QUANTITATIVE TECHNIQUES

M-228

Self Learning Material

उत्तिष्ठत जाग्रत प्राप्य वरान्निबोधत

Directorate of Distance Education

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QUANTITATIVE TECHNIQUES

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Unit - III

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UNIT I OPERATIONS RESEARCH

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★ STRUCTURE ★

1.0 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- explain operation research (OR) as a subject of study and understanding the history.
- discuss the use of OR in different fields as a tool of decision-making
- describe concept of mathematical modeling and development of a model
- understanding the steps in application of OR.
- describe classification of OR techniques.
- discuss the limitations in the use of OR.
- explain relationship of OR and management.

1.1 HISTORY AND BACKGROUND OF OPERATIONS RESEARCH

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In the books of management one often finds a specific period of the development of management thought, called the *Period of Scientific Management.* It was in 1885 that Fredrick W. Taylor, "father of scientific management", developed the scientific management theories. It was also called the Modern era when rapid development of concepts, theories and techniques of management took place. During World War II, production bottlenecks forced the government of Great Britain to look up to scientists and engineers to help achieve maximum military production. These scientists and engineers created mathematical models to find the solution of the problems about increasing production of military equipments. This branch of study was called *Operations Research* (OR). Since it was used in the research in war operations of armed forces. These problems of the armed forces seemed to be similar to those that occurred in production systems. Because of the success of OR in military operations and approach to war problems it began to be used in industry as well.

1.2 WHY STUDY OPERATIONS RESEARCH ?

We basically helps in determining the best (optimum) solution (course of action) to problems where decision has to be taken under the restriction of limited resources. It is possible to convert any real life problem into a mathematical model. The basic feature of OR is to formulate a real world problem as a mathematical model. Since in the production industry, most of the manufacturers want to lower their labour or production costs to achieve higher profits, OR can be very usefully implied to real life production problems.

OR should be seen as a problem-solving technique. Like management, OR is also a both Science and an Art. The Science part of OR is using mathematical techniques for solving decision problems. The Art part of OR is the ability of the OR team to develop good rapport with those supplying information and those who have to implement the recommended solutions. It is important that both the Science and the Art parts of the OR are understood properly as a system of problem-solving.

In India, OR society was formed in 1950's. The Journal of Operations Research has the mission to serve entire OR community including practitioners, researchers, educators and students. It celebrated its 50th Anniversary of Operations Research and published Anniversary issue in Jan.-Feb. 2001. Industry has become quite aware of the potential of OR as a technique and many industrial and business houses have OR teams working to find solutions to their problems. Particularly, Railways, Indian Airlines, Defence Forces, Telco, DCM, etc., are using OR to their advantage. As a matter of fact, some techniques of OR like Programme Evaluation and Reviewing Techniques (PERT) and Critical Path Method (CPM) are frequently used by many organizations for effective planning and control of the construction projects.

OR helps in taking decisions which optimize (maximize) the interest of the organization, it is a decision-making tool and should be seen as such. Many individuals and organizations see it as a management fad, which has limited use to them. At the same time, tendency of some organizations to force-fit OR to prove that they use managerial techniques whether their functioning needs demand OR or not must be curbed. However, this is also true that complex real life problems can be solved for the advantage of the organizations by using OR techniques.

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1.3 DEFINITION OF OPERATIONS RESEARCH

Many authors have given different interpretation to the meaning of Operations Research as it is not possible to restrict the scope of Operations Research in a few sentences. Students must understand that there is no need to single definition of Operations Research which is acceptable to everyone. Two of the widely accepted definitions are provided below for understanding the concept of Operations Research.

> *"Operations Research is concerned with scientifically deciding how best to design and operate man-machine system usually under conditions requiring the allocation of scarce resources."*

– Operations Research Society of America

The salient features of the above definition are :

(*a*) It is a scientific decision-making technique.

(*b*) It deals with optimizing (maximizing) the results.

(*c*) It is concerned with man-machine systems.

(*d*) The resources are limited.

> *"Operations Research is a scientific approach to problem solving for executive management."*

– HM Wagner

The above definition lays emphasis on :

(*a*) OR being a scientific technique.

(*b*) It is a problem-solving technique.

(*c*) It is for the use of executives who have to take decisions for the organizations.

