the pedal equation of the evolute of the curve $p = f(r)$.

REMARK

- In above formulation the relation between p' and r' gives the evolute of the curve $p = f(r)$.

SOLVED EXAMPLES

Example 1. *Find the evolute of the hyperbola $x^2/a^2 - y^2/b^2 = 1$.*

Solution. Let $P(a \sec \theta, b \tan \theta)$ be any point on the hyperbola

$$x^2/a^2 - y^2/b^2 = 1.$$

The equation of the normal at P to the given hyperbola is

$$ax \cos \theta + by \cot \theta = a^2 + b^2. \qquad \ldots (1)$$

Differentiating (1) partially w.r.t. θ, we get

$$-ax \sin \theta - by \operatorname{cosec}^2 \theta = 0$$

or
$$\sin^3 \theta = -\frac{by}{ax}$$

or
$$\sin \theta = \left(-\frac{by}{ax}\right)^{1/3}$$

$$\therefore \quad \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{by}{ax}\right)^{2/3}}$$

and
$$\cot \theta = \frac{\sqrt{1 - \left(\frac{by}{ax}\right)^{2/3}}}{\left(-\frac{by}{ax}\right)^{1/3}}$$

Putting the values of $\cos \theta$ and $\cot \theta$ in (1), we get

$$ax\left[\sqrt{1-\left(\frac{by}{ax}\right)^{2/3}}\right]+by\frac{\sqrt{1-\left(\frac{by}{ax}\right)^{2/3}}}{\left(-\frac{by}{ax}\right)^{1/3}}=(a^2+b^2)$$

or $$\frac{ax}{(ax)^{1/3}}\sqrt{(ax)^{2/3}-(by)^{2/3}}-\frac{by}{(by)^{1/3}}\sqrt{(ax)^{2/3}-(by)^{2/3}}=(a^2+b^2)$$

or $$\sqrt{(ax)^{2/3}-(by)^{2/3}}\,[(ax)^{2/3}-(by)^{2/3}]=(a^2+b^2)$$

or $$[(ax)^{2/3}-(by)^{2/3}]^{3/2}=(a^2+b^2)$$

or $$(ax)^{2/3}-(by)^{2/3}=(a^2+b^2)^{2/3}.$$

This is the required evolute of the given curve.

Example 2. Show that the evolute of an equiangular spiral is an equiangular spiral.

Solution. Since the pedal equation of an equiangular spiral is

$$p = r\sin\alpha \qquad \ldots(1)$$

so that $$\frac{dp}{dr}=\sin\alpha.$$

$\therefore$ $$\rho = r\frac{dr}{dp}=r\,.\,\frac{1}{\sin\alpha}=r\operatorname{cosec}\alpha$$

or $$\rho = r\operatorname{cosec}\alpha. \qquad \ldots(2)$$

Let (p', r') be any point on the evolute corresponding to the point (p, r) on the curve (1). Then we have,

$$r'^2 = r^2+\rho^2-2\rho p$$
$$= r^2+r^2\operatorname{cosec}^2\alpha-2r\operatorname{cosec}\alpha\,.\,r\sin\alpha$$
$$= r^2\operatorname{cosec}^2\alpha - r^2$$
$$r'^2 = r^2\cot^2\alpha. \qquad \ldots(3)$$

Also, we have

$$p'^2 = r^2-p^2=r^2-r^2\sin^2\alpha$$
$$= r^2(1-\sin^2\alpha)$$
$$p'^2 = r^2\cos^2\alpha. \qquad \ldots(4)$$

Divding (4) by (3), we get

$$\frac{p'^2}{r'^2}=\frac{r^2\cos^2\alpha}{r^2\cot^2\alpha}=\sin^2\alpha.$$

$\therefore$ $$p'^2 = r'^2\sin^2\alpha$$

or $$p' = r'\sin\alpha.$$

Thus the locus of the point (p', r') is $p = r\sin\alpha$, which is an equiangular spiral.

• SUMMARY

- **Family of curves :**
  (i) $F(x, y, \lambda) = 0$ is a family of curves with one parameter λ.
  (ii) $F(x, y, \lambda, \mu) = 0$ is a family of curves with two parameters λ and μ.
- **Envelope of $F(x, y, \lambda) = 0$**
  The equation obtained by eliminating λ between $F(x, y, \lambda) = 0$ and $\frac{\partial F}{\partial \alpha} = 0$ is called envelope.
- **Envelope of $F(x, y, \lambda, \mu) = 0$**
  The equation obtained by eliminating λ and μ between $F(x, y, \lambda, \mu) = 0$, $\frac{\partial F}{\partial \lambda} = 0$, $\frac{\partial F}{\partial \mu} = 0$, is called envelope.
- **Evolute of $F(x, y, \lambda) = 0$**
  If $\phi(x, y, a) = 0$ be the equation of the normal to the curve $F(x, y, \lambda) = 0$, then the envelope of $\phi(x, y, a) = 0$ is called evolute of $F(x, y, \lambda) = 0$.

• STUDENT ACTIVITY

1. Find the envelope of the circles drawn on the radii vectors of the parabola $y^2 = 4ax$ as diameter.

2. Find the evolute of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

• TEST YOURSELF–2

1. Find the equation of the evolute of the parabola $y^2 = 2ax$.
2. Show that the equation of the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is
$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$
3. Find the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.
4. Show that the whole length of the evolkute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $4\left(\frac{a^2}{b} - \frac{b^2}{a}\right)$.
5. Find th evolute of the parabola $y^2 = 4ax$.

ANSWERS

1. $27ay^2 = 8(x-a)^3$ 3. $(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$ 5. $3x^2 + 4ay - 4ax + 4a^2 = 0$

OBJECTIVE EVALUATION

Fill in the Blanks :

1. If the equation of the family of curves is $A\lambda^2 + B\lambda + C = 0$, where A, B, C we are functions of x, y; then its envelope is ....
2. The envelope of a family of curves .... each member of the family.
3. $xm^2 = 2ym + a = 0$ is a family of straight lines, where m being the parameter, then its envelope is ....
4. The envelope of the normals to the curve is ....

True or False :

Write 'T' for true and 'F' for false :

1. The equation $F(x, y, \lambda) = 0$ represents a family of curve with one parameter. *(T/F)*
2. The envelope of a family of curves intersects each member of the family. *(T/F)*

3. If the equation of a family of curves is $A\lambda^2 = B\lambda + C = 0$ then its envelope is $B^2 - 4AC = 0$. (T/F)

Multiple Choice Questions :

1. The envelope of a family of curves .... each member of the family :
   (a) intersect (b) touches (c) is perpendicula to (d)None of these
2. The envelope of the family of curves $xm^2 - 2ym + a = 0$, m being the parameter is :
   (a) $y^2 = 4ax$ (b) $y^2 = 2ax$ (c) $y^2 = ax$ (d) $x^2 = ay$
3. The locus of the centre of curvature for a curve is :
   (a) envelope (b) evolute (c) radius of curvature (d) none of these

ANSWERS

Fill in the Blanks :

1. $B^2 - 4AC = 0$ **2.** Touches **3.** $y^2 = ax$ **4.** Evolute

True or False :

1. T **2.** F **3.** T

Multiple Choice Questions :

1. (b) **2.** (c) **3.** (b).

UNIT

7

MAXIMA AND MINIMA OF FUNCTIONS OF TWO AND THREE VARIABLES

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- How to find the maximum and minimum values of a function of two or more than two independent variables ?
- What are Lagrange's multipliers and using these multiplies how to find the maximum and minimum values ?

7.1. MAXIMA AND MINIMA OF A FUNCTION OF SINGLE INDEPENDENT VARIABLES

Let $f(x, y)$ be a function of two independent variables x and y. If $f(x, y)$ is continuous and finite for all values of x and y in the neighbourhood of their values $x = a$ and $y = b$ respectively, then $f(a, b)$ is said to have a maixmum or a minimum values of $f(x, y)$ according as $f(a + h, b + k)$ is less than or greater than $f(a, b)$ for all values of h and k (where h and k are sufficiently small may be positive or negative), provided both are not equal to zero.

7.2. MAXIMA AND MINIMA OF A FUNCTION OF SEVERAL INDEPENDENT VARIABLES

Let $f(x, y, z, \ldots)$ be a function of several independent variables $x, y, z, \ldots$. If f is continuous and finite for all values of $x, y, z, \ldots$ in the neighbourhood of $x = a, y = b, z = c, \ldots$ respectively, then the value of $f(a, b, c, \ldots)$ is said to be a maximum or minimum if $f(a + h, b + k, c + l, \ldots)$ is less than or greater than $f(a, b, c, \ldots)$ for all values of $h, k, l, \ldots$ (where $h, k, l, \ldots$) are sufficiently small, may be positive or negative) provided they are not all zero.

Or

In other words we can say, the value of $f(a, b, c, \ldots)$ is said to be a maximum or minimum if $f(a + h, b + k, c + l \ldots) - f(a, b, c, \ldots)$ maintain an invariant sign (may be positive or negative) for all values of $h, k, l, \ldots$, positive or negative provided they are taken sufficiently small and finite.

Stationary and Extreme Points.

A point $(a_1, a_2, \ldots, a_n)$ is called a stationary point, if all the first order partial derivative of the function $f(x_1, x_2, \ldots, x_n)$ vanish at the point. A stationary point, if it is maximum or minimum is known as **extreme point** and the value of the function at an extreme point is known as an **extreme value**.

REMARK

- A stationary point may be a maximum or minimum or neither of these two.

• 7.3. NECESSARY CONDITION FOR THE EXISTENCE OF MAXIMA OR MINIMA

Let $f(x, y, z, \ldots)$ be a function of several independent variable $x, y, z \ldots$. It is clear from the definition of maxima and minima that maximum or minimum of $f(x, y, z, \ldots)$ will occur for those values of $x, y, z, \ldots$ for which the expression

$$f(x+h, y+k, z+l \ldots) - f(x, y, z, \ldots)$$

maintain an invariant sign for all sufficiently small and finite values of $h, k, l, \ldots$, positive or negative.

Now, expanding $f(x+h, y+k, z+l, \ldots)$ by Taylor's theorem, we have

$$f(x+h, y+k, z+l \ldots) = f(x, y, z, \ldots) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + \ldots\right)$$

+ terms of second and higher order.

$$\Rightarrow \quad f(x+h, y+k, z+l, \ldots) - f(x, y, z, \ldots) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + \ldots\right)$$

+ terms of second and higher orders. ...(1)

Now, since $h, k, l, \ldots$ are sufficiently small, the first degree expression

$$\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + \ldots\right)$$

of the equation (1) can be made to govern the sign of right hand side and hence, of the left hand side as well. Thus, by changing the sign of the left hand side of the equation (1) will also change.

Since, left hand side is to preserve an invariable sign for maxima or minima, therefore, as a necessary condition for maximum and minimum values, we must have

$$h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + \ldots = 0.$$

Now, since $h, k, l, \ldots$ are arbitrary and independent of each other, we must have

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0 \ldots \text{etc.} \qquad \ldots(3)$$

If the number of independent variables be n, we shall get n simultaneous equations in these n variables, which will give the values $a, b, c, \ldots$ of the n variables $x, y, z, \ldots$ respectively for which $f(x, y, z, \ldots)$ will have a maximum or a minimum value.

REMARKS

- The necessary condition for a function $f(x, y, z, \ldots)$ of the independent variables $x, y, z, \ldots$ to be maximum or minimum is given by

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0, \ldots$$

- The conditions given above is only a necessary condition for the maxima and minima of the funtion $f(x, y, z, \ldots)$. These conditions are not sufficient.

Maxima and Minima for a Function of Two Independent Variables.

(1) To find the condition which governs the sign of a quadratic expression.

Let us suppose, there is a binary expression

$$I = ax^2 + 2hxy + by^2$$

of two variables x and y. Then I can be written as

$$I = ax^2 + 2hxy + by^2$$
$$= \frac{1}{a}[(ax+hy)^2 + (ab-h^2)y^2].$$

If $(ab - h^2)$ is positive, the sign of I will be the same as that of a.

But if $(ab - h^2)$ is negative, then. the expression within the brackets may be positive or negative and so therefore we can not say anything about the sign of expression I.

Stationary and Extreme Points (For the Function of Two Independent Variables).

Let $f(x, y)$ be a function of two independent variables x and y. A point (a, b) is called a stationary point, if both the first order partial derivatives $\left(\frac{\partial f}{\partial a} \text{ and } \frac{\partial f}{\partial b}\right)$ of the function $f(x, y)$ at (a, b) vanish.

A stationary point which is either a maximum or minimum is called an extreme point.

REMARKS

- A stationary point is not necessarily an extreme point, hence a stationary point may be a maximum or a minimum or neither of these two.
- The value of the function at extreme point is called extreme value.
- A point at which function is neither maximum nor minimum, is known as **saddle points.**

Necessary Condition for Maxima or Minima.

Let $f(x, y)$ be a function of two independent variables x and y.

Then, it is clear that, we have the maximum or minimum of $f(x, y)$ at $x = a$ and $x = b$ if the expression

$$f(a + h, b + k) - f(a, b)$$

is of invariable sign for all sufficiently small independent variables h and k provided both of them are not equal to zero.

We observe that,

(i) If the sign of $f(a + h, b + k) - f(a, b)$ is negative, then we have a maximum of $f(x, y)$ at $x = a, y = b$.

(ii) If the sign of $f(a + h, b + k) - f(a, b)$ is positive, we have a minimum of $f(x, y)$ at $x = a$, $y = b$.

Expand $f(a + h, b + k)$ by Taylor's theorem, we have

$$f(a + h, b + k) = f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}} + \ldots$$

$$\Rightarrow \quad f(a + h, b + k) - f(a, b) = h\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\y=b}} + k\left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}}$$

+ term of the second and higher orders in h and k.

Now, since h and k are sufficiently small, the expression

$$h\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\y=b}} + k\left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}}$$

of the equation (1) can be made to govern the sign of right hand side and hence of the left hand side as well. Thus by changing the sign of h and k, the sign of the left hand side of the equation (1) with also change.

Since L.H.S. is to preserve an invariable sign for maximum or minimum, therefore, as a necessary condition for maximum and minimum values, we must have

$$h\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\y=b}} + k\left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}} = 0. \qquad \ldots(2)$$

If $k = 0$, we find that if $\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\y=b}} \neq 0$, the R.H.S. of (2) changes sign when h changes sign.

Therefore $f(x, y)$ can not have a maximum or minimum at $x = a$, $y = b$ if $\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\y=b}} \neq 0$.

Similarly, taking $h = 0$, we see that $f(x, y)$ can not have a maximum or a minimum at $x = a$, $y = b$ if $\left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}} \neq 0$.

Thus, a set of necessary conditions that $f(x, y)$ should have a maximum or minimum at $x = a$, $y = b$ is that

$$\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\y=b}} = 0 \text{ and } \left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}} = 0.$$

• 7.4. SUFFICIENT CONDITION FOR MAXIMA OR MINIMA : THE LAGRANGE'S CONDITION

Let $f(x, y)$ be a function of two variables x and y.

Let $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x\, \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at $x = a$ and $y = b$.

As a set of necessary conditions for a maximum or minimum, at (a, b) we have

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0 \text{ at } (a, b)$$

then $$f(a + h, b + k) - f(a, b) = \frac{1}{2\,!}[rh^2 + 2shk + tk^2] + R$$

where R consists of terms of third and higher order of small quantities h and k.

Now, by taking h and k sufficiently small, the second degree terms in R.H.S. of (1) may be made to govern the sign of R.H.S. and therefore of the L.H.S. also *i.e.* for sufficiently small values of h and k, the sign of

$$\frac{1}{2}(rh^2 + 2shk + tk^2) + R$$

is same as that of

$$rh^2 + 2shk + tk^2.$$

If the sign is negative, then the function is maximum at (a, b) and if the sign is positive, then the function is minimum at (a, b).

Now, there are following three cases :

Case (i) If $(rt - s^2) > 0$.

Here, neither r nor t can be zero. Hence, we can write

$$rh^2 + 2shk + tk^2 = \frac{1}{r}[r^2h^2 + 2rshk + rtk^2]$$

$$= \frac{1}{r}[(rh + sk)^2 + (rt - s^2)k^2]$$

since $rt - s^2 > 0$, therefore

$$(rh + sk)^2 + (rt - s^2)k^2 > 0$$

for all values of h and k except when $rh + sk = 0$, $k = 0$ *i.e.* at $h = 0$, $k = 0$, which is not possible.

Hence in this case the expression $rh^2 + 2shk + tk^2$ will have the same sign for all values of h and k, and the sign is determined by the sign of r.

Thus, the function $f(x, y)$ will have a maximum or minimum at $x = a$ and $y = b$. If $rt - s^2 > 0$. Further, the function $f(x, y)$ is maximum or minimum according as r is negative or positive.

Case (ii) $(rt - s^2) < 0$.

If $rt - s^2$ is negative, we are not sure about the sign of second degree term of R.H.S. of (1) and hence there is neither a maximum nor a minimum value.

Case (iii) $rt - s^2 = 0$.

If $rt = s^2$, then quadratic expression

$$rh^2 + 2shk + tk^2$$

becomes $\frac{1}{r}(hr+ks)^2$.

So that, the quadratic expression will be of the same sign as that of r or t unless

$$\frac{h}{k} = -\frac{s}{r} = \alpha \text{ (say)}$$

i.e., $rh + sk = 0.$

If this condition is satisfied, then the second degree expression in R.H.S. of (1) vanishes and hence, the sign of the R.H.S. of (1) depends upon third degree expression in h and k, which change sign with the change of sign of h and k and hence, the sign of L.H.S. of (1) will also change and hence, there will be neither maximum nor minimum. Thus, the necessary condition for the existence of maxima and minima now is that the cubic terms must vanish collectively in R.H.S. of (1) when $\frac{h}{k} = -\frac{s}{r} = \alpha$; and then the biquadratic terms of R.H.S. of (1) must collectively of the same sign as r and t, when

$$\frac{h}{k} = -\frac{s}{r} = \alpha$$

i.e., $hr + ks = 0.$

Hence, the case is doubtful.

Thus, if $rt - s^2 = 0$, the case is doubtful and further, investigation is needed to determine the maxima and minima of $f(x, y)$ at (a, b).

Working procedure. To discuss the maxima and minima at $x = a$, $y = b$, we must find

$$r = \left(\frac{\partial^2 u}{\partial x^2}\right)_{\substack{x=a\\y=b}}, \quad s = \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_{\substack{x=a\\y=b}}, \quad t = \left(\frac{\partial^2 u}{\partial y^2}\right)_{\substack{x=a\\y=b}}.$$

Then, **calculate** $rt - s^2$.

Now following cases arise :

(i) **If $rt - s^2 > 0$,** then

(A) If r is negative then, $f(x, y)$ is maximum at $x = a$, $y = b$..

(B) If r is positive then, $f(x, y)$ is minimum at $x = a$, $y = b$.

(ii) **If $rt - s^2 < 0$,** $f(x, y)$ is neither maximum nor minimum at $x = a$, $y = b$.

(iii) **If $rt - s^2 = 0$** the case is doubtful, and further investigation will be required.

An important identity. While solving problems, we frequently used the identity, given by **Lagrange**

$$\{(a^2+b^2+c^2)(p^2+q^2+r^2) - (ap+bq+cr)^2\}$$
$$= \{(br-cq)^2 + (cp-ar)^2 + (aq-bp)^2\}.$$

SOLVED EXAMPLES

Example 1. *Find all maxima or minima values of the function* $f(x, y) = y^2 + x^2y + x^4$.

Solution. Since we have

$$f(x, y) = y^2 + x^2y + x^4$$

$\therefore$ $$\frac{\partial f}{\partial x} = 2xy + x^3$$

and $$\frac{\partial f}{\partial y} = 2y + x^2.$$

For a maximum or minimum of $f(x, y)$, we must have

$$\frac{\partial f}{\partial x} = 0 \quad \text{or} \quad \frac{\partial f}{\partial y} = 0$$

$\therefore$ $$\frac{\partial f}{\partial x} = 0 \Rightarrow 2xy + 4a^3 = 0$$

$$\Rightarrow 2x(y + x^2) = 0 \quad \ldots (1)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y + x^2 = 0 \quad \ldots (2)$$

Solving (1) and (2), we get

$$x = 0,\ y = 0$$

Thus (0, 0) is the only point o maximum or minimum.

Now
$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = [2y + 12x^2]_{(0,0)} = 0$$

$$s = \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{(0,0)} = [2x]_{(0,0)} = 0$$

$$t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = [2]_{(0,0)} = 2$$

$$\therefore \quad rt - s^2 = 0\,(2) - 0^2 = 0.$$

Thyus, the case is doubtful and further investigation will be required.

Example 2. *Find the maximum or minimum values of the function* $x^3y^2(1 - x - y)$.

Solution. Let $\quad u = x^3y^2(1 - x - y)$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 3x^2y^2(1 - x - y) - x^3y^2$$

and
$$\frac{\partial u}{\partial y} = 2x^3y(1 - x - y) - x^3y^2$$

For a maximum or minimum of u, we must have

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \quad 3x^2y^2(1 - x - y) - x^3y^2 = 0 \qquad \text{...(1)}$$

and
$$2x^3y(1 - x - y) - x^3y^2 = 0. \qquad \text{...(2)}$$

Now, subtracting (2) from (1), we have

$$x^2y(1 - x - y)(3y - 2x) = 0$$

which gives
$$y = \frac{2}{3}x.$$

Putting the value of y in (1), we get

$$x = \frac{1}{2}$$

so $\left(\frac{1}{2}, \frac{1}{3}\right)$ be the point of maxima or minima.

Now
$$r = \frac{\partial^2 u}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$= -\frac{1}{9}, \text{ at } \left(\frac{1}{2}, \frac{1}{3}\right)$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2x^3 - 2x^4 - 6x^2y$$

$$= -\frac{3}{8}, \text{ at } \left(\frac{1}{2}, \frac{1}{3}\right)$$

$$s = \frac{\partial^2 u}{\partial x\,\partial y} = -6x^2y - 8x^3y - 9x^2y^2$$

$$= -\frac{1}{12} \text{ at } \left(\frac{1}{2}, \frac{1}{3}\right)$$

Now, $\quad rt - s^2 =$ positive.

Also, r is negative, hence the function u has a maximum at $x = \frac{1}{2}, y = \frac{1}{3}$.

The maximum value is $= \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}.$

Example 3. *Find the maximum and minimum values of* $xy(a - x - y)$.

Solution. Let $\quad u = xy(a - x - y)$

Then $$\frac{\partial u}{\partial x} = ay - 2xy - y^2$$

and $$\frac{\partial u}{\partial y} = ax - x^2 - 2xy$$

For a maximum or minimum of u, we have

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0.$$

Thus, we have

$$ay - 2xy - y^2 = 0 \Rightarrow y(a - 2x - y) = 0 \quad ...(1)$$

$$ax - x^2 - 2xy = 0 \Rightarrow x(a - x - 2y) = 0. \quad ...(2)$$

Solving (1) and (2), we get the following pairs of values x and y which makes the function stationary

$$(0, 0), (0, a), (a, 0), \left(\frac{1}{3}a, \frac{1}{3}a\right).$$

Here $$r = \frac{\partial^2 u}{\partial x^2} = -2y,$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = a - 2x - 2y$$

and $$t = \frac{\partial^2 u}{\partial y^2} = -2x.$$

For (0, 0). $r = 0, s = a, t = 0.$

$\Rightarrow$ $rt - s^2$ is negative.

$\therefore$ We have neither a maximum nor a minimum of u at (0, 0).

For (0, *a*). $r = -2a, s = -a, t = 0$

$\Rightarrow$ $rt - s^2$ is negative.

$\therefore$ We have, neither a maximum nor a minimum of u at $(0, a)$.

Similarly, we have, neither a maximum nor a minimum of u at $(a, 0)$.

For $\left(\frac{1}{3}a, \frac{1}{3}a\right)$.

$$r = -\frac{2}{3}a, s = -\frac{1}{3}a, t = -\frac{2}{3}a$$

$\Rightarrow$ $rt - s^2$ is positive.

Since, $rt - s^2 > 0$.

$\therefore$ u has an extreme value at $\left(\frac{1}{3}a, \frac{1}{3}a\right)$.

$\Rightarrow$ u has a maximum if r is negative, *i.e.* if a is positive and u has a minimum if r is positive, *i.e.* if a is negative.

Example 4. *Show that the minimum value of* $u = xy + \left(\frac{a^3}{x}\right) + \left(\frac{a^3}{y}\right)$ *is* $3a^2$.

Solution. Here, we have

$$u = xy + \left(\frac{a^3}{x}\right) + \left(\frac{a^3}{y}\right)$$

$\Rightarrow$ $$\frac{\partial u}{\partial x} = y - \frac{a^3}{x^2}$$

and $$\frac{\partial u}{\partial y} = x - \frac{a^3}{y^2}.$$

For a maximum or minimum of u, we must have

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = 0 \Rightarrow y - \frac{a^3}{x^2} = 0 \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow x - \frac{a^3}{y^2} = 0. \qquad ...(2)$$

Solving (1) and (2), we get $x = a$, $y = a$

Now $$r = \frac{\partial^2 u}{\partial x^2} = \frac{2a^3}{x^3}, \quad s = \frac{\partial^2 u}{\partial x \partial y} = 1$$

and $$t = \frac{\partial^2 u}{\partial y^2} = \frac{2a^3}{y^3}.$$

At $x = y = a$. We have

$$r = 2, s = 1, t = 2$$

$\Rightarrow$ $$rt - s^2 = 3 > 0.$$

Thus, at (a, a), $rt - s^2 > 0$ and $r > 0$. Therefore u is minimum at $x = a$, $y = a$.

The minimum value of $u = a \cdot a + \left(\frac{a^3}{a}\right) + \left(\frac{a^3}{a}\right)$

$$= 3a^2.$$

Example 5. *Determine the points where a function $x^3 + y^3 - 3axy$ has a maximum or minimum.*

Solution. Here, we have

$$u = x^3 + y^3 - 3axy$$

$\Rightarrow$ $$\frac{\partial u}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3ax.$$

For a maximum or minimum of u, we must have

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0$$

which gives, $$x^2 - ay = 0 \qquad ...(1)$$

and $$y^2 - ax = 0. \qquad ...(2)$$

Solving (1) and (2), we get

$$x = 0, y = 0; x = a, y = a.$$

Thus (0, 0) and (a, a) are the stationary points of u.

Now $$r = \frac{\partial^2 u}{\partial x^2} = 6x, \quad s = \frac{\partial^2 u}{\partial x \partial y} = -3a, \quad t = \frac{\partial^2 u}{\partial y^2} = 6y.$$

For $x = 0, y = 0$. $r = 0$, $s = -3a$ and $t = 0$

$\therefore$ $$rt - s^2 = -9a^2 < 0, \text{ for all values of } a.$$

$\Rightarrow$ u is neither maximum nor minimum at $x = 0$, $y = 0$.

For $x = a, y = a$.

$$r = 6a, s = -3a \text{ and } t = 6a$$

$\Rightarrow$ $$rt - s^2 = 27a^2 > 0, \text{ for all values of } a.$$

Also $r = 6a$, which is positive if $a > 0$.

Thus **(i)** u is maximum at $x = a$, $y = a$ if $a < 0$

and **(ii)** u is minimum at $x = a$, $y = a$ if $a > 0$.

Example 6. *Discuss the maxima and minima of the function u is given by*

$$u = \sin x \sin y \sin (x + y).$$

Solution. Here, we have

$$u = \sin x \sin y \sin (x + y)$$

$\Rightarrow$ $$\frac{\partial u}{\partial x} = \sin y [\sin x \cos (x + y) + \cos x \sin (x + y)]$$

and $$\frac{\partial u}{\partial y} = \sin x\,[\sin y \cos (x + y) + \cos y \sin (x + y)].$$

For a maxima and minima of u, we must have

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0.$$

$\Rightarrow$ $$\sin y\,[\sin x \cos (x + y) + \cos x \sin (x + y)] = 0$$

and $$\sin x\,[\sin y \cos (x + y) + \cos y \sin (x + y)] = 0.$$

Equation (1) and (2) gives

$$\left.\begin{array}{ll} \tan (x + y) = -\tan x & \ldots(1) \\ \tan (x + y) = -\tan y & \ldots(2) \end{array}\right] \begin{array}{l} \Rightarrow \tan x = \tan y \\ \Rightarrow \quad x = y \end{array}$$

From (1) and (2), we have

$$\tan 2x = -\tan x = \tan (\pi - x)$$

$\Rightarrow$ $$2x = \pi - x$$

$$3x = \pi$$

$$x = \frac{\pi}{3} = y.$$

Moreover, $\frac{\partial u}{\partial x} = 0$, gives $\sin y = 0 \Rightarrow y = 0$

and $\frac{\partial u}{\partial y} = 0$ gives $\sin x = 0 \Rightarrow x = 0$.

Thus, we get the following pair of values, which makes the function u stationary.

$$(0, 0), \left(\frac{\pi}{3}, \frac{\pi}{3}\right).$$

Now $$r = \frac{\partial^2 u}{\partial x^2} = 2 \sin y \cos (2x + y),$$

$$s = \frac{\partial^2 u}{\partial x\, \partial y} = \sin 2 (x + y),$$

and $$t = \frac{\partial u}{\partial y^2} = 2 \sin x \cos (2y + x).$$

For (0, 0). $r = 0, s = 0; t = 0$

$\Rightarrow$ $rt - s^2 = 0.$

$\therefore$ this case is doubtful and need further investigation.

For $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$r = 2 \sin \frac{1}{3}\pi \,.\, \cos \pi = -\sqrt{3},$$

$$s = \sin\left(\frac{4\pi}{3}\right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2},$$

and $$t = 2 \sin \frac{1}{3}\pi \cos \pi = -\sqrt{3}.$$

$\therefore$ $rt - s^2 = \frac{9}{4}$ = positive.

Also $r = -\sqrt{3}.$

Hence, u has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Example 7. *Find the maximum value of $x^2 + y^2 + z^2$ when $ax + by + cz = p$.*

Solution. Here, we have

$$u = x^2 + y^2 + z^2 \quad \ldots(1)$$

given that $ax + by + cz = p$

$\Rightarrow$ $$z = \frac{p - ax - by}{c}.$$

Put this value of z in equation (1), we get

$$u = x^2 + y^2 + \frac{(p - ax - by)^2}{c^2}$$

$$\Rightarrow \qquad \frac{\partial u}{\partial x} = 2x - \frac{2a}{c^2}(p - ax - by)$$

and $$\frac{\partial u}{\partial y} = 2y - \frac{2b}{c^2}(p - ax - by).$$

For a maxima and minima of u, we must have $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$

$$\Rightarrow \qquad x = \frac{ap}{a^2 + b^2 + c^2} \text{ and } y = \frac{bp}{a^2 + b^2 + c^2}.$$

Now, $$r = \frac{\partial^2 u}{\partial x^2} = 2 + \frac{2a^2}{c^2},$$

$$s = \frac{\partial^2 u}{\partial x \, \partial y} = \frac{2ab}{c^2}$$

and $$t = \frac{\partial^2 u}{\partial y^2} = 2 + \frac{2b^2}{c^2}$$

$$\Rightarrow \qquad rt - s^2 = 4\left(1 + \frac{a^2}{c^2}\right)\left(1 + \frac{b^2}{c^2}\right) - \frac{4a^2b^2}{c^4}$$

$$= 4\left(1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}\right)$$

$$= \text{positive.}$$

Since r is positive and $rt - s^2 > 0$, therefore u is minimum for the above values of x and y.

The minimum value is $= \frac{p^2}{a^2 + b^2 + c^2}$.

• TEST YOURSELF–1

1. Find the points (x, y) where the function $f(x, y) = xy(1 - x - y)$ is maximum or minimum. Also find the maximum value of $f(x, y)$.
2. Discuss the maxima and minima of the function
$$f(x, y) = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}.$$
3. Discuss the maxima and minima of the function $f(x, y) = x^4 + 2x^2y - x^2 + 3y^2$.
4. Examine for maximum and minimum values of the function $f(x, y) = x^2 - 3xy + y^2 + 2x$.
5. Examine the function $f(x, y) = x^2y - y^2x - x + y$ for maxima and minima.
6. Discuss the maxima and minima of the function
$$f(x, y) = 2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y) + \cos (x + y).$$
7. Find points on $z^2 = xy + 1$ nearest to the origin.

ANSWERS

1. $f(x, y)$ is maximum at the point $\left(\frac{1}{3}, \frac{1}{3}\right)$; maximum value $= \frac{1}{27}$.
2. $f(x, y)$ is minimum at $(1, 1)$.
3. $f(x, y)$ is minimum for $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{4}\right)$.
4. Stationary point is $x = \frac{4}{5}, y = \frac{6}{5}$. The function $f(x, y)$ is neither maximum nor minimum at $\left(\frac{4}{5}, \frac{6}{5}\right)$.

5. At (1, 1) and (− 1, − 1) function is neither maximum nor minimum.
6. $x = y = 2n\pi \pm \pi/2$

   $x = y = n\pi + (-1)^n \pi/6$.
7. (0, 0, 1) and (0, 0, − 1).

• 7.5. MAXIMA AND MINIMA OF THE FUNCTION OF THREE INDEPENDENT VARIABLES

(1) To find the condition, which governs the sign of the quadratic equation of three independent variables.