A close observation of the essential aspects of the above two definitions will make it clear that both are in reality conveying the same meaning. Other definitions of OR also converge on these essential features. One need not remember the definitions word by word but understand the true meaning of the definition provided by different authors. The emphasis has to be on the application of technique so that organizations are benefitted. Hence, the real work of any managerial technique is the ability of the organizations to take advantage for meeting their objectives.

1.4 SALIENT FEATURES OF OPERATIONS RESEARCH

After having understood the basic concept of OR and the need, one can easily understand its salient features.

1. **System Approach:** OR is a systematic approach as is clear from the conceptual model of OR explained above. It encompasses all the sub-

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systems and departments of an organization. Since it is a technique that effects the entire organization, optimizing results of one part of the organization is not the proper use of OR. Before applying OR techniques the management must understand its impact and implications on the entire organization.

2. **OR is both a Science and an Art:** OR has the scientific orientation because of its inherent methodology and scientific methods are used for problem-solving. But its implementation needs the art of taking the entire organization along. OR does not perform experiment but helps in finding out solutions. OR must take into account the human factor which is the most important factor in implementing any technique/methods of problem-solving.
3. **Interdependency Approach:** Problem of organizations could be related with economics, engineering, infrastructure related with markets, management of human resources and so on. If OR has to find a solution to problems related to diverse fields, the OR team must be constituted of members with background disciplines of science, management and engineering etc. Only then, practical solutions which can be implemented, can be found to the advantage of organizations.
4. **Management Decision-making:** Management of any organization has to make decision, which has, impact on its profitability. All business organizations exist to make profits. Non-business organizations like hospitals, educational institutions, NGOs etc., generate profits by reducing the inputs and increasing the outputs through effective and efficient management. Decision-making involves generating different alternatives and selecting the best under the given situation. OR helps in making the right decisions.
5. **Quantitative Technique:** OR is a quantitative technique, which uses mathematical models and finds rational quantitative solutions to the managerial problems. The management may use the OR inputs and take into account the quantitative analysis of the problem in finding the solution in the best interest of the organization.
6. **Use of Information Technology (IT):** OR extensively uses the IT for complex mathematical problems to its advantage. OR approach to decision-making depends heavily on the use of computers.

1.5 OPERATION RESEARCH MODELS

What is a Model ?

It is very difficult to represent the exact real life situations on a piece of paper. A model attempts to represent reality of the situation by identifying all factors of situation and by establishing some relationship between them. In real life situations, there are so many uncertainties and complexities, which cannot be exactly reproduced. Model helps in identifying such uncertainties and complexities in terms of different factors.

Types of Operation Research Models

Following are some important types of operations research models:

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Symbolic or Mathematical Models

This is the most important type of model. Mathematical modelling focuses on creating a mathematical representation of management problems in organisations. All the variables in a particular problem are expressed mathematically. The model then provides different outcomes, which will result from the different choices the management wishes to use. The best outcome in a particular situation will help the management in decision-making. These models use set of mathematical symbols and hence are also called *symbolic models*.

The variables in many business and industry situations can be related together by mathematical equations. To understand the concepts of symbolic or mathematical model, visualise a balance sheet or profit and loss account as a symbolic representation of the budget. Similarly, the demand curve in economics can be seen as symbolic representation of the buyers' behaviour at varying price levels.

Simulation Models

In simulation model, the behaviour of the system under study is 'initiated over a period of time'. Simulation models do not need mathematical variables to be related in the form of equations, normally, these models are used for solving such problems that cannot be solved mathematically. Simulation is a general technique, which helps us in developing dynamic models, which are similar to the real process. Developing good simulation models is difficult because 'creating' a real life situation to perfection is extremely difficult.

Iconic Models

These models represent the physical simulations to the real life system under study. Physical dimensions are scaled up or down to simplify the actual characteristics and specifications of the system. Preparation of prototype models for say, an automobile or 3-D plant layout are some examples of iconic models.

These are the physical replica of a system and are based on a smaller scale than the original. The models have all the operating features of the actual system. Flight simulators, missile firing simulators, etc., are also examples of iconic models.

Analog Models

They are not the exact replica. Like the iconic models, these are smaller, simple physical systems as compared to the real life systems which are complex. These models are used to explain an actual system by analogy.