Let I be the expression of three independent variables x, y and z given by

$$I = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

I can be written as

$$I = \frac{1}{a}[a^2x^2 + aby^2 + acz^2 + 2afyz + 2agzx + 2ahxy]$$

$$= \frac{1}{a}[a^2x^2 + 2ax(gz + hy) + aby^2 + acz^2 + 2afyz]$$

$$= \frac{1}{a}[(ax + hy + gz)^2 + aby^2 + acz^2 + 2afyz - (gz + hy)^2]$$

$$= \frac{1}{a}[(ax + hy + gz)^2 + (ab - h^2)y^2 + 2yz(af - gh) + (ac - g^2)z^2]$$

Here, we observe, that I be of the same sign as a provided the expression within the square brackets is positive which will of course be so if

$ab - h^2$ and $\{(ab - h^2)(ac - g^2) - (af - gh)^2\}$ are positive *i.e.*, if

$ab - h^2$ and $a[abc + 2fgh - af^2 - bg^2 - ch^2]$ are both positive.

Hence, I will be positive if

$$a, \begin{vmatrix} a & h \\ h & b \end{vmatrix}, \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

be all positive and will be negative if these three expression are alternately negative and positive.

• 7.6. MAXIMA AND MINIMA FOR A FUNCTION OF THREE INDEPENDENT VARIABLES : THE LAGRANGE'S CONDITION

Let $f(x, y, z)$ be a given function of three independent variables x, y and z.

Let A, B, C, F, G, H stand for $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial y\,\partial z}, \frac{\partial^2 f}{\partial z\,\partial x}, \frac{\partial^2 f}{\partial x\,\partial y}$ respectively.

Let a set of the values of x, y, z obtained by solving the equations

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

be a, b, c.

By Taylor's theorem, we have

$$f(a + h, b + k, c + l) - f(a, b, c)$$

$$= \frac{1}{2!}[Ah^2 + Bk^2 + Cl + 2Fkl + 2Glh + 2Hhk] + R \qquad ...(1)$$

where, remainder term R consist of third and higher order of same quantity (*i.e.*, h, k, l).

Now, by taking h, k, l sufficiently small the second term of R.H.S. of (1) can be made to govern the sign of R.H.S. and therefore of L.H.S. also.

If for all such values of h, k and l, these terms be of permanent sign, then we shall have a maximum or minimum of $f(x, y, z)$ according as that sign is negative or positive.

Hence, the function will be **minimum** if the expression

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \text{ be all positive.}$$

The function will have a **maximum** value, if the above three quantities are alternately negative and positive If these conditions are not satisfied, we have neither a maximum nor a minimum.

Working Procedure. Let us suppose $f(x, y, z)$ be a function of three independent variables x, y and z. Find the values of triads (a, b, c) of the value x, y and z by putting $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial z} = 0$. The values of triads (a, b, c) will give the stationary values of $f(x, y, z)$.

Now, to discuss maximum and minimum values, at (a, b, c) we find the following six partial derivatives of second order

$$A = \frac{\partial^2 f}{\partial x^2}, B = \frac{\partial^2 f}{\partial y^2}, C = \frac{\partial^2 f}{\partial z^2}, F = \frac{\partial^2 f}{\partial y \, \partial z}, G = \frac{\partial^2 f}{\partial z \, \partial x} \text{ and } H = \frac{\partial^2 f}{\partial x \, \partial y}.$$

Now, we have the following cases :

Case (i) The function $f(x, y, z)$ will be minimum at (a, b, c) if the expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \text{ be all positive at } (a, b, c).$$

Case (ii) The function $f(x, y, z)$ will be maximum at $(a. b, c)$ if the expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be alternately negative and positive.

Case (iii) If the expression, using in case (i) and (ii) neither be all positive nor having alternately negative and positive sign at (a, b, c). Then $f(x, y, z)$ is neither maximum nor minimum at (a, b, c).

REMARK

- To find the maximum and minimum of the function at stationary point, it is sufficient to find the value of a second order partial derivative of function with respect to any of the independent variables. Then, the value of the function is maximum or minimum according as the value of this second order partial derivative at the stationary point under consideration is negative or positive.

SOLVED EXAMPLES

Example 1. *Find the maximum value of u, where*

$$u = \frac{xyz}{(a+x)(x+y)(y+z)(z+b)}.$$

Solution. Here, we have

$$u = \frac{xyz}{(a+x)(x+y)(y+z)(z+b)}.$$

Taking, log of both the sides, we have

$$\log u = \log x + \log y + \log z - \log (a+x) - \log (x+y) - \log (y+z) - \log (z+b).$$

Differentiating w.r.t. x, we have

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} - \frac{1}{a+x} - \frac{1}{x+y} = \frac{ay - x^2}{x(a+x)(x+y)}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{(ay - x^2)\, u}{x(a+x)(x+y)}.$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{(xz - y^2)\, u}{y(x+y)(y+z)}$$

and

$$\frac{\partial u}{\partial z} = \frac{(by - z^2)\, u}{z(y+z)(z+b)}.$$

For, a maxima and minima of u, we must have

$$\frac{\partial u}{\partial x} = 0 \Rightarrow ay - x^2 = 0$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow xz - y^2 = 0$$

and
$$\frac{\partial u}{\partial z} = 0 \Rightarrow by - z^2 = 0.$$

Here, we observe that $x^2 = ay$, $y^2 = xz$, $z^2 = by$ which implies that a, x, y, z and b are in G.P. Let r be the common ratio of this G.P.

Then $\quad ar^4 = b \quad \text{or} \quad r = \left(\frac{b}{a}\right)^{1/4}.$

Also $\quad x = ar, \; y = ar^2, \; z = ar^3.$

Hence, we have

$$u = \frac{ar \, . \, ar^2 \, . \, ar^3}{a\,(1+r)\, ar\,(1+r)\, ar^2\,(1+r)\, ar^3\,(1+r)}$$

$$= \frac{1}{a\,(1+r)^4} = \frac{1}{a\left[1 + \left(\frac{b}{a}\right)^{1/4}\right]^4} = \frac{1}{(a^{1/4} + b^{1/4})^4}.$$

which gives a stationary value of u. Now, to decide whether this value of u is a maximum or a minimum, we proceed to find the second order partial derivative of u such that

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2ux}{x\,(a+x)\,(x+y)} + (ay - x^2)\,\frac{\partial}{\partial x}\left[\frac{u}{x\,(a+x)\,(x+y)}\right].$$

$\therefore$ When $x = ar$, $y = ar^2$, $z = ar^3$, we have

$$A = \frac{\partial^2 u}{\partial x^2} = -\frac{2u}{a^2 r\,(1+r^2)} < 0.$$

Hence, the above stationary value of u is maximum.

Example 2. *Find the maxima and minima value of the function*

$$u = \sin x \sin y \sin z$$

where x, y and z are the vertex angles of a triangle.

Solution. Here, we have

$$u = \sin x \sin y \sin z \, . \qquad \text{...(1)}$$

where $\quad x + y + z = \pi$

$\therefore \quad y = \sin x \sin y \sin\,[\pi - (x+y)]$

$\quad = \sin x \sin y \sin\,(x+y)$

$$\therefore \quad \frac{\partial u}{\partial x} = \cos x \sin y \sin\,(x+y) + \sin x \sin y \cos\,(x+y)$$

$$= \sin y \sin\,(2x+y) \qquad \text{...(2)}$$

Similarly
$$\frac{\partial u}{\partial y} = \sin x \sin\,(2y+x)\,. \qquad \text{...(3)}$$

For a maxima and minima, we must have

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = 0 \Rightarrow \sin y \sin\,(2x+y) = 0$$

$\Rightarrow \sin y = 0 \;\text{ or }\; \sin\,(2x+y) = 0$

$\Rightarrow y = 0 \;\text{ or }\; \sin\,(x+x+y) = 0$

$\Rightarrow y = 0$ or $\sin x \cos\,(x+y) + \cos x \sin\,(x+y) = 0$

$\Rightarrow \tan\,(x+y) = -\tan x$

$\Rightarrow \tan\,(x+y) = \tan\,(-x) = \tan\,(\pi - x) \qquad \text{...(4)}$

$\Rightarrow x + y = \pi - x$

$\Rightarrow 2x + y = \pi. \qquad \text{...(5)}$

Similarly, from (3)

$$y = 0 \text{ or } \tan(x + y) = -\tan y. \qquad \ldots(6)$$

Now, by (4) and (6), we have

$$\tan x = \tan y \Rightarrow x = y.$$

Hence, by (5), we have

$$3y = \pi \quad \Rightarrow \quad y = \frac{\pi}{3} \text{ and } x = \frac{\pi}{3}.$$

Here, the stationary points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and (0, 0).

For (0, 0).

$$u = 0$$

For $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$r = \frac{\partial^2 u}{\partial x^2} = 2 \sin y \cos(2x + y)$$

$$= 2 \sin \frac{\pi}{3} \cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right) = -\sqrt{3} < 0,$$

$$s = \frac{\partial^2 u}{\partial x \, \partial y} = \sin(2x + 2y) \quad \sin\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right)$$

$$= \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} < 0$$

and

$$t = \frac{\partial^2 u}{\partial y^2} = 2 \sin x \cos(x + 2y)$$

$$= 2 \sin \frac{\pi}{3} \cos \pi = -\sqrt{3} < 0.$$

Now

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{2} > 0.$$

Thus $\quad rt - s^2 > 0$ and $r < 0$.

Hence, the function u will be maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$.

• TEST YOURSELF–2

1. Prove that the function $u = x^2 + y^2 + z^2 + x - 2z - xy$ is minimum at $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$.
2. Find the maximum and minimum values of $u = y^2 + 2z^2 - 5x^4 + 4x^5$.

ANSWER

2. Minimum at (1, 0, 0), neither maximum nor minimum at (0, 0, 0).

• 7.7. LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Let $u = f(x_1, x_2, \ldots, x_n)$ be a function of n variables $x_1, x_2, \ldots, x_n$.

Let us suppose, these, variables $x_1, x_2, \ldots, x_n$ are connected by k equations

$$g_1(x_1, x_2, \ldots, x_n) = 0$$
$$g_2(x_1, x_2, \ldots, x_n) = 0$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$g_k(x_1, x_2, \ldots, x_n) = 0$$

so that there are $n - k$ independent variables out of these n variables.

For the maxima and minima of u, we find

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots + \frac{\partial u}{\partial x_n} dx_n = 0 \qquad \ldots(1)$$

Also
$$dg_1 = \frac{\partial g_1}{\partial x_1} dx_1 + \frac{\partial g_1}{\partial x_2} dx_2 + \ldots + \frac{\partial g_1}{\partial x_n} dx_n = 0 \qquad \ldots(2)$$

$$dg_2 = \frac{\partial g_2}{\partial x_1} dx_1 + \frac{\partial g_2}{\partial x_2} dx_2 + \ldots + \frac{\partial g_2}{\partial x_n} dx_n = 0 \qquad \ldots(3)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$dg_k = \frac{\partial g_k}{\partial x_1} dx_1 + \frac{\partial g_k}{\partial x_2} dx_2 + \ldots + \frac{\partial g_k}{\partial x_n} dx_n = 0. \qquad \ldots(k+1)$$

Multiplying equation (1), (2), (3) ...$(k + 1)$ by $1, l_1, l_2, \ldots, l_k$ respectively and adding, we get the result, which can be written as

$$P_1\, dx_1 + P_2\, dx_2 + P_3\, dx_3 + \ldots + P_n\, dx_n = 0 \qquad \ldots(4)$$

where
$$P_k = \frac{\partial u}{\partial x_k} + l_1 \frac{\partial g_1}{\partial x_k} + l_2 \frac{\partial g_2}{\partial x_k} + \ldots + l_k \frac{\partial g_k}{\partial x_k}.$$

Now we have at our choice k multiple *viz* $l_1, l_2, \ldots, l_k$ and can be chosen such that

$$P_1 = 0, P_2 = 0, \ldots, P_k = 0.$$

Then, the equation (4) reduces to

$$P_{k+1}\, dx_{k+1} + P_{k+2}\, dx_{k+2} + \ldots + P_n\, dx_n = 0. \qquad \ldots(5)$$

Now, let us suppose that out of n variables, the following $(n - k)$ variables $x_{k+1}, x_{k+2} \ldots, x_n$ are independent.

Then, since $n - k$ quantities $dx_{k+1}, dx_{k+2} \ldots dx_n$ are independent so their coefficients must be separately zero. Hence, we have

$$P_{k+1} = 0, P_{k+2} = 0, \ldots, P_n = 0.$$

Thus, we $k + n$ equations

$$\mathbf{P_1 = 0, P_2 = 0, \ldots, P_n = 0}$$

and
$$\mathbf{g_1 = 0, g_2 = 0, \ldots, g_k = 0.}$$

Hence, we get $(n + k)$ equations which determine the k multipliers $l_1, l_2, \ldots, l_k$ and get the possible value of u.

REMARKS

- The Lagrange's method of undetermined multipliers is very convenient to apply.
- It gives the maximum and minimum values of the function without actually determining the values of the multipliers $l_1, l_2, \ldots, l_k$.
- It does not determine the nature of stationary point, which is the only drawback of this method.

Application of the Method of Undetermined Multipliers.

The Lagrange's method of undetermined multipliers can be applied to determine the extreme values of the given functions, it does not detemine the nature of stationary point. Now, it is more convenient to find out the extreme values of a function F with the help of new function, given by

$$V = g + l_1 f_1 + l_2 f_2 + \ldots + l_m f_m$$

and use the following method. Here, we give the method for four variables x, y, u, v connected by the following two relations.

Let $F = g(x, y, u, v)$ be subjected to the conditions

$$f_1(x, y, u, v) = 0 \qquad \ldots(1)$$

and
$$f_2(x, y, u, v) = 0. \qquad \ldots(2)$$

For the maxima and minima of F, we have

$$dF = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial v}dv = 0. \qquad ...(3)$$

Now, from (1) and (2), we have

$$df_1 = \frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial u}du + \frac{\partial f_1}{\partial v}dv = 0 \qquad ...(4)$$

and
$$df_2 = \frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial u}du + \frac{\partial f_2}{\partial v}dv = 0. \qquad ...(5)$$

Multiplying (4) by l_1, (5) by l_2 and adding their sum to (3), we get

$$\left(\frac{\partial g}{\partial x} + l_1\frac{\partial f_1}{\partial x} + l_2\frac{\partial f_2}{\partial x}\right)dx + \left(\frac{\partial g}{\partial y} + l_1\frac{\partial f_1}{\partial y} + l_2\frac{\partial f_2}{\partial y}\right)dy$$
$$+\left(\frac{\partial g}{\partial u} + l_1\frac{\partial f_1}{\partial u} + l_2\frac{\partial f_2}{\partial u}\right)du + \left(\frac{\partial g}{\partial v} + l_1\frac{\partial f_1}{\partial v} + l_2\frac{\partial f_2}{\partial v}\right)dv = 0. \qquad ...(6)$$

Here, we have l_1 and l_2 are arbitrary, therefore we can choose them to satisfy the two linear equations

$$\frac{\partial g}{\partial x} + l_1\frac{\partial f_1}{\partial x} + l_2\frac{\partial f_2}{\partial x} = 0 \qquad ...(7)$$

and
$$\frac{\partial g}{\partial y} + l_1\frac{\partial f_1}{\partial y} + l_2\frac{\partial f_2}{\partial y} = 0. \qquad ...(8)$$

Using (7) and (8), equation (6) reduces to

$$\left(\frac{\partial g}{\partial u} + l_1\frac{\partial f_1}{\partial u} + l_2\frac{\partial f_2}{\partial u}\right)du + \left(\frac{\partial g}{\partial v} + l_1\frac{\partial f_1}{\partial v} + l_2\frac{\partial f_2}{\partial v}\right)dv = 0.$$

Since, the given function contains four variables (namely x, y, u and v) and we are given two equations of conditions, so therefore, only two of the variables are independent and it is immaterial which two of the four variables are regarded as independent. Let them be u and v then du and dv are also independent, therefore, their coefficients must separately zero. Thus

$$\frac{\partial g}{\partial u} + l_1\frac{\partial f_1}{\partial u} + l_2\frac{\partial f_2}{\partial u} = 0 \qquad ...(9)$$

$$\frac{\partial g}{\partial v} + l_1\frac{\partial f_1}{\partial v} + l_2\frac{\partial f_2}{\partial v} = 0. \qquad ...(10)$$

Now, we have six equations namely (1), (2), (7), (8), (9) and (10) to determine the two multipliers l_1, l_2 and values of the four variables x, y, u and v for which maximum and minimum values of F are possible.

Now, defined a new function $V(x, y, u, v)$ such that

$$V(x, y, u, v) = g(x, y, u, v) + l_1 f_1(x, y, u, v) + l_2 f_2(x, y, u, v).$$

Assuming that x, y, u, v are now all independent variables. Hence, for the maxima and minima of V, we must have

$$\frac{\partial V}{\partial x} = \frac{\partial g}{\partial x} + l_1\frac{\partial f_1}{\partial x} + l_2\frac{\partial f_2}{\partial x} = 0 \qquad ...(11)$$

$$\frac{\partial V}{\partial y} = \frac{\partial g}{\partial y} + l_1\frac{\partial f_1}{\partial y} + l_2\frac{\partial f_2}{\partial y} = 0 \qquad ...(12)$$

$$\frac{\partial V}{\partial u} = \frac{\partial g}{\partial u} + l_1\frac{\partial f_1}{\partial u} + l_2\frac{\partial f_2}{\partial u} = 0 \qquad ...(13)$$

and
$$\frac{\partial V}{\partial v} = \frac{\partial g}{\partial v} + l_1\frac{\partial f_1}{\partial v} + l_2\frac{\partial f_2}{\partial v} = 0. \qquad ...(14)$$

Equations (11), (12), (13) and (14) are exactly the same as the equations (7), (8), (9) and (10).

Hence, the maxima and minima of $V(x, y, u, v)$ are same as those of $F(x, y, u, v)$ assuming that $V(x, y, u, v)$ the variables x, y, u, v are now all independent.

Now, we proceed to find whether the values of F obtained with the help of above equations are maximum or minimum. For this, adopt the procedure, which is discussed below.

From (3), we get

$$d^2F = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial u}du + \frac{\partial}{\partial v}dv\right)^2 g + \left(\frac{\partial g}{\partial x}d^2x + \frac{\partial g}{\partial y}d^2y + \frac{\partial g}{\partial u}d^2u + \frac{\partial g}{\partial v}d^2v\right) \quad ...(15)$$

Also

$$d^2f_1 = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial u}du + \frac{\partial}{\partial v}dv\right)^2 f_1 + \frac{\partial f_1}{\partial x}d^2x + \frac{\partial f_1}{\partial y}d^2y + \frac{\partial f_1}{\partial u}d^2u + \frac{\partial f_1}{\partial v}d^2v = 0 \quad ...(16)$$

and $$d^2f_2 = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial u}du + \frac{\partial}{\partial v}dv\right)^2 f_2 + \frac{\partial f_2}{\partial x}d^2x + \frac{\partial f_2}{\partial y}d^2y + \frac{\partial f_2}{\partial u}d^2u + \frac{\partial^2 f}{\partial v}d^2v = 0. \quad ...(17)$$

Mutiplying (16) by l_1 and (17) by l_2 and adding their sum to (15) and using the result (11), (12), (13) and (14), we have

$$d^2F = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial u}du + \frac{\partial}{\partial v}dv\right)^2 (g + l_1f_1 + l_2f_2)$$

$$= \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial u}du + \frac{\partial}{\partial v}dv\right)^2 V$$

$$= d^2V.$$

Hence d^2F is equal to d^2V, where d^2V is obtained by assuming all the variables x, y, u and v as independent. Therefore, it is clear that d^2V and d^2F have the same sign. Hence, F will be minimum or maximum according as V is minimum or maximum.

REMARK

- This method has the advantage over the Lagrange's methods that it enables us to decide whether the values are maximum or minimum.

SOLVED EXAMPLES

Example 1. *Find the maxima and minima of $x^2 + y^2 + z^2$ subject to the conditions* :

$$ax^2 + by^2 + cz^2 = 1$$

and $$lx + my + nz = 0.$$

Solution. Here, we have

$$u = x^2 + y^2 + z^2 \quad ...(1)$$

where, the relations between the variables x, y and z are given by

$$ax^2 + by^2 + cz^2 = 1 \quad ...(2)$$

and $$lx + my + nz = 0. \quad ...(3)$$

For the maxima and minima of u, we must have

$$du = 0$$

$$\Rightarrow \quad 2x\,dx + 2y\,dy + 2z\,dz = 0$$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0 \quad ...(4)$$

From (2) and (3), we get

$$ax\,dx + by\,dy + cz\,dz = 0 \quad ...(5)$$

and $$l\,dx + m\,dy + n\,dz = 0. \quad ...(6)$$

Now, multiplying (4) by 1, (5) by l_1 and (6) by l_2 and adding, we get

$$(x\,dx + y\,dy + z\,dz) + l_1(ax\,dx + by\,dy + cz\,dz) + l_2(l\,dx + m\,dy + n\,dz) = 0$$

$$\Rightarrow (x + al_1x + ll_2)\,dx + (y + bl_1y + ml_2)\,dy + (z + cl_1z + nl_2)\,dz = 0.$$

Now equating the coefficient of dx, dy, dz to zero, we get

$$x + l_1ax + l_2l = 0 \quad ...(7)$$

$$y + l_1by + l_2m = 0 \quad ...(8)$$

and $$z + l_1cy + l_2n = 0. \quad ...(9)$$

Multiplying the equations (7), (8) and (9) by x, y and z respectively, and adding we get

$$x^2 + y^2 + z^2 + l_1 (ax^2 + by^2 + cz^2) + l_2 (lx + my + nz) = 0$$

or $$u + l_1 \,.\, 1 + l_2 \,.\, 0 = 0 \qquad \text{[by using (1), (2) and (3)]}$$

$\Rightarrow$ $$l_1 = -u.$$

Substituting for l_1 in the equations (7), (8) and (9), we get

$$x = \frac{l_2 l}{au - 1}, \; y = \frac{l_2 m}{bu - 1}, \; z = \frac{l_2 n}{cu - 1}. \qquad ...(10)$$

Now from (10) and (3), we get

$$\frac{l_2 l^2}{au - 1} + \frac{l_2 m^2}{bu - 1} + \frac{l_2 n^2}{cu - 1} = 0$$

or $$\frac{l^2}{au - 1} + \frac{m^2}{bu - 1} + \frac{n^2}{cu - 1} = 0 \qquad ...(11)$$

which gives the maximum and minimum of $u = x^2 + y^2 + z^2$.

Example 2. *Find the maxima and minima of $x^2 + y^2 + z^2$, where*

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1.$$

Solution. Here, we have

$$u = x^2 + y^2 + z^2 \qquad ...(1)$$

where the relation between the variables x, y and z is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1. \qquad ...(2)$$

For a maximum or minima of u, we must have

$$du = 0$$

$\Rightarrow$ $$x\,dx + y\,dy + z\,dz = 0. \qquad ...(3)$$

From (2), we have

$$2ax\,dx + 2by\,dy + 2cz\,dz + 2fy\,dz + 2fz\,dy + 2gz\,dx + 2gx\,dz + 2hx\,dy + 2hy\,dx = 0$$

$$\Rightarrow (ax + hy + gz)\,dx + (hx + by + fz)\,dy + (gx + fy + cz)\,dz = 0. \qquad ...(4)$$

Now, multiplying (3) by 1 and (4) by l_1, and adding, and then equating the coefficient of dx, dy, dz to zero, we have

$$x + l_1 (ax + hy + gz) = 0 \qquad ...(5)$$

$$y + l_1 (hx + by + fz) = 0 \qquad ...(6)$$

and $$z + l_1 (gx + fy + cz) = 0 \qquad ...(7)$$

Multiplying (5) by x, (6) by y, (7) by z and adding, we get

$$x^2 + y^2 + z^2 + l_1 (ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0$$

$\Rightarrow$ $$u + l_1 \,.\, 1 = 0 \qquad \text{[From (1) and (2)]}$$

$\therefore$ $$l_1 = -u.$$

Hence, from (5), we have

$$x - u (ax + hy + gz) = 0$$

$\Rightarrow$ $$\left(a - \frac{1}{u}\right) x + hy + gz = 0. \qquad ...(8)$$

Similarly from (6) and (7), we get

$$hx + \left(b - \frac{1}{u}\right) y + fz = 0 \qquad ...(9)$$

and $$gx + fy + \left(c - \frac{1}{u}\right) z = 0 \qquad ...(10)$$

Eliminating x, y, z from (8), (9) and (10), we get

$$\begin{vmatrix} \left(a - \frac{1}{u}\right) & h & g \\ h & \left(b - \frac{1}{u}\right) & f \\ g & f & \left(c - \frac{1}{u}\right) \end{vmatrix} = 0 \qquad ...(11)$$

Hence, the maximum or minimum values of u are the roots of the equation (11).

Example 3. *Find the maximum and minima of $u = x^2 + y^2$ subject to the condition.*

$$ax^2 + 2hxy + by^2 = 1.$$

Solution. Here, we have

$$u = x^2 + y^2 \qquad ...(1)$$

where the relation between the variables x and y is

$$ax^2 + 2hxy + by^2 = 1. \qquad ...(2)$$

For the maxima and minima of u, we must have

$$du = 0$$

$$\Rightarrow \quad 2x\,dx + 2y\,dy = 0$$

$$\Rightarrow \quad x\,dx + y\,dy = 0. \qquad ...(3)$$

Now, from (2), we get

$$2ax\,dx + 2hx\,dy + 2hy\,dx + 2by\,dy = 0$$

$$\Rightarrow \quad (ax + hy)\,dx + (hx + by)\,dy = 0. \qquad ...(4)$$

Now, multiplying (3) and (1), (4) by l_1 and adding, then equating the coefficients of dx, dy to zero, we have

$$x + l_1\,(ax + hy) = 0 \qquad ...(5)$$

and

$$y + l_1\,(hx + by) = 0. \qquad ...(6)$$

Multiplying (5) by x, (6) by y and adding, we get

$$x^2 + y^2 + l_1\,(ax^2 + 2hxy + by^2) = 0$$

$$\Rightarrow \quad u + l_1 \,.\, 1 = 0 \qquad \text{[Using (1) and (2)]}$$

$$\Rightarrow \quad u = -\,l_1.$$

Therefore, from (5), we have

$$x - u\,(ax + hy) = 0$$

$$\Rightarrow \quad \left(a - \frac{1}{u}\right)x + hy = 0. \qquad ...(7)$$

Similarly from (6), we have

$$hx + \left(b - \frac{1}{u}\right)y = 0. \qquad ...(8)$$

Eliminating x and y from (7) and (8), we get

$$\begin{vmatrix} a - \frac{1}{u} & h \\ h & b - \frac{1}{u} \end{vmatrix} = 0. \qquad ...(9)$$

Hence, the maximum or minimum values of u are the roots of the equation (9).

Example 4. *Show that the maximum and minimum values of*

$$u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

subject to the conditions

$$lx + my + nz = 0$$

and

$$x^2 + y^2 + z^2 = 1$$

are given by the equation

$$\begin{vmatrix} a - u & h & g & l \\ h & b - u & f & m \\ g & f & c - u & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

Solution. Here, we have

$$u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy. \qquad ...(1)$$

The variables x, y and z are connected by the relation

$$lx + my + nz = 0 \qquad ...(2)$$

$$x^2 + y^2 + z^2 = 1. \quad ...(3)$$

Differentiating (1), (2) and (3), we get

$$du = 2\,(ax + gz + hy)\,dx + 2\,(by + fz + hx)\,dy + 2\,(cz + fy + gx)\,dz \quad ...(4)$$

$$l\,dx + m\,dy + n\,dz = 0 \quad ...(5)$$

and

$$x\,dx + m\,dy + n\,dz = 0. \quad ...(6)$$

For the maxima and minima of u we must have

$$du = 0$$

$$\Rightarrow (ax + hy + gz)\,dx + (by + fz + hx)\,dy + (cz + fy + gx)\,dz = 0. \quad ...(7)$$

Now multiplying (7) by 1, (5) by l_1 and (6) by l_2, and adding then equating the coefficients of dx, dy and dz to zero, we get

$$(ax + hy + gz) + ll_1 + l_2x = 0 \quad ...(8)$$

$$(hx + by + fz) + l_1m + l_2y = 0 \quad ...(9)$$

and

$$(gx + fy + cz) + l_1n + l_2z = 0. \quad ...(10)$$

Now, multiplying (8), (9) and (10) by x, y and z respectively then adding and using (1), (2) and (3), we get

$$u + l_1 \,.\, 0 + l_2 \,.\, 1 = 0$$

$$\Rightarrow \qquad l_2 = -u$$

Now putting $l_2 = -u$ in (8), (9) and (10), we get

$$(a - u)\,x + hy + gz + l_1l = 0 \quad ...(11)$$

$$hx + (b - u)\,y + fz + l_1m = 0 \quad ...(12)$$

$$gx + fy + (c - u)\,z + l_1n = 0. \quad ...(13)$$

Now eliminating x, y, z and l_1 from (2), (11), (12) and (13), we get

$$\begin{vmatrix} a-u & h & g & l \\ h & b-u & f & m \\ g & f & c-u & n \\ l & m & n & 0 \end{vmatrix} = 0$$

which gives the required maximum and minimum value of u.

Example 5. *In a plane triangle ABC, find the maximum value of*

$$u = \cos A \cos B \cos C.$$

Solution. Here, we have

$$u = \cos A \cos B \cos C. \quad ...(1)$$

Since, we know that the sum of the angles of a triangle is always 180°.

$\therefore$ The variables A, B and C are connected by the relation

$$A + B + C = \pi. \quad ...(2)$$

From (1), we get

$$\log u = \log \cos A + \log \cos B + \log \cos C$$

$$\Rightarrow \qquad \frac{1}{u}\,du = -\tan A\,dA - \tan B\,dB - \tan C\,dC.$$

For the maxima and minima of u, we must have

$$du = 0$$

$$\Rightarrow \qquad \tan A\,dA + \tan B\,dB + \tan C\,dC = 0. \quad ...(3)$$

Also from (2),

$$dA + dB + dC = 0. \quad ...(4)$$

Now, multiply (3) by 1, (4) by l and adding, equating the coefficients of dA, dB and dC to zero, we get

$$\tan A + l = 0$$

$$\tan B + l = 0$$

$$\tan C + l = 0$$

$$\Rightarrow \qquad l = -\tan A = -\tan B = -\tan C$$

$$\Rightarrow \qquad A = B = C.$$

Now from (2), $A = B = C = \frac{\pi}{3}$ *i.e.* the triangle is equilateral.

Now to show that the stationary value of u given by

$$A = B = C = \frac{\pi}{3} \text{ is maximum.}$$

Now, let C be a function of A and B, regarding A and B as independent variables. From (1),

$$\log u = \log \cos A + \log \cos B + \log \cos C$$

$$\Rightarrow \qquad \frac{1}{u}\frac{du}{dA} = -\tan A - \tan C\,\frac{\partial C}{\partial A}.$$

Now, differentiating (2), partially w.r.t. A, we get

$$1 + \frac{\partial C}{\partial A} = 0 \Rightarrow \frac{\partial C}{\partial A} = -1$$

$$\therefore \qquad \frac{1}{u}\frac{\partial u}{\partial A} = -\tan A + \tan C$$

$$\Rightarrow \qquad \frac{1}{u}\frac{\partial^2 u}{\partial^2 A} - \frac{1}{u^2}\left(\frac{\partial u}{\partial A}\right)^2 = -\sec^2 A + \sec^2 C \cdot \frac{\partial C}{\partial A}$$

$$= -(\sec^2 A + \sec^2 C).$$

At stationary point $\qquad \frac{\partial u}{\partial A} = 0.$

$$\therefore \qquad \frac{\partial^2 u}{\partial^2 A} = -u(\sec^2 A + \sec^2 C)$$

$$= -\text{ ve for } A = B = C = \frac{\pi}{3}.$$

Hence, u is maximum at $A = B = C = \frac{\pi}{3}$ and the maximum value is given by

$$u = \left(\cos\frac{\pi}{3}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

• SUMMARY

- Let $f(x, y) = 0$ be a functon of two variables. For maxima or minima of $f(x, y) = 0$, we must have

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0.$$

Suppose at the point (a, b), $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, then we calculate

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,\,b)}, \quad s = \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{(a,\,b)}, \quad t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,\,b)}$$

We have following cases :

Case I : If $rt - s^2 > 0$, then

(i) $f(x, y)$ is maximum at (a, b) if $r < 0$.

(ii) $f(x, y)$ is minimum at (a, b) if $r > 0$.

Case II : If $rt - s^2 < 0$, then

$f(x, y)$ is neither maximum nor minimum at (a, b).

Case III : If $rt - s^2 = 0$, then this case is doubtful and further investigation will be required.

- Let $f(x, y, z) = 0$ be a function of three variables :

Suppose at (a, b, c), $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$, then we calculate the values at (a, b, c) :

$$A = \frac{\partial^2 f}{\partial y^2}, \quad B = \frac{\partial^2 f}{\partial y^2}, \quad C = \frac{\partial^2 f}{\partial z^2}$$

$$F = \frac{\partial^2 f}{\partial y\,\partial z}, \quad G = \frac{\partial^2 f}{\partial z\,\partial x}, \quad H = \frac{\partial^2 f}{\partial x\,\partial y}$$

We have following cases :

Case I : $f(x, y, z)$ will be minimum at (a, b, c) if

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \text{ are all positive at } (a, b, c).$$

Case II : $f(x, y, z)$ will be maximum at (a, b, c) if

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \text{ are alternately negative and positive.}$$

Case III : $f(x, y, z)$ is neither maximum nor minimum if

- $$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \text{ are neither all positive nor fitted with alternative signs.}$$

- If $u = f(x_1, x_2, \ldots\ldots, x_n)$ be a function of n variables $x_1, x_2, \ldots., x_n$ and suppose that $x_1, x_2, x_3, \ldots\ldots, x_4$ are connected by k equations :

$$g_1(x_1, x_2, \ldots\ldots, x_n) = 0$$
$$g_2(x_1, x_2, \ldots\ldots, x_n) = 0$$
$$g_k(x_1, x_2, \ldots\ldots, x_n) = 0$$

so that there are $n - k$ independent variables.