Deterministic Models

When the change of one variable has a certain or definite change in the outcome, the model is called a Deterministic model. In fact, everything is absolutely clearly defined and the results are known. Economic Order Quantity (EOQ) is a deterministic model, as economic lot size can be exactly known, with change in one of the variables in the EOQ formula.

1.6 METHODOLOGY OF OPERATION RESEARCH

There are many steps involved in application of OR. The methodology to be adopted involves the following :

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1. **Observations of the Operating Environment :** OR is a problem-solving technique. First step in solving the problem is the formulation of the problem. This is done through observation of the system and its environment. As much of information regarding the problem as possible is generated by using the researchers, observers etc.
2. **Formulation of the Problem :** Any problem has many interconnected factors related to the situation. The factors may or may not be in the control of the management. The factors which are relevant to the situation of the problem and are under the control of management must be identified. Once the problem area is known, different variables considered responsible for the problem are listed. Now, it is possible to define the problem in terms of the variables and their relationship.
3. **Selecting and Developing a Suitable Model :** At this stage, a suitable model which best represents the real life situation has to be selected. The model is developed to show the relations and interrelation between a cause and effect. Normally, the model is fully tested and modified to ensure that OR technique applied is able to solve the problem.
4. **Collecting the Data :** Next step is to collect the data required by the selected model. The process of the OR model in finding the solution depends to a large extent on the quantity and quality of data. More the data and lesser the errors in data, the quality of managerial decisions will be better. The required information can be obtained through observations or from recorded data or even based on experience and maturity of the OR team.
5. **Finding the Solution :** Once the model has been developed, it is possible to find the solution. OR solutions are under a particular situation and under certain assumptions. Many assumptions have to be made by the OR team to simplify the model. The solution is valid only under these assumptions. Once the solution by the OR techniques is found, certain input variables are changed to see the output. By this method the best possible solution can be found.
6. **Presenting the Solution to the Management :** The OR team has to present the solution to the management in a proper manner. The conditions under which the solution can be used and the conditions under which solution cannot work must be explained to the management. The assumptions made at arriving the solution and the weakness of the solutions should also be explained to the management.
7. **Implementing the Solution :** This is the last step in the OR application methodology. The solution provided by the OR technique is scientific but the application of this technique involves many behavioral aspects. This is the 'art' part of OR and is of utmost importance. Any gap between the perception of the management and the approach of OR team must be removed.

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1.7 TOOLS OF OPERATION RESEARCH

Operation Research is a very versatile science and has many tools/techniques, which can be used for problem solving. However, it is not possible to list all these techniques as everyday new methods in the use of OR are being developed. Some of the tools of OR are discussed in the succeeding paragraphs :

1. **Linear Programming (LP) :** Most of the industrial and business organisations have the objectives of minimizing costs and maximizing the profits. LP deals with maximizing a given objective. Since the objective function and boundry conditions are linear in nature, this mathematical model is called *Linear Programming Model*. It is a mathematical technique used to allocate limited resources amongst competing demands in an optimal manner. The application of LP requires that there must be a well-defined objective function (like maximizing profits and minimizing costs) and there must be constraints on the amount and extent of resources available for satisfying the objective function.
2. **Queuing Theory :** In real life situations, the phenomenon of waiting is involved whether it is the people waiting to buy goods in a shop, patients waiting outside an Out Patient Department (OPD), vehicles waiting to be serviced in a garage and so on. Because in general, customer's arrival and his service time is not known in advance; hence a queue is formed. Queuing or waiting line theory aims at minimizing the overall cost due to servicing and waiting. How many servicing facilities can be added at what cost to minimize the time in queue is the aim in the application of this theory.
3. **Network Analysis Technique :** A network can be used to present or depict the activities necessary to complete a project. This helps us in planning, scheduling, monitoring and control of large and complex projects. The project may be developing a new battle tank, construction of dam or a space flight. The project managers are interested in knowing the total project completion time, probability that a project can be completed by a particular time, and the least cost method of reducing the total project completion time. Techniques like Programme Evaluation and Reviewing Technique (PERT) and Critical Path Method (CPM) are part of network analysis. These are popular techniques and widely used in project management.
4. **Replacement Theory Model :** All plants, machinery and equipment needs to be replaced at some point of time, either because there is deterioration in their efficiency or because new and better equipment is available and the old one has become obsolete. Sooner or later the equipment needs to be replaced. The decision to be taken by the management involves consideration of the cost of new equipment which is to be purchased and what can be recovered from the old equipment through its sale, or its scrap value, the residual life of the old equipment and many other related aspects. These are important decisions involving investment of capital and need to be taken very carefully.