For the maxima and minima of u, we define

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots. \frac{\partial u}{\partial x_n} dx_n = 0$$

$$dg_1 = \frac{\partial g_1}{\partial x_1} \partial x_1 + \frac{\partial g_1}{\partial x_2} dx_2 + \ldots. + \frac{\partial g_1}{\partial x_n} = 0$$

....

$$dg_k = \frac{\partial g_k}{\partial x_1} \partial x_1 + \frac{\partial g_k}{\partial x_2} dx_2 + \ldots. + \frac{\partial g_k}{\partial x_n} dx_n = 0$$

• STUDENT ACTIVITY

1. Find the maximum and minimum values of $xy(a - x - y)$.

2. Find the maxima and minima of $x^2 + y^2 + z^2$ subject to the conditons :

$$ax^2 + by^2 + cz^2 = 1$$

and $$lx + my + nz = 0.$$

• TEST YOURSELF-3

Using Lagrange's method of undetermined multipliers :

1. Find the maximum and minimum values of

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$$

where $lx+my+nz=0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

2. Find the maximum and minimum values of

$$f=a^2x^2+b^2y^2+c^2z^2$$

where $x^2+y^2+z^2=1$ and $lx+my+nz=0.$

3. Show that the maximum and minimum values of

$$u=x^2+y^2+z^2$$

subject to the conditions

$$px+qy+rz=0 \quad \text{and} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

are given by

$$\frac{a^2p^2}{u-a^2}+\frac{b^2q^2}{u-b^2}.$$

4. Find the minimum value of

$$u=x+y+z$$

subject to the condition

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1.$$

5. Find the minimum value of $u=x^2+y^2+z^2$, subject to the condition

$$ax+by+cz=p.$$

ANSWERS

1. The maximum and minimum values of the given function is given by the equation

$$\frac{l^2a^4}{a^2u-1}+\frac{m^2b^4}{b^2u-1}+\frac{n^2c^4}{c^2u-1}=0.$$

2. The maximum and minimum values of the given function is given by

$$\frac{l^2}{u-a^2}+\frac{m^2}{u-b^2}+\frac{n^2}{u-c^2}=0.$$

4. Stationary points are

$$x=\sqrt{a}\,(\sqrt{a}+\sqrt{b}+\sqrt{c}),\quad y=\sqrt{b}\,(\sqrt{a}+\sqrt{b}\,(+\sqrt{a}\sqrt{c}+)\,\sqrt{b}+\sqrt{c})$$

minimum value is $(\sqrt{a}+\sqrt{b}+\sqrt{c})^2$

5. Minimum value is $\frac{p^2}{(a^2+b^2+c^2)}$.

OBJECTIVE EVALUATION

Fill in the Blanks :

1. For a function $f(x, y, z)$. to be a maximum or minimum, it is .... that $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$ and $\frac{\partial f}{\partial z}=0.$
2. In any triangle ABC, the maximum value of $\cos A \cos B \cos C$ is ....
3. If the Lagrange's condition $rt-s^2>0$ and $r>0$ are satisfied then function is ....
4. If the Lagrange's condition $rt-s^2>0$ and $r<0$ are satisfied then function is ....

True or False :

Write T for true and F for false :

1. The value of the function at extreme point is always called maximum value. (T/F)
2. The value of the function at extreme point is always called minimum value. (T/F)
3. The value of the function at extreme point is always called extreme value. (T/F)
4. The stationary value may be a maximum or minimum. (T/F)
5. The stationary point can be obtained by solving the simultaneous equations

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0.$$ (T/F)

Multiple Choice Questions :

Choose the most appropriate one :

1. If the Lagrange's condition $rt - s^2 < 0$ is satisfied, then the function is :
   (a) maximum (b) minimum
   (c) neither maximum nor minimum (d) none of these.
2. For the maxima and minima of a function u it is necessary that :
   (a) $du = 0$ (b) $du \neq 0$ (c) $du > 0$ (d) $du < 0$.
3. The value of the function at extreme point is called :
   (a) extreme value (b) maximum value
   (c) minimum value (d) none of these.
4. The stationary value is :
   (a) always maximum (c) always minimum
   (c) either maximum or minimum (d) neither maximum nor minimum.
5. In any triangle ABC, the maximum value of $\cos A \cos B \cos C$ is equal to :
   (a) 1 (b) π (c) $\frac{1}{8}$ (d) 0.

ANSWERS

Fill in the Blanks :

1. Necessary 2. $\frac{1}{8}$ 3. Minimu 4. Maximum

True or False :

1. F 2. F 3. T 4. T 5. T

Multiple Choice Questions :

1. (c) 2. (a) 3. (a) 4. (c) 5. (c)

❑❑❑

UNIT 8

BETA AND GAMMA FUNCTIONS

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- What are Gamma and Beta functions ?
- How to find the relation btween Beta and Gamma functions.
- How to find the solutions of the concerned problems using Beta and Gamma functions?

• 8.1. GAMMA FUNCTION

(1) The definite integral

$$\int_0^\infty e^{-x} x^{n-1}\, dx, \text{ for } n > 0$$

is known as the gamma function and is denoted by $\Gamma(n)$ ['read as Gamma n']. Gamma function is also called the Eulerian integral of second kind.

REMARK

➤ The integral is valid only for $n > 0$ because it is for just those values of m and n that the above integral are convergent.

• 8.2. PROPERTIES OF GAMMA FUNCTIONS

(1) *To show that* $\Gamma(1) = 1$.

Solution. We have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx, \; n > 0.$$

Put $n = 1$ in equation of gamma function

$$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1}\, dx$$

$$= \int_0^\infty e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 1$$

$\therefore \quad \Gamma(1) = 1.$

(2) *To show that* $\Gamma(n+1) = n\,\Gamma(n)$, $n > 0$.

Solution. We have

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx,\ n > 0$$

replacing n by $(n+1)$, we have

$$\Gamma(n+1) = \int_0^\infty e^{-x}x^{n+1-1}\,dx$$

$$= \int_0^\infty e^{-x}x^n\,dx$$

$$= \left[x^n . (-e^{-x})\right]_0^\infty - \int_0^\infty (nx^{n-1})(-e^{-x})\,dx \quad \text{[on integrating by parts]}$$

$$\therefore \quad \Gamma(n+1) = -\lim_{x\to\infty} \frac{x^n}{e^x} + 0 + n\int_0^\infty e^{-x}x^{n-1}\,dx \quad ...(1)$$

$$\left(\because \operatorname{Lim}_{x\to 0} x^n e^{-x} = 0 \text{ as } n > 0\right)$$

But
$$\operatorname{Lim}_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{x^n}{1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}}$$

$$= \operatorname{Lim}_{x\to\infty} \frac{1}{\frac{1}{x^n} + \frac{1}{1!\,x^{n-1}} + \ldots + \frac{1}{n!} + \frac{x}{(n+1)} + \ldots}$$

$$= 0. \quad ...(2)$$

Also, by definition, we have

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx. \quad ...(3)$$

Using (2) and (3), (1) reduces to

$$\Gamma(n+1) = n\,\Gamma(n)$$

(3) *If n is a non-negative integer, then* $\mathbf{\Gamma(n+1) = n\,!}$.

Solution. We know that for $n > 0$,

$$\Gamma(n+1) = n\,\Gamma(n)$$
$$= n\,\Gamma(n-1+1)$$
$$= n\,(n-1)\,\Gamma(n-1) \quad \text{[by property 2]}$$
$$= n\,(n-1)\,(n-2)\,\Gamma(n-2) \quad \text{[by (1)]}$$
$$= n\,(n-1)\,(n-2)\ldots 3\,.\,2\,.\,1\,.\,\Gamma(1)$$
$$= n\,!. \quad [\because\ \Gamma(1) = 1]$$

(4) *To show that* $\mathbf{\Gamma(1/2) = \sqrt{\pi}}$.

Solution. By definition, we have

$$\Gamma(n) = \int_0^\infty e^{-t}t^{n-1}\,dt n > 0. \quad ...(1)$$

Replacing n by $1/2$ in equation (1), we get

$$\Gamma(1/2) = \int_0^\infty e^{-t}t^{-1/2}\,dt = 2\int_0^\infty e^{-u^2}\,du \quad ...(2)$$

[Putting $t = u^2$, so that $dt = 2u\,du$]

$$\therefore \quad \Gamma(1/2) = 2\int_0^\infty e^{-x^2}\,dx \text{ and } \Gamma(1/2) = 2\int_0^\infty e^{-y^2}\,dy. \quad ...(3)$$

(limits remaining same)

Multiplying the corresponding sides of two equations of (3), we get

$$[\Gamma(1/2)]^2 = \left(2\int_0^\infty e^{-x^2}\,dx\right)\left(2\int_0^\infty e^{-y^2}\,dy\right)$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

Now, changing the variables to polar co-ordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow x^2 + y^2 = r^2$ and $dx\,dy = r\,d\theta\,dr$ we have

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,d\theta\,dr.$$

The area of integration in the positive quadrant of plane is given by

$$= 2\int_0^{\pi/2}\left\{\int_0^\infty 2e^{-r^2}\, r\,.\,dr\right\} d\theta.$$

Putting $r^2 = v$, so that $2r\,dr = dv$

$$= 2\int_0^{\pi/2}\left[-e^{-v}\right]_0^\infty d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\,[\,\theta\,]_0^{\pi/2} = \pi$$

these $[\Gamma(1/2)]^2 = \pi$ so that $\Gamma(1/2) = \sqrt{\pi}$.

(5) *To show that* $\Gamma(n) = \int_0^1 (\log 1/y)^{n-1}\,dy$.

Solution. By definition of gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx,\ n > 0.$$

Putting $x = \log(1/y)$ in gamma function we get

$$\Gamma(n) = -\int_1^0 (\log 1/y)^{n-1}\,dy = \int_0^1 (\log 1/y)^{n-1}\,dy.$$

• 8.3. SOME TRANSFORMATION OF GAMMA FUNCTIONS

Gamma function is given by

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx. \qquad ...(1)$$

(*i*) *Show that* $\dfrac{\Gamma(n)}{a^n} = \int_0^\infty e^{-ay}y^{n-1}\,dy$.

Solution. We have

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx,\ n > 0.$$

Put $x = ay$, so that $dx = a\,dy$.
When $x = 0$, $y = 0$ and when $x \to \infty$, $y \to \infty$.

$$\therefore \quad \Gamma(n) = \int_0^\infty e^{-ay}(ay)^{n-1}\,.\,a\,dy.$$

Hence, $$\int_0^\infty e^{-ay}y^{n-1}\,dy = \frac{\Gamma(n)}{a^n}.$$

(*ii*) *Show that*

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}}\,dx,\ n > 0.$$

Solution. We have

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx,\ n > 0.$$

Put $x^n = t$.

So that $nx^{n-1}\,dx = dt$, then (1) gives

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-t^{1/n}}\,dt$$

$$\Rightarrow \qquad \Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}}\,dx. \qquad \text{[By the property of definite integral]}$$

(iii) *Show that* $\Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}\,dx,\ n > 0.$

Solution. We have $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx.$

Put $x = t^2$ so that $dx = 2t\,dt$

$$\Gamma(n) = \int_0^\infty e^{-t^2}(t^2)^{n-1}\,2t\,dt$$

or

$$\Gamma(n) = 2\int_0^\infty e^{-t^2}t^{2n-1}\,dt$$

$$\Rightarrow \qquad \Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}\,dx.$$

SOLVED EXAMPLES

Example 1. *Evaluate* :

(*i*) $\int_0^\infty e^{-x}x^4\,dx$ (*ii*) $\int_0^\infty x^6 e^{-2x}\,dx.$

Solution. (i) We have $\int_0^\infty e^{-x}x^4\,dx$

$$= \int_0^\infty e^{-x}x^{5-1}\,dx, \qquad \text{[by definition of gamma function]}$$

$$= \Gamma(5)$$

$$= (4)! = 24.$$

(ii) Let $I = \int_0^\infty x^6 e^{-2x}\,dx.$ ...(1)

Put $2x = t$, so that $dx = 1/2\,dt$ then

$$I = \int_0^\infty \left(\frac{t}{2}\right)^6 e^{-t}\cdot\frac{1}{2}\,dt = \frac{1}{2^7}\int_0^\infty e^{-t}t^{7-1}\,dt$$

$$= \frac{1}{2^7}\Gamma(7) \qquad \text{[by definition of gamma function]}$$

$$= \frac{1}{2^7}\times(6\,!) = \frac{45}{8}.$$

Example 2. *Show that* $\int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \sqrt{\pi}$.

Solution. We know that

$$\Gamma(n) = \int_0^1 (-\log x)^{n-1}\, dx.$$

Putting $n = 1/2$, we have

$$\Gamma(1/2) = \int_0^1 (-\log x)^{(1/2)-1}\, dx$$

or $$\sqrt{\pi} = \int_0^1 (-\log x)^{-1/2}\, dx$$

or $$\sqrt{\pi} = \int_0^1 \frac{dx}{\sqrt{(-\log x)}}.$$

Example 3. *Prove that*

(*a*) $\int_0^\infty xe^{-\alpha x} \cos \beta x\, dx = \frac{\alpha^2 - \beta^2}{(\alpha^2 + \beta^2)^2}$, $\alpha > 0$

(*b*) $\int_0^\infty xe^{-\alpha x} \sin \beta x\, dx = \frac{2\alpha\beta}{(\alpha^2 + \beta^2)}$, $\alpha > 0$.

Solution. We know that

$$\int_0^\infty e^{-kx} x^{n-1}\, dx = \frac{\Gamma(n)}{k^n}, \quad n > 0, k > 0. \qquad ...(1)$$

Putting $k = \alpha - i\beta$ and $n = 2$ in (1), we get

$$\int_0^\infty e^{-(\alpha - i\beta)x}\, x\, dx = \frac{\Gamma(2)}{(\alpha - i\beta)^2}$$

or $$\int_0^\infty xe^{-\alpha x} e^{i\beta x}\, dx = \frac{(\alpha + i\beta)^2}{(\alpha - i\beta)^2 (\alpha + i\beta)^2} \quad \text{as } \Gamma(2) = 1$$

$$= \int_0^\infty xe^{-\alpha x} e^{i\beta x}\, dx = \frac{\alpha^2 - \beta^2 + 2i\alpha\beta}{[(\alpha + i\beta)(\alpha - i\beta)]^2}$$

$$= \int_0^\infty xe^{-\alpha x} (\cos \beta x + i \sin \beta x)\, dx$$

$$= \frac{\alpha^2 - \beta^2 + 2i\alpha\beta}{(\alpha^2 + \beta^2)^2}$$

or $$\int_0^\infty xe^{-\alpha x} \cos \beta x\, dx + i \int_0^\infty xe^{-\alpha x} \sin \beta x\, dx = \frac{\alpha - \beta^2}{(\alpha^2 + \beta^2)^2} + i\frac{2\alpha\beta}{(\alpha^2 + \beta^2)^2}.$$

Equating real and imaginary parts of both sides, we get

$$\int_0^\infty xe^{-\alpha x} \cos \beta x\, dx = \frac{\alpha^2 - \beta^2}{(\alpha^2 + \beta^2)^2}$$

and $$\int_0^\infty xe^{-\alpha x} \sin \beta x = \frac{2\alpha\beta}{(\alpha^2 + \beta^2)^2}.$$

Example 4. *Show that* $\int_0^\infty \frac{x^c}{c^x}\, dx = \frac{\Gamma(c + 1)}{(\log c)^{c+1}}$, $c > 0$.

Solution. $$\int_0^\infty \frac{x^c}{c^x}\,dx = \int_0^\infty x^c c^{-x}\,dx$$

$$= \int_0^\infty x^c\,[e^{\log_e c}]^{-x}\,dx \qquad [\because\ c = e^{\log_e c} \text{ if } c \geq 0]$$

$$= \int_0^\infty x^{(c+1)-1}\,e^{-x\log_e c}\,dx$$

$$= \frac{\Gamma(c+1)}{(\log_e c)^{c+1}} \qquad \left[\because\ \int_0^\infty x^{n-1}e^{-kx}\,dx = \frac{\Gamma(n)}{k^n}\ n>0, k>0\right]$$

• 8.4. BETA FUNCTION

Definition. The definite integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \text{ for } m > 0, n > 0$$

is known as the Beta function and denoted by $B(m, n)$ which is read as "Beta m, n", where m, n are positive number or integers. Thus

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

Beta function is also called the Eulerian integral of first kind.

• 8.5. PROPERTIES OF BETA FUNCTION

(i) **Symmetry of beta function** *i.e.*, $B(m, n) = B(n, m)$.

By the definition of beta function, we have

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx \qquad \left[\because\ \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{m-1}x^{n-1}\,dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= B(n, m) \qquad \text{[By the definition of beta function]}$$

$$B(m, n) = B(n, m)$$

i.e., the interchange of position of m and n does not change the value of beta function. This is the fundamental property of beta function and also called **symmetry property** of bēta function.