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5. **Inventory Control :** Inventory includes all the stocks of material, which an organization buys for production/manufacture of goods and services for sale. It will include raw material; semi-finished and finished products, spare parts of machines, etc. Managers face the problems of how much of raw material should be purchased, when should it be purchased and how much should be kept in stock. Overstocking will result in locked capital not available for other purposes, whereas under-stocking will mean stock-out and idle manpower and machine resulting in reduced output. It is desirable to have just the right amount of inventory at the right time. Inventory control models can help us in finding out the optimal order size, reorder level, etc., so that the capital resources are conserved and maximum output ensured.

6. **Integer Programming :** Integer programming deals with certain situations in which the variable assumes non-negative integer (complete or whole number) values only. In LP models the variable may take even a fraction value and the figures are rounded off to the nearest integer to get the solution, *i.e.*, number of vehicles available in a problem cannot be in fractions. When such rounding off is done the solution does not remain an optimal solution. In integer programming the solution containing unacceptable and fractional values are ruled out and the next best solution using whole numbers is obtained. An integer programming may be called *mixed* or *pure depending* on whether some or all the variables are restricted to integer values.

7. **Transportation Problems :** Transportation problems are basically LP model problems. This model deals with finding out the minimum transportation cost for transporting the single commodity from a number of sources to number of destinations. Typical problem involves transportation of some manufactured products (say cars in 3 different plants) and these have to be sent to the warehouses of various dealers in different parts of country. This may be understood as a special case of simplex method developed for LP problems, allocating scare resources to competing demands. The main purpose of the transportation is to schedule the dispatch of the single product from different sources like factories to different destinations as total transportation cost is minimized.

8. **Decision Theory and Games Theory :** Information for making decisions is the most important factor. Many models of OR assume availability of perfect information which is called *decision-making under certainty*. However, in real life situations, only partial or imperfect information is available. In such a situation we have two cases, either decision under risk or decision under uncertainty. Hence from the point of view of availability of information, there are three cases, certainty and uncertainty, the two extreme cases and risk is the "in-between" case.

   Games theory is concerned with decision-making in a conflict situation where two or more intelligent opponents try to optimize their own decision. In Games theory, an opponent is referred to as a player and each player

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has a number of choices. The Games theory helps the decision maker to analyse the course of action available to his opponent. In decision theory, we use decision tree which can be graphically represented to solve the decision-making problems.

9. **Assignment Problems :** We have the problem of assigning a number of tasks to a number of persons who may use machines. The objective is to assign the jobs to the machines in such a way that the cost is least. This may be considered a special case of LP transportation model. Here jobs may be treated as 'services' and machines may be considered the 'destinations'. Assignment of a particular job to a particular person so that all the jobs can be completed in shortest possible time hence incurring the least cost, is the assignment problem.
10. **Markov Analysis :** Markov analysis is used to predict future conditions. It assumes that the occurrence of a future state depends upon the immediately preceding state and only on it. It is based on the probability theory and predicts the change in a system over a period of time if the present behaviour of the system is known. Predicting market share of the companies in future as also whether a machine will function properly or not in future, are examples of Markov analysis.
11. **Simulation Techniques :** Since all real life situations cannot be represented mathematically, certain assumptions are made and dynamic models which act like the real processes are developed. It is very difficult to develop simulation models which can give accurate solutions to the problems, but this is a good method of problem solving, when the problems are very complex and cannot be solved otherwise.

1.8 IMPORTANT APPLICATIONS OF OPERATION RESEARCH

In today's world where decision-making does not depend on intuition, managerial techniques are widely used. All the applications of OR cannot be listed because OR as a tool finds new application everyday. It finds typical applications in many activities related to work planning.

Some important applications of OR are :

1. **Manufacturing/Production**
   - Production planning and control
   - Inventory management.
2. ***Facilities Planning***
   - Design of logistic systems
   - Factory/building location and size decisions
   - Transportation, loading and unloading
   - Planning warehouse locations.