(ii) **Beta function $B(m, n)$ can be evaluated in an explicit form if m or n is a positive integer.**

Case I. When 'n' is a positive integer.

If $n = 1$, then by definition of beta function

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad ...(1)$$

$$\Rightarrow \qquad B(m, 1) = \int_0^1 x^{m-1}(1-x)^{1-1}\,dx$$

$$= \int_0^1 x^{m-1}\,dx = \left[\frac{x^m}{m}\right]_0^1 = \frac{1}{m}. \qquad \ldots(2)$$

Now, let $n > 1$, then from (1), we have

$$B(m, n) = \int_0^1 (1-x)^{n-1} x^{m-1}\,dx$$

$$= \left[(1-x)^{n-1} \cdot \frac{x^m}{m}\right]_0^1 - \int_0^1 (n-1)(1-x)^{n-2} \cdot (-1)\frac{x^m}{m}\,dx.$$

Integrating by parts taking x^{m-1} as second function, we have

$$= 0 + \frac{n-1}{m}\int_0^1 x^m (1-x)^{n-2}\,dx \qquad [\because\ n > 1]$$

$$\text{since } \lim_{x \to 0} (1-x)^{n-1}\frac{x^m}{m} = 0$$

$$= \frac{n-1}{m}\int_0^1 x^{(m+1)-1}(1-x)^{(n-1)-1}\,dx$$

$$= \frac{n-1}{m} B(m+1, n-1).$$

Thus $$B(m, n) = \frac{n-1}{m} B(m+1, n-1). \qquad \ldots(3)$$

Now replacing m by $m + 1$ and n by $n - 1$ in (3) then we get

$$B(m+1, n-1) = \frac{n-1-1}{m+1} B(m+2, n-2). \qquad \ldots(4)$$

Using equation (4), the equation (3) becomes

$$B(m, n) = \frac{n-1}{m} \cdot \frac{n-2}{m+1} B(m+2, n-2). \qquad \ldots(5)$$

After applying the above process successively, we get

$$B(m, n) = \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots \frac{1}{m+n-2} B(m+n-1, 1) \qquad \ldots(6)$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots \frac{1}{m+n-2}\int_0^1 x^{m+n-2}(1-x)^0\,dx$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots \frac{1}{m+n-2}\left[\frac{x^{m+n-1}}{m+n-1}\right]_0^1$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots \frac{1}{m+n-2} \cdot \frac{1}{m+n-1}$$

$$B(m, n) = \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots \frac{1}{m+n-2} \cdot \frac{1}{m+n-1}$$

$$B(m, n) = \frac{(n-1)!}{m(m+1)(m+2)\ldots(m+n-2)(m+n-1)}. \qquad \ldots(7)$$

Case II. When m is a positive integer.

Since the beta function is symmetrical in m and n *i.e.*, $B(m, n) = B(n, m)$ therefore by case 1 interchanging m and n in case 1 equation and we get

$$B(m, n) = \frac{(m-1)!}{n(n+1)(n+2)\ldots(n+m-2)(n+m-1)}. \qquad \ldots(8)$$

Case III. When both m and n are positive integers.

We have, by case 1

$$B(m, n) = \frac{(n-1)!}{m(m+1)(m+2)-(m+n-2)(m+n-1)}$$

$$= \frac{[1\,.\,2\,.\,3\,\ldots\,(m-1)]\,(n-1)\,!}{1\,.\,2\,.\,3\,\ldots\,m\,(m+1)\,(m+2)\,\ldots\,(m+n-2)\,(m+n-1)}.$$

Multiplying both numerator and denominator by $1\,.\,2\,.\,3\,\ldots\,(m-1)\,!$, we get

$$B(m,n) = \frac{(m-1)\,!\,(n-1)\,!}{(m+n-1)\,!}.$$

• 8.6. TRANSFORMATION OF BETA FUNCTION

The Beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad \ldots(A)$$

can be transformed into many forms given below :

(I) $\boldsymbol{B(m,n) = \int_0^\infty \frac{x^{n-1}\,dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx.}$

Solution. Put $x = \frac{1}{(1+y)}$ and $dx = -\frac{dy}{(1+y)^2}$ and [$y \to 0$ when $x = 1$, $y \to \infty$, when $x = 0$].

$$\therefore \qquad B\,(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left[1-\frac{1}{1+y}\right]^{n-1}\left[\frac{-dy}{(1+y)^2}\right]$$

$$= \int_0^\infty \frac{(y)^{n-1}}{(1+y)^{m+1}}\left(\frac{1}{1+y}\right)^{n-1}.\,dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

or

$$B\,(m,n) = \int_0^\infty \frac{x^{n-1}\,dx}{(1+x)^{m+n}}. \qquad \ldots(1)$$

Since m and n are interchangeable in beta function by symmetry property therefore (1) gives

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

thus

$$B(m,n) = \int_0^\infty \frac{x^{n-1}\,dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}}.$$

(II) $\boldsymbol{B(m,n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta.}$

Solution. Put $x = \sin^2\theta$ and $dx = 2\sin\theta\cos\theta\,d\theta$
and when $x = 0$, $\theta = 0$, $\theta = \pi/2$ when $x = 1$.

$$\therefore \qquad B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta$$

[by symmetry property of beta function]

(III) $\boldsymbol{B(m,n) = \frac{1}{a^{m+n-1}}\,.\int_0^a x^{m-1}(a-x)^{n-1}\,dx.}$

Solution. Put $x = y/a$, *i.e.* $dx = \frac{1}{a}\,dy$

and when $x \to 0$, then $y \to 0$, when $x = 1$ then $y \to a$.

So
$$B(m, n) = \frac{1}{a^{m+n-1}} \int_0^a y^{m-1} (a - y)^{n-1} \, dy$$

$$= \frac{1}{a^{m+n-1}} \int_0^a x^{m-1} (a - x)^{n-1} \, dx$$

(IV) $\dfrac{B(m, n)}{a^n (1 + a)^m} = \displaystyle\int_0^1 \frac{x^{m-1} (1 - x)^{n-1} \, dx}{(x + a)^{m+n}}$.

Solution. Let $\dfrac{x}{1 + a} = \dfrac{t}{t + a}$

$$dx = a (1 + a) \frac{dt}{(t + a)^2}$$

then we have

$$B(m, n) = \int_0^1 (1 + a)^{m-1} \left(\frac{t}{t + a}\right)^{m-1} a^{n-1} \left(\frac{1 - t}{a + t}\right)^{n-1} \frac{a (a + 1)}{(t + a)^2} \, dt$$

$$= a^n (1 + a)^m \int_0^1 \frac{t^{m-1} (1 - t)^{n-1}}{(t + a)^{m+n}} \, dt$$

$$= a^n (1 + a)^m \int_0^1 \frac{x^{m-1} (1 - x)^{n-1}}{(x + a)^{m+n}} \, dx$$

Hence,
$$\frac{B(m, n)}{a^n (1 + a)^m} = \int_0^1 \frac{x^{m-1} (1 - x)^{n-1}}{(x + a)^{m+n}} \, dx$$

(V) $B(m, n) (a - b)^{m+n-1} = \displaystyle\int_b^a (x - b)^{m-1} (a - x)^{n-1} \, dx$.

Solution. Put
$$x = \frac{t - b}{a - b}$$

so that
$$dx = \frac{dt}{a - b}.$$

Put all these values in the formula of Beta function, we get

$$B(m, n) = \int_b^a \left(\frac{t - b}{a - b}\right)^{m-1} \left(\frac{a - t}{a - b}\right)^{n-1} \frac{dt}{a - b}$$

$$= \frac{1}{(a - b)^{m+n-1}} \int_b^a (t - b)^{m-1} (a - t)^{n-1} \, dx$$

$$= \frac{1}{(a - b)^{m+n-1}} \int_b^a (x - b)^{m-1} (a - x)^{n-1} \, dx$$

$$\therefore \; B(m, n) (a - b)^{m+n-1} = \int_b^a (x - b)^{m-1} (a - x)^{n-1} \, dx$$

(VI) $\dfrac{1}{a^n b^m} B(m, n) = \displaystyle\int_0^1 \frac{x^{m-1} (1 - x)^{n-1} \, dx}{\{a + (b - a) x\}^{m+n}}$

Solution. We put

$$\frac{a}{y} - \frac{b}{x} = a - b. \qquad \text{...(1)}$$

$$\therefore \quad \frac{b}{x} = \frac{a}{y} + (b - a) = \frac{a + (b - a) y}{y}$$

$$\Rightarrow \quad x = \frac{by}{a + (b - a) y} \qquad \text{...(2)}$$

$$\Gamma n \, . \, x^{m-1} e^{-x} = \int_0^\infty x^{n+m-1} y^{n-1} e^{-(y+1)x} \, dy.$$

Integrating both sides with respect to x within limits $x = 0$ to $x = \infty$, we have

$$\Gamma(n) \int_0^\infty x^{m-1} e^{-x} \, dx = \int_0^\infty \left[\int_0^\infty x^{n+m-1} e^{-(y+1)x} \, dx \right] y^{n-1} \, dy \qquad ...(3)$$

But $\int_0^\infty x^{(n+m)-1} e^{-(y+1)x} dx = \dfrac{\Gamma(n+m)}{(1+y)^{m+n}}$.

Hence with the help of this result and (2), we get from (3)

$$\Gamma(n)\,\Gamma(m) = \int_0^\infty \Gamma(n+m) \frac{y^{n-1}}{(1+y)^{n+m}} dy$$

$$= \Gamma(n+m) \int_0^\infty \frac{y^{n-1}}{(1+y)^{n+m}} dy = \Gamma(n+m)\, B(m, n) \qquad (ii)$$

or
$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(n+m)}.$$

Cor. 1. $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$, *where* $0 < n < 1$.

Proof. We know that

$$B(m, n) = \int_0^\infty \frac{x^{n-1} \, dx}{(1+x)^{m+n}}; \; m > 0, n > 0.$$

Therefore the relation between beta and gamma functions becomes

$$\int_0^\infty \frac{x^{n-1} \, dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Taking $m + n = 1$, so that $m = 1 - n$, we get

$$\int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)}, \quad 0 < n < 1.$$

[$\because$ $m > 0 \Rightarrow 1 - n > 0 \Rightarrow n < 1$ Also $n > 0$]

But we know that

$$\int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi} \text{ and } \Gamma(1) = 1.$$

$$\therefore \quad \frac{\pi}{\sin n\pi} = \Gamma(1-n)\,\Gamma(n), \; 0 < n < 1.$$

Cor. 2. *To show that* $\Gamma(1/2) = \sqrt{\pi}$.

Proof. We have just proved that

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}. \qquad ...(1)$$

Putting $n = 1/2$ in (1), we obtain

$$\Gamma(1/2)\,\Gamma(1 - 1/2) = \frac{\pi}{\sin \pi/2}$$

or
$$[\Gamma(1/2)]^2 = \pi$$
$$\Gamma(1/2) = \sqrt{\pi}.$$

Aliter. We know $B(m, n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$.

Putting $m = n = 1/2$ in it, we get

$$B(1/2, 1/2) = \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1/2 + 1/2)} = \frac{\{\Gamma(1/2)\}^2}{\Gamma(1)}$$

or $$\{\Gamma(1/2)\}^2 = B(1/2, 1/2) = \int_0^1 x^{(1/2)-1}(1-x)^{(1/2)-1}\,dx$$

$$= \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx$$

$$= \int_0^1 \frac{dx}{\sqrt{x}\sqrt{1-x}}$$

$$= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\,d\theta}{\sin\theta\sqrt{(1-\sin^2\theta)}} \qquad \text{putting } x = \sin^2\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\,[\,\theta\,]_0^{\pi/2} = 2\,(\pi/2) = \pi$$

$$= \{(\Gamma(1/2)\}^2 = \pi \quad \Gamma(1/2) = \sqrt{\pi}.$$

Cor. 3. *To show that* $\int_0^1 e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$.

Proof. $$\int_0^1 e^{-x^2}\,dx = \int_0^1 e^{-y}\,.\frac{1}{2\sqrt{y}}\,dy, \text{ putting } x^2 = y,\ 2x\,dx = dy$$

$$= \frac{1}{2}\int_0^1 e^{-y}\,y^{-1/2}\,dy$$

$$= \frac{1}{2}\int_0^1 e^{-y}\,y^{(1/2)-1}\,dy$$

$$= \frac{1}{2}\Gamma(1/2) \qquad \left[\because \int_0^{\infty} e^{-x}\,.\,x^{n-1}\,dx = \Gamma n\right]$$

$$= \frac{1}{2}\sqrt{\pi} \qquad [\because \Gamma(1/2) = \sqrt{\pi}]$$

$$\Rightarrow \quad \int_0^{\infty} e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}.$$

Cor. 4. *To prove that*

$$\int_0^{\pi/2} \cos^m\theta\,\sin^n\theta\,d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}.$$

For all values of m and n such that $m > -1, n > -1$.

Proof. We put

$$\sin^2\theta = x,$$

$$\Rightarrow \quad 2\sin\theta\cos\theta\,d\theta = dx$$

$$\Rightarrow \quad 2\sin\theta\,.\sqrt{(1-\sin^2\theta)}\,d\theta = dx$$

$$\Rightarrow \quad 2x^{1/2}\sqrt{1-x}\,d\theta = dx$$

$$\therefore \quad d\theta = \frac{dx}{2x^{1/2}(1-x)^{1/2}}.$$

when $\theta = \pi/2, x = 1$ and $\theta = 0, x = 0$.

Putting these values in L.H.S. of the given equation, we get

$$\int_0^{\pi/2} \cos^m\theta\,\sin^n\theta\,d\theta = \int_0^{\pi/2} (1-\sin^2\theta)^{m/2}\,\sin^n\theta\,d\theta$$

So that $dx = \frac{1}{6} \cdot y^{-5/6}\, dy$

$$\therefore \quad I = \frac{1}{6}\int_0^{\infty} \frac{y^{1/6} \cdot y^{-5/6}}{1+y}\, dy$$

$$= \frac{1}{6}\int_0^{\infty} \frac{y^{-2/3}}{1+y}\, dy$$

$$= \frac{1}{6}\int_0^{\infty} \frac{y^{(1/3)-1}}{(1+y)^{2/3+1/3}}\, dy = \frac{1}{6} B(1/3, 2/3)$$

$$= \frac{1}{6}\frac{\Gamma(1/3)\,\Gamma(2/3)}{\Gamma(1/2+2/3)} = \frac{1}{6}\frac{\Gamma(1/3)\,\Gamma(1-1/3)}{\Gamma 1} = \frac{1}{6}\frac{\pi}{\sin\frac{\pi}{3}}$$

$$\left[\because \Gamma n\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{6} \cdot \frac{\pi}{(\sqrt{3}/2)} = \frac{1}{6} \cdot \frac{2\pi}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}.$$

Example 2. *Show that* $\int_0^2 (8-x^3)^{-1/3}\, dx = \frac{2\pi}{3\sqrt{3}}$

Solution. Let $x^3 = 8t$, then $x = 2t^{1/3}$.

$$\therefore \quad dx = \frac{2}{3} t^{-2/3}\, dt,$$

and when $x = 0$ to $x = 2$, $t = 0$ to $t = 1$.

$$\int_0^2 (8-x^3)^{-1/3}\, dx = \int_0^1 (8-8t)^{-1/3} \cdot \frac{2}{3} t^{-2/3}\, dt$$

$$= (8)^{-1/3} \cdot \frac{2}{3}\int_0^1 t^{-2/3}(1-t)^{-1/3}\, dt$$

$$= \frac{1}{3}\int_0^1 t^{(1/3)-1}(1-t)^{2/3)-1}(1-t)^{(2/2)-1}\, dt$$

$$= \frac{1}{3} B\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)}$$

$$= \frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)}{\Gamma(1)}$$

$$= \frac{1}{3}\frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}.$$

• TEST YOURSELF–1

1. Show that $\int_0^{\infty} e^{-4x} x^{3/2}\, dx = \frac{3\sqrt{\pi}}{128}$.

2. Show that $\int_0^{\infty} e^{-x^2} \cdot x^2\, dx = \frac{\sqrt{\pi}}{4}$.

3. Show that $\int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \sqrt{\pi}$.

4. Show that $\int_0^1 x^{n-1}\left(\log\frac{1}{x}\right)^{m-1} dx = \frac{\Gamma(m)}{n^m}$ $m > 0$, $n > 0$.

5. Show that $\int_0^1 \left(\frac{1}{x}-1\right)^{1/4} dx = B(5/4, 3/4) = \frac{\pi}{2\sqrt{2}}$.

• 8.8. DUPLICATION FORMULA

To prove that $\Gamma(n)\,\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$, $n > 0$.