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3. **Accounting**
   — Credit policy decisions
   — Cash flow and fund flow planning.
4. **Construction Management**
   — Allocation of resources to different projects in hand
   — Workforce/labour planning
   — Project management (scheduling, monitoring and control).
5. **Financial Management**
   — Investment decisions
   — Portfolio management.
6. **Marketing Management**
   — Product-mix decisions
   — Advertisement/Promotion budget decisions
   — Launching new product decisions.
7. **Purchasing Decisions**
   Inventory management (optimal level of purchase), Optimal re-ordering.
8. **Personnel Management**
   — Recruitment and selection of employees
   — Designing training and development programmes
   — Human Resources Planning (HRP).
9. **Research and Development**
   — Planning and control of new research and development projects.
   — Product launch planning.

1.9 PITFALLS IN THE USE OF OPERATION RESEARCH FOR DECISION-MAKING

The first stage of OR application after collecting data/information through observation is the formulation of the problem. It is the most important and most difficult task in OR application. Have the OR team been able to identify the right problem for finding the solution ? Has the problem been accurately defined in unambiguous manner ? Selecting and developing a suitable model is not an easy task. The model must represent the real life situation as far as possible. Collection of data needs a lot of time by a number of people. It is time-consuming and expensive process. Collection of data is done either by observation or from the previous recorded data. When a system is being observed by the OR team, it effects the behaviour of the persons performing the task. The very fact that the workers know that they are being observed is likely to change their work behaviour. The second method of data collection, the records, are never reliable and do not provide sufficient information which is required.

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As OR problem-solving techniques is very time-consuming, the quality of decision-making may become a causality. The management has to make a decision either way. Decision based on insufficient or incomplete information will not be the best decision. A reasonably good solution without the use of OR may be preferred by the management as compared to a slightly better solution provided by the use of OR which is very expensive in time and money.

Due to the above reasons, many OR specialists try and fit the solution they have, to the problem. This is dangerous and unethical and organizations must guard against this.

1.10 LIMITATIONS OF OPERATIONS RESEARCH

Operation Research is an extremely powerful tool in the hands of a decision-maker and to that extent the advantage of OR techniques are immense. Some of them are :

(*a*) *It helps in optimum use of resources.* LP techniques suggest many methods of most effective and efficient ways of optimally using the production factors.

(*b*) *Quality of decision can be improved by suitable use of OR techniques.* If a mathematical model representing the real life situation is well-formulated representing the real life situation, the computation tables give a clear picture of the happenings (changes in the various elements *i.e.*, variables) in the model. The decision-maker can use it to his advantage, specially if computerised software can be used to make changes in variables as per requirement.

The limitations of OR emerge only out of the time and cost involved as also the problem of formulating a suitable mathematical model, otherwise, as suggested above, it is a very powerful medium of getting the best out of limited resources. So, the problem is its application rather than its utility, which is beyond doubt. Some of the limitations are :

(*a*) *Large number of cumbersome computations.* Formulation of mathematical models which takes into account all possible factors which define a real life problem is difficult. Because of this, the computations involved in developing relationships in very large variables need the help of computers. This discourages small companies and other organisations from getting the best out of OR techniques.

(*b*) *Quantification of problems.* All the problems cannot be qualified properly as there are a large number of intangible factors, such as human emotions, human relationship and so on. If these intangible elements/variables are excluded from the problem even though they may be more important than the tangible ones, the best solution cannot be determined.

(*c*) *Difficult to conceptualize and use by the managers.* OR applications is a specialist's job, these persons may be mathematicians or statisticians

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who understand the formulation of models, finding solution and recommending the implementation. The managers really do not have the hang of it. Those who recommend a particular OR technique may not understand the problem well enough and those who have to use may not understand the 'why' of that recommendation. This creates a 'gap' between the two and the results may not be optimal.

1.11 TIPS ON FORMULATING LINEAR PROGRAMMING MODELS

(*a*) *Read the statement of the problem carefully.*

(*b*) *Identify the decision variables.* These are the decisions that are to be made. What set of variables has a direct impact on the level of achievement of the objectives and can be controlled by the decision-maker ? Once these variables are identified, list them providing a written definition (*e.g.*, x_1 = number of units produced and sold per week of product 1, x_2 = number of units produced and sold per week of product 2).

(*c*) *Identify the objective.* What is to be maximised or minimised ? (*e.g.*, maximize total weekly profit from producing product 1 and 2).