Proof. We know that

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \text{ where } m > 0, n > 0. \quad ...(1)$$

Now putting $m = n$ in equation (1), we get

$$B(n, n) = \frac{[\Gamma(n)]^2}{\Gamma(2n)}. \quad ...(2)$$

By definition of beta function, we get

$$B(n, n) = \int_0^1 x^{n-1}(1-x)^{n-1}\,dx.$$

Putting $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$ in (1), we get

$$B(n, n) = \int_0^{\pi/2} (\sin^2\theta)^{n-1}(1-\sin^2\theta)^{n-1} . 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2n-1}\,d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2n-1} d\theta$$

$$= \frac{1}{2^{2n-2}}\int_0^{\pi/2} \sin^{2n-1} 2\theta\,d\theta$$

$$= \frac{1}{2^{2n-2}}\int_0^{\pi} \sin^{2n-1}\phi\,\frac{d\phi}{2} \qquad \text{By putting } 2\theta = \phi, \therefore d\theta = \tfrac{1}{2}d\phi$$

$$= \frac{1}{2^{2n-1}}\int_0^{\pi} \sin^{2n-1}\phi\,d\phi$$

$$= \frac{1}{2^{2n-2}}\int_0^{\pi/2} \sin^{2n-1}\phi\,d\phi$$

$$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ when } f(2a-x) = f(x)\right]$$

$$= \frac{1}{2^{2n-2}}\int_0^{\pi/2} \sin^{2n-1}\phi\,(\cos\phi)^0\,d\phi$$

$$= \frac{1}{2^{2n-2}}\,\frac{\Gamma\left(\frac{2n-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{2n-1+0+2}{2}\right)}$$

$$\therefore \quad B(n, n) = \frac{1}{2^{2n-1}} \cdot \frac{\Gamma(n)\sqrt{\pi}}{\Gamma(n+\frac{1}{2})} \text{ as } \Gamma(1/2) = \sqrt{\pi}.$$

Equating two values of $B(n, n)$ given by (2) and (3), we obtain

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = \frac{1}{2^{2n-1}} \frac{\Gamma(n)\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}$$

or

$$\Gamma(n)\,\Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n) \qquad \text{...(4)}$$

SOLVED EXAMPLES

Example 1. *Express* $\Gamma(1/6)$ *in terms of* $\Gamma(1/3)$.

Solution. By duplication formula, we have

$$\Gamma(n)\,\Gamma(n+1/2) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n) \qquad \text{...(1)}$$

Putting $n = 1/6$ in (1), we get

$$\Gamma(1/6)\,\Gamma(2/3) = \frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}} \Rightarrow \Gamma(1/6) = \frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}\,\Gamma(2/3)}. \qquad \text{...(2)}$$

Also, we know that

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \qquad \text{...(3)}$$

Putting $n = 1/3$ in (3), we get

$$\Gamma(1/3)\,\Gamma(2/3) = \frac{\pi}{\sin(\pi/3)} = 2\pi/\sqrt{3}.$$

$$\Gamma(2/3) = \frac{2\pi}{\sqrt{3}\,\Gamma(1/3)}.$$

Substituting the value of $\Gamma(2/3)$ given by (4) in (2), we get

$$\Gamma(1/6) = \frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}} \cdot \frac{\sqrt{3}\,\Gamma(1/3)}{2\pi} = \frac{\sqrt{3}}{2^{1/3}\sqrt{\pi}}\,[\Gamma(1/3)]^2$$

Example 2. *Prove that* $\int_{-\infty}^{\infty} \cos\frac{\pi}{2}x^2\,dx = 1.$

Solution. Let $\quad I = \int_{-\infty}^{\infty} \cos\frac{1}{2}\pi x^2\,dx.$

Since $\cos\frac{1}{2}\pi x^2$ is an even function therefore (1) gives

$$I = 2\int_0^{\infty} \cos\frac{1}{2}\pi x^2\,dx. \qquad \text{...(2)}$$

Putting $x^2 = t$ so that $x = t^{1/2}$ and $dx = (1/2)\,t^{-1/2}\,dt$ then equation (2) reduces to

$$I = 2\int_0^{\infty} \cos\frac{1}{2}\pi t \,.\, \frac{1}{2}t^{-1/2}\,dt$$

$$= \int_0^{\infty} (t)^{1/2-1}\cos\frac{1}{2}\pi t\,dt = \frac{\Gamma(1/2)}{(\pi/2)^{1/2}}\cos\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$\left[\because \int_0^{\infty} x^{m-1}\cos bx\,dx = \frac{\Gamma(m)}{b^m}\cos\frac{m\pi}{2}, \text{ Here } m = \frac{1}{2}, b = \frac{\pi}{2}\right]$$

$$= \frac{\Gamma(1/2)}{(\pi/2)^{1/2}}\cos\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{\pi}/\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = 1.$$

Example 2. *Prove that* $\int_0^\infty \cos (bx^{1/n})\, dx = \frac{\Gamma(n+1)}{b^n} \cos \frac{n\pi}{2}$

Solution. Let $$I = \int_0^\infty \cos (bx^{1/n})\, dx \quad \ldots(1)$$

Putting $x = t^n$, so that $dx = nt^{n-1}\, dt$ the (1) gives

$$I = n \int_0^\infty \cos (bt) \,.\, t^{n-1}\, dt = \frac{n\, \Gamma(n)}{b^n} \cos \frac{n\pi}{2}$$

$$= \frac{\Gamma(n+1)}{b^n} \cos \frac{n\pi}{2}.$$

Aliter. $\int_0^1 \cos (bz^{1/n})\, dz = \frac{1}{b^2} \Gamma(n+1) \cos n\pi/2.$

Solution. Put $\quad z^{1/n} = x$

So that $\quad dz = nx^{n-1}\, dx.$

$$\therefore \int_0^\infty \cos (bz^{1/n})\, dz = \int_0^\infty \cos (bx) \,.\, nx^{n-1}\, dx$$

$$= n \int_0^\infty x^{n-1} \cos (bx)\, dx$$

$$= \text{real part of } n \int_0^\infty e^{-bxi} x^{n-1}\, dx$$

$$= \text{real part of } n \frac{\Gamma(n)}{(bi)^n}$$

$$= \text{real part of } \frac{n\,\Gamma(n)}{b^n} (\cos \pi/2 + i \sin \pi/2)^{-n}$$

$$= \text{real part of } \frac{\Gamma(n+1)}{b^n} \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right)$$

$$= \frac{1}{b^n} \Gamma(n+1) \cos (n\pi/2).$$

• SUMMARY

- Gamma function :

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0$$

Also, (i) $\Gamma(1) = 1$

(ii) $\Gamma(n+1) = n\, \Gamma(n)$

(iii) $\Gamma(n+1) = n!$ n is a non-negative integer

(iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

- Beta function :

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx, \quad m > 0, \quad n > 0$$

Also (i) $B(m, n) = B(n, m)$

(ii) $B(m, n) = \int_0^\infty \frac{x^{n-1}\, dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

(iii) $B(m, n) = 2 \int_0^{\pi/2} \cos^{2m-1} \theta \sin^{2n-1} \theta\, d\theta$

ANSWER

Fill in the Blanks :

1. 1 2. n 3. $\sqrt{\pi}$ 4. $\Gamma(n)$

True or False :

1. T 2. F 3. T 4. F

Multiple Choice Questions :

1. (b) 2. (a) 3. (c) 4. (b)

U N I T

9

MULTIPLE INTEGRALS

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- What are double and triple integrals ?
- How can we use these integrals in the application of concerned fields such as areas, surfaces and volumes
- How to change the variable to other variables ?
- How to change the order of integration ?

• 9.1. DOUBLE INTEGRALS

Double integral is an extension of a definite integral in two-dimensional space. Let $f(x, y)$ be a single valued function of x and y, bounded and defined in the region R of XY-plane. Let A be the area of region R and let R be divided in any manner into n-sub regions $\alpha_1, \alpha_2, \ldots, \alpha_n$ whose areas are $\delta s_1, \delta s_2, \ldots, \delta s_n$ respectively. Let $p_r(\xi_r, \eta_r)$ be any point inside the region $\alpha_n \cdot \beta_n = f(\xi_1)$.

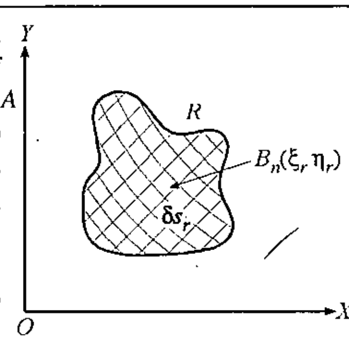

Fig. 1

Let $B_n = \sum_{r=1}^{n} f(\xi_r, \mu_r)\, \delta s_r$ then the limits of B_n which is assumed to be existed as $n \to \infty$ such that every $\alpha_r \to 0$ in all its dimensions is known as double integral of $f(x, y)$ over the region R and is denoted by

$$\int_R f(x, y)\, ds$$

or

$$\iint_R f(x, y)\, dx\, dy.$$

Hence, the area R is called the region or field of integration for the double integral and ds is called element of area.

• 9.2. PROPERTIES OF A DOUBLE INTEGRAL

(I) When the region R is partitioned into two parts say R_1 and R_2 then

$$\iint_R f(x, y)\, dx\, dy = \iint_{R_1} f(x, y)\, dx\, dy + \iint_{R_2} f(x, y)\, dx\, dy.$$

Similarly, we divide the region into three or more parts.

(II) The double integral of a algebraic sum of a fixed number of functions is equal to the algebraic sum of double integrals taken for each term separately. Thus

$$\iint_R [f_1(x, y) + f_2(x, y) + f_3(x, y) + \ldots]\, dx\, dy$$

$$= \iint_R f_1(x, y)\, dx\, dy + \iint_R f_2(x, y)\, dx\, dy + \iint_R f_3(x, y)\, dx\, dy + \ldots$$

(III) A constant factor may be taken outside the integral sign. Thus

$$\iint_R m f(x, y)\, dx\, dy = m \iint_R f(x, y)\, dx\, dy$$

where m is a constant.

• 9.3. EVALUATION OF DOUBLE INTEGRALS

(i) Over a rectangular region *R*. If the region R be given by the inequalities $a \le x \le b$, $c \le y \le d$, then the double integral

$$\iint_R f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dx\, dy = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx. \qquad \ldots(1)$$

We first evaluate $\int_c^d f(x, y)\, dy$ *i.e.*, integrate $f(x, y)$ with respect to y regarding x as constant and then resulting function of x is to be integrated with respect to x between the limits a and b

or $$\iint_R f(x, y)\, dx\, dy = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy. \qquad \ldots(2)$$

Here, we integrate $\int_a^b f(x, y)\, dx$ and then integrate with respect to y.

(ii) Over the regions which are not rectangular. Let the region R be described by $a \le x \le b$ and $\phi_1(x) \le y \le \phi_2(x)$ so that $y = \phi_1(x)$ and $y = \phi_2(x)$ respectively, the boundary of R then

$$\iint_R f(x, y)\, dx\, dy = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy\right] dx.$$

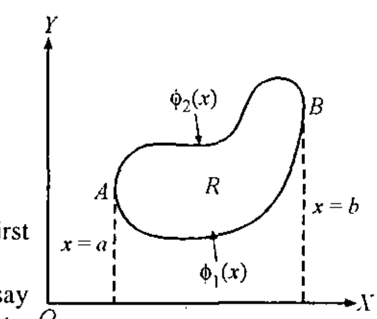

Fig. 2

Here, the inner integral $\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy$ is integrated first and in this integral the result of integration is a function of x, say $\phi_1(x)$, then $\phi_1(x)$ is integrated with respect to x between the limits a and b to obtain the value of double integral.

In a similar way, if R can be described by

$$c \le y \le d, \quad \phi_3(y) \le x \le \phi_4(y)$$

then we get

$$\iint_R f(x, y)\, dx\, dy = \int_c^d \left[\int_{\phi_3(y)}^{\phi_4(y)} f(x, y)\, dx\right] dy.$$

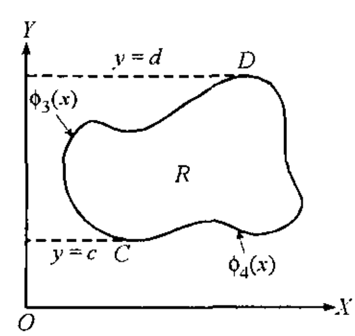

Fig. 3

Here, the result of integration

$$\int_{\phi_3(y)}^{\phi_4(y)} f(x, y)\, dx,$$

which is evaluated first, is a function of y say $\phi_2(y)$, then $\phi_2(y)$ is integrated with respect to y between the limits c to d.

Working Procedure. While evaluating double integrals, first integrate with respect to variable having variable limits and treating the other variable as constant and then integrate with respect to variable with constant limits. In case the limits of integration of both the variables are constants.

SOLVED EXAMPLES

Example 1. *Evaluate* $\int_1^2 \int_0^{y/2} y\, dy\, dx$.

Solution. We have

$$\int_1^2 \int_0^{y/2} y\, dy\, dx = \int_1^2 y\, [x]_0^{y/2}\, dy = \int_1^2 y\left(\frac{1}{2} y\right) dy$$

$$= \frac{1}{2}\int_1^2 y^2\, dy = \frac{1}{2}\left[\frac{1}{3}y^3\right]_1^2 = \frac{1}{6}(2^3 - 1^3)$$

$$= 7/6.$$

Example 2. *Evaluate* $\int_1^2 \int_0^x \frac{1}{x^2+y^2}\, dx\, dy$.

Solution. We have

$$\int_1^2 \int_0^x \frac{dx\, dy}{x^2+y^2} = \int_1^2 \left[\int_0^x \frac{dy}{x^2+y^2}\right] dx$$

$$= \int_1^2 \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_{y=0}^{x} dx$$

$$= \int_1^2 \left[\frac{1}{x}(\tan^{-1} 1 - \tan^{-1} 0)\right] dx$$

$$= \frac{\pi}{4}\int_1^2 \frac{dx}{x} = \frac{\pi}{4}[\log x]_1^2$$

$$= \frac{\pi}{4}[\log 2 - \log 1] = \frac{1}{4}\pi \log 2.$$

Example 3. *Show that* $\int_1^2 \int_0^{y/2} y\, dy\, dx = \int_1^2 \int_0^{x/2} x\, dx\, dy$.

Solution. We have

$$\int_1^2 \int_0^{y/2} y\, dy\, dx = \int_1^2 y\left[\int_0^{y/2} dx\right] dy$$

$$= \int_1^2 y\, [x]_0^{y/2}\, dy = \int_1^2 y\, [y/2 - 0]\, dy$$

$$= \frac{1}{2}\int_1^2 y^2\, dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_1^2 = \frac{7}{6}$$

Again $\int_1^2 \int_0^{x/2} x\, dx\, dy = \int_1^2 x\left[\int_0^{x/2} dy\right] dx$

$$= \int_1^2 x\, [y]_0^{x/2}\, dx$$

$$= \int_1^2 x\left[\frac{x}{2} - 0\right] dx = \frac{1}{2}\int_1^2 x^2\, dx$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}.$$

• TEST YOURSELF–1

1. Evaluate $\int_2^3 dx \int_0^1 (x^2 + 3y^2)\, dy$.

2. Evaluate $\int_0^2 \int_0^{\sqrt{4+x^2}} \frac{dx\, dy}{(4 + x^2 + y^2)}$.

3. Evaluate $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x + y)\, dx\, dy$.

4. Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} x\, dx\, dy$.

5. Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\, dx\, dy$.

6. Evaluate $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}}$.

7. Evaluate $\iint e^{2x+3y}\, dx\, dy$ over the triangle bounded by $x = 0$, $y = 0$ and $x+y = 1$.

8. Evaluate $\iint_p x \sin(x + y)\, dx\, dy$, where p is a rectangle $[0 \le x \le \pi, 0 \le y \le \pi/2]$.

9. Show that $\int_1^2 \int_3^4 (xy + e^y)\, dx\, dy = \int_3^4 \int_1^2 (xy + e^y)\, dy\, dx$.

10. Evaluate $\iint x^2y^2\, dx\, dy$ over the region bounded by $x = 0$, $y = 0$ and $x^2 + y^2 = 1$.

11. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration.

12. Show that by double integration that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4by$ is $(16/3)\, ab$.

ANSWERS

1. $\frac{22}{3}$	2. $\frac{\pi}{4} \log(1 + \sqrt{2})$	3. -2	4. $\frac{\pi}{2}$
5. $\frac{1}{2}$	6. $\frac{\pi^2}{4}$	7. $\frac{1}{6}(e-1)^2(2e+1)$	
8. $\pi - 2$	9. $\frac{21}{4} + e^4 - e^3$	10. $\pi/96$	11. πab

• 9.4. APPLICATIONS OF DOUBLE INTEGRATION

Double integration is generally used in area of curves, volume and surface of solids of revolution.

(a) Area of curves. Let AD be an arc of curve $y = f(x)$.

Let area $ABCD$ be divided into sub-area by drawing lines parallel to X and Y axis respectively such that distance between two adjoining lines drawn parallel to Y-axis be δx and those drawn parallel to X-axis be δy.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points on the curve AD. PN and QM are the co-ordinates at P and Q respectively. Then the area of element shown by shadded lines is $\delta x\, \delta y$.