(*d*) *Identify the constraints.* What conditions must be satisfied when we assign values to the decision variables? You may like to write a verbal description of the restriction before writing the mathematical representation (*e.g.*, total production of product 1 > 100 units).

(*e*) *Write out the mathematical model.* Depending on the problem, you might start by defining the objective function on the constraints. Do not forget to include the non-negativity constraints.

SUMMARY

- Operations Research is concerned with scientifically deciding how best to design and operate man-machine system usually under conditions requiring the allocation of scarce resources.
- Operations Research is a scientific approach to problem solving for executive management
- Economic Order Quantity (EOQ) is a deterministic model, as economic lot size can be exactly known, with change in one of the variables in the EOQ formula.
- Operation Research is a very versatile science and has many tools/techniques, which can be used for problem solving.
- LP deals with maximizing a given objective. Since the objective function and boundry conditions are linear in nature, this mathematical model is called *Linear Programming Model.*
- Operation Research is an extremely powerful tool in the hands of a decision-maker and to that extent the advantage of OR techniques are immense.

REVIEW QUESTIONS

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1. What is the concept of Operation Research ? Write a detailed note on its development.
2. Discuss significance and scope of OR in business and industry.
3. What are the different phases of OR ? How is OR helpful in decision-making ?
4. Discuss briefly various steps involved in solving an OR problem. Illustrate with one example from the functional area of your choice.
5. Explain applications of Operations Research in business.
6. What is the significance and scope of Operation Research in the development of Indian Economy ?
7. What is the role of OR in modern day business ? Give examples in support of your answer.
8. Discuss the meaning, significance and scope of Operations Research. Describe some methods of OR.
9. Illustrate and explain various features of OR.
10. Define Operations Research in your own words and explain various tools of OR.
11. What is a model ? What are the types of models you are familiar with ? What are the advantages and pitfalls of models ?
12. Define an OR model. Give examples from industry and business to explain the use of models.
13. Define OR and discuss its scope.
14. Discuss the significance and scope of Operations Research in modern management.
15. Write a detailed note on the use of models for decision-making. Your answer should specifically cover the following :
    (*i*) Need for model building
    (*ii*) Type of model appropriate to the situation
    (*iii*) Steps involved in the construction of a model
    (*iv*) Setting up criteria for evaluating different alternatives
    (*v*) Role of random numbers.

FURTHER READINGS

- **Operations Research:** Col. D.S. Cheema, University Science Press.
- **Introductory Operations Research:** Theory & Applications 3e: Kasana, Springer
- **Operations Research:** N.P. Agarwal, Indus Valley Publication.
- **Operations Research:** Jaya Banerjee, Shree Niwas.
- **Operations Research for Library and Information Professionals:** Dariush Alimohammadi, Ess Ess Publications.

UNIT II LINEAR PROGRAMMING

NOTES

★ STRUCTURE ★

2.0 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- explain to Linear Programming Problems (LPP)
- know about graphical method
- define formulation problems
- describe about duality theorems
- define about economic interpretation of dual variable.

2.1 INTRODUCTION TO LINEAR PROGRAMMING PROBLEMS (LPP)

I. When a problem is identified then the attempt is to make an mathematical model. In decision making all the decisions are taken through some variables which are known as decision variables. In engineering design, these variables are known as design vectors. So in the formation of mathematical model the following **three phases** are carried out :

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(*i*) Identify the decision variables.

(*ii*) Identify the objective using the decision variables and

(*iii*) Identify the constraints or restrictions using the decision variables.

Let there be n decision variable $x_1, x_2, ..., x_n$ and the general form of the mathematical model which is called as Mathematical programming problem under decision-making can be stated as follows :

Maximize/Minimize $\quad z = f(x_1, x_2, ..., x_n)$

Subject to, $\quad g_i(x_1, x_2, ..., x_n) \{\le, \ge \text{ or } =\} b_i$

$i = 1, 2, ..., m.$

and the type of the decisions *i.e.*, $x_j \ge 0$

or, $\quad x_j \le 0$ or x_j's are unrestricted

or $\quad$ combination types decisions.

In the above, if the functions f and g_i $(i = 1, 2, ..., m)$ are all linear, then the model is called "*Linear Programming Problem (LPP)*". If