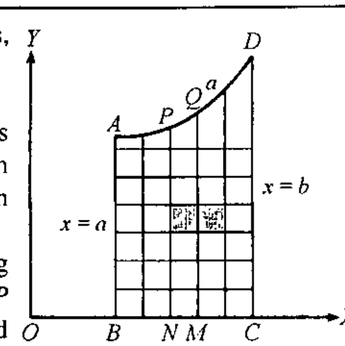

Fig. 5

Therefore, the area of strip PN

$$= \int_{y=0}^{f(x)} dx\, dy \quad \text{where } y = f(x).$$

The required area

$$ABCD = \int_{x=a}^{b} \int_{y=0}^{f(x)} dx\, dy.$$

(ii) We can find the area bounded by the two curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = a$ and $x = b$

$$= \int_{x=a}^{b} \int_{y=f_2(x)}^{f_1(x)} dx\, dy.$$

(b) Volume of a solid. Consider the area $dy\, dz$ on the plane $x = 0$ through each point on the boundary of this small area. Draw the lines parallel to X-axis and thus construct a small cylinder whose base is area to X-axis. This cylinder cuts the given surface, and volume of this cylinder

$$= x\, dy\, dz.$$

$\therefore$ Volume of solid $= \iint x\, dy\, dz.$

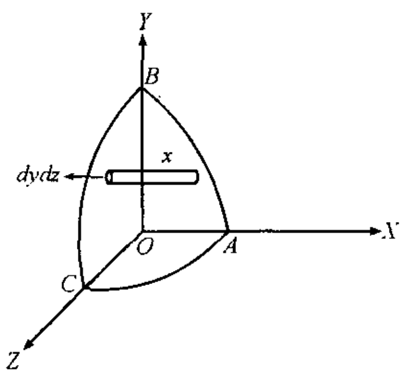

Fig. 6

REMARKS

- By considering area $dx\, dy$ on plane $z = 0$ the volume of solid $= \iint z\, dx\, dy.$
- By considering area $dx\, dz$ on plane $y = 0$ the volume of solid $= \iint y\, dx\, dz.$

(c) Area of surface of a solid. Let the equation of surface be $z = f(x, y)$. Consider a point $P(x, y, z)$ on this surface surrounding this point P. Consider an element of area δs of the surface. Let $\delta x\, \delta y$ be the projection of this area δs on the plane $z = 0$, then we have

$$\delta x\, \delta y = \delta s \cos \alpha \qquad ...(1)$$

where α is the angle between the tangent plane to the given surface at $P(x, y, z)$ and the plane $z = 0$ then by co-ordinate geometry, we have

$$\sec \alpha = \sqrt{\left[1 + \left\{\frac{\partial z}{\partial x}\right\}^2 + \left\{\frac{\partial z}{\partial y}\right\}^2\right]}. \qquad ...(2)$$

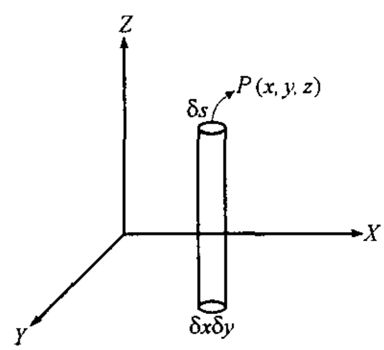

Fig. 7

From (1) we have $\delta s = \delta x\, \delta y \sec \alpha$

$$= \delta x\, \delta y \sqrt{\left[1 + \left\{\frac{\partial z}{\partial x}\right\}^2 + \left\{\frac{\partial z}{\partial y}\right\}^2\right]} \qquad \text{[From (2)]}$$

$\therefore$ the required area of surface

$$= s = \iint \sqrt{\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]}\, dx\, dy.$$

SOLVED EXAMPLES

Example 1. *Find the area of ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Solution. Required area of ellipse

$= 4$ (area of quadrants $OABO$ of ellipse)

$$= 4 \int_{x=0}^{a} \int_{y=0}^{f(x)} dx\, dy, \qquad \text{where } y = f(x) = \frac{b}{a}\sqrt{(a^2 - x^2)}$$

$$= 4\int_0^a [y]_0^{f(x)}\,dx = 4\int_0^a f(x)\,dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{(a^2-x^2)}\,dx$$

$$= \frac{4b}{a}\left[\frac{1}{2}x\sqrt{(a^2-x^2)} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{2b}{a}[0 + a^2\sin^{-1}(1)] = \frac{2b}{a}a^2\,.\,\frac{\pi}{2} = ab\pi$$

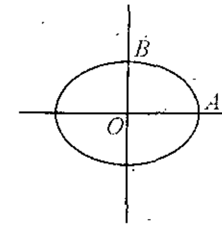

Fig. 8

Example 2. *Find the whole area of curve $a^2x^2 = y^3(2a - y)$.*

Solution. The shape of curve is shown in fig. 9. The required area

$$= 2 \times \text{area } OAB$$

$$= 2\int_{y=0}^{2a}\int_{x=0}^{f(y)} dy\,dx$$

where $x = f(y)$ *i.e.*, $x = y^{3/2}\dfrac{\sqrt{2a-y}}{a}$ is equation of curve.

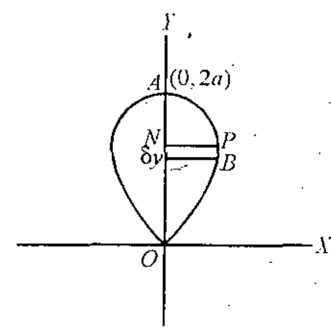

Fig. 9

∴ the required area

$$= 2\int_{y=0}^{2a} [x]_0^{f(y)}\,dy$$

$$= 2\int_0^{2a} f(y)\,dy$$

$$= 2\int_0^{2a} \frac{y^{3/2}\sqrt{2a-y}}{a}\,dy \qquad \left[\because\ f(y) = x = y^{3/2}\frac{\sqrt{2a-y}}{a}\right]$$

Put $y = 2a\sin^2\theta$

$dy = 4a\sin\theta\cos\theta\,d\theta$

at $y = 0,\ \theta = 0$

$y = 2a,\ \theta = \pi/2$

∴ Required area $= \dfrac{2}{a}\displaystyle\int_0^{\pi/2} (2a\sin^2\theta)^{3/2}\sqrt{(2a-2a\sin^2\theta)}\,4a\sin\theta\cos\theta\,d\theta$

$$= 32a^2\int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta$$

$$= \frac{32a^2\,\Gamma(5/2)\,\Gamma(3/2)}{2\,\Gamma 4}$$

$$= \frac{32a^2\,.\,3/2\,.\,1/2\sqrt{\pi}\ 1/2\sqrt{\pi}}{2\,.\,3\,.\,2\,.\,1} = \pi a^2.$$

Example 3. *Find by double integration the area between $y = \dfrac{3x}{(x^2+2)}$ and $4y = x^2$.*

Solution. We have

$4y = x^2,\qquad y = \dfrac{3x}{(x^2+2)}$

$\Rightarrow\ 4y = \dfrac{12x}{(x^2+2)},\qquad 4y = x^2$

$\Rightarrow\ x^2 = \dfrac{12x}{(x^2+2)}\qquad \Rightarrow\ x^4 + 2x^2 - 12x = 0$

$\Rightarrow\ x(x^3 + 2x - 12) = 0$

$x = 0, 2.$

$\therefore$ Required area $= \int_{x=0}^{2} \int_{y=x^2/4}^{3x/(x^2+2)} dx\, dy$

$$= \int_0^2 [y]_{x^2/4}^{3x/(x^2+2)} dx$$

$$= \int_0^2 \left[\frac{3x}{x^2+2} - \frac{x^2}{4}\right] dx$$

$$= \frac{3}{2}\int_0^2 \frac{2x\,dx}{x^2+2} - \frac{1}{4}\int_0^2 x^2\,dx$$

$$= \frac{3}{2}\left[\log (x^2+2)\right]_0^2 - \frac{1}{4}\left(\frac{1}{3}x^3\right)_0^2$$

$$= \frac{3}{2}[\log (6) - \log (2)] - \frac{1}{12}(8-0)$$

$$= \frac{3}{2}\log 3 - \frac{2}{3}.$$

Example 4. *Find the volume bounded by co-ordinates planes and the plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

Solution. The plane cuts X, Y and Z-axis at point $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively. The surface $ABCD$ of co-ordinates planes will be equal to $c\,(1 - x/a - y/b)$

$$\int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} dx\, dy\, dz$$

$$= \int_0^a \int_0^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx$$

$$= c\int_0^a \int_0^{b(1-x/a)} \left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx$$

$$= c\int_0^a \left[y - \frac{x}{a}.y - \frac{y^2}{2b}\right]_0^{b(1-x/a)} dx$$

$$= c\int_0^a \left[b\left(1 - \frac{x}{a}\right) - \frac{x}{a}.b\left(1 - \frac{x}{a}\right) - \frac{1}{2b}b^2\left(1 - \frac{x}{a}\right)^2\right] dx = \frac{1}{6}abc.$$

• TEST YOURSELF–2

1. Find by double integration, the area of the region enclosed by curves
   (a) $y = 4x - x^2$, $y = x$
   (b) $(x^2 + 4a^2)\,y = 8a^3$, $2y = x$ and $x = 0$
   (c) $y = \frac{3x}{(x^2+2)}$, $4y = x^2$.
2. Find by double integration the area included between the parabola $x^2 = 4ay$ and the curve $y = \frac{8a^3}{(x^2 + 4a^2)}$.

ANSWERS

1. (a) $= \frac{9}{2}$ (b) $(\pi - 1)\,a^2$ (c) $\frac{3}{2}\log 3 - \frac{2}{3}$ **2.** $\left(2\pi - \frac{4}{3}\right)a^2$

9.5. TRIPLE INTEGRAL

Definition. Let $f(x, y, z)$ be a single-valued function of the independent variables x, y, z in finite region v. Divide the region v into n subregions $\delta v_1, \delta v_2, \delta v_3, \ldots$. Let P be any point on the boundary or inside.

Take a point in each part and form the sum

$$= s_n = f(x_1, y_1, z_1)\,\delta v_1 + f(x_2, y_2, z_2)\,\delta v_2 + \ldots + f(x_n, y_n, z_n)\,\delta v_n.$$

$$= \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta v_r \qquad \ldots(1)$$

when n tends to infinity. The limit of sum (1) tends to zero is called the triple integral of function $f(x, y, z)$ over the region v and is denoted by

$$\iiint_v f(x, y, z)\, dv$$

the triple integral can be utilised in evaluating a number of physical quantities like, $f(x, y, z) = 1$.

We find the volume, $V = \iiint_v dV$, and putting $f(x, y, z) = \rho$

we get, mass $= \iiint_{v} \rho\, dV$.

Evaluation of Triple Integrals :

The region v divide into elementary cuboids by drawing parallel co-ordinate planes. The volume V can then be considered as the sum of a number of columns parallel to z-axis extending from the lower surface of V say $z = z_1(x, y)$ to the upper surface of V say $z = z_2(x, y)$ the bases of these as column (only one column has been shown in fig. 10) are the elementary area δs_r Which cover a certain area S in x-y plane *i.e.*, plane $z = 0$.

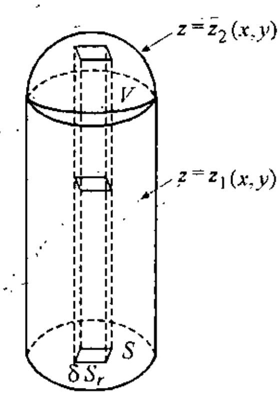

Fig. 10

$\therefore$ Summing up over the elementary cuboids in the same column first and then taking the sum of all such columns we can write

$$\sum_{r=1}^{n} f(x_r, y_r, z_r) \text{ as } \sum_r \left[\sum_m f(x_r, y_r, z_m)\,\delta z\right] \delta s_r$$

where (x_r, y_r, z_r) is a point in the mth cuboid.

When δS_r and δz tend to zero this becomes equal to

$$\iint_S \left\{ \int_{z = z_1(x, y)}^{z_2(x, y)} f(x, y, z)\, dz \right\} ds.$$

(a) If the region V be specified by inequalities
$a \le x \le b,\ c \le y \le d,\ e \le z \le f$
then triple integral

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \int_a^b \int_c^d \int_e^f f(x, y, z)\, dx\, dy\, dz$$

$$= \int_a^b dx \int_c^d dy \int_e^f f(x, y, z)\, dz.$$

Here we integrate first with respect to z keeping x and y constant and then the remaining integration is done as in the case of double integrals.

(b) If the limits of z are function of x and y and y as function of x and x takes the constant values say from $x = a$ to $x = b$ then

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\, dz.$$

The integration with respect to z is performed first regarding x and y as constant then integration w.r. to y regarding x as a constant and then integrate w.r. to x.

SOLVED EXAMPLES

Example 1. *Evaluate* $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dy \, dx \, dz.$

Solution. We have

$$I = \int_0^1 \int_{y^2}^1 (z)_0^{1-x} x \, dy \, dx$$

$$= \int_0^1 \int_{y^2}^1 x (1-x) \, dy \, dx$$

$$= \int_0^1 \int_{y^2}^1 (x - x^2) \, dy \, dx = \int_0^1 \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{y^2}^1 dy$$

$$= \int_0^1 \left[\left\{\frac{1}{2}(1)^2 - \frac{1}{3}(1)^3\right\} - \left\{\frac{1}{2}(y^2)^2 - \frac{1}{3}(y^2)^3\right\}\right] dy$$

$$= \int_0^1 \left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2}y^4 - \frac{1}{3}y^6\right)\right] dy$$

$$= \int_0^1 \left(\frac{1}{6} - \frac{1}{2}y^4 + \frac{1}{3}y^6\right) dy = \left(\frac{1}{6}y - \frac{1}{10}y^5 + \frac{1}{21}y^7\right)_0^1$$

$$= \frac{1}{6} - \frac{1}{10} + \frac{1}{21} = \frac{4}{35}.$$

Example 2. *Evaluate*

$$\int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz \, dx \, dy \, dz.$$

Solution. The given integral

$$I = \int_{x=0}^1 \int_0^{\sqrt{1-x^2}} xy \left(\frac{1}{2}z^2\right)_0^{\sqrt{1-x^2-y^2}} dx \, dy$$

$$= \frac{1}{2} \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} xy (1 - x^2 - y^2) \, dx \, dy$$

$$= \frac{1}{2} \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} x [y(1-x^2) - y^3] \, dx \, dy$$

$$= \frac{1}{2} \int_{x=0}^1 x \left[\frac{1}{2}(1-x^2)y^2 - \frac{1}{4}y^4\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_0^1 x \left[\frac{1}{2}(1-x^2)(1-x^2) - \frac{1}{4}(1-x^2)^2\right] dx$$

$$= \frac{1}{2} \int_0^1 x \left(\frac{1}{2} - \frac{1}{4}\right)(1-x^2)^2 \, dx$$

$$= \frac{1}{8} \int_0^1 (x - 2x^3 + x^5) \, dx = \frac{1}{8}\left[\frac{1}{2}x^2 - \frac{1}{2}x^4 + \frac{1}{6}x^6\right]_0^1$$

$$= \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}.$$

Example 3. *Evaluate* $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dz \, dx \, dy.$

Example 3. *By using the transformation $x + y = u$, $y = vu$, show that*

$$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dx\, dy = \frac{1}{2}(e - 1).$$

Solution. We have $dx\, dy = u\, du\, dv$.

The region of integration is bounded by the lines

$$y = 0,\ y = 1 - x,\ x = 0 \text{ and } x = 1.$$

Changing these equations to new variables u and v by using the relation

$$x = u - y = u - uv = u(1 - v)$$

and $y = uv$, we have

$$uv = 0,\ uv = 1 - u(1 - v),\ u(1 - v) = 0$$

and

$$u(1 - v) = 1$$

giving

$$v = 0 \text{ to } v = 1,\ u = 0 \text{ to } u = 1.$$

Therefore for the given region v varies from 0 to 1 and u varies from 0 to 1 and

$$e^{y/(x+y)} = e^{uv/u} = e^v.$$

Changing the variables to u, v the given integral becomes

$$I = \int_0^1 \int_0^1 e^v u\, du\, dv = \int_0^1 \left[e^v\right]_0^1 u\, du \quad \ldots(1)$$

$$= \int_0^1 (e^1 - e^0)\, u\, du$$

$$= (e - 1)\int_0^1 u\, du = (e - 1)\left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}(e - 1).$$

TEST YOURSELF–5

1. Transform $\int_0^a \int_0^{a-x} f(x, y)\, dx\, dy$, by the substitution $x + y = u$, $y = uv$.
2. By using the transformation $x + y = u$, $y = uv$ show that $\iint \{xy(1 - x - y)\}^{1/2}\, dx\, dy$ taken over the area of the triangle bounded by lines $x = 0$, $y = 0$, $x + y = 1$ is $\frac{2\pi}{105}$.
3. Transform the integral
$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} y\sqrt{x^2 + y^2}\, dx\, dy$$
by changing to polar co-ordinates and hence solve it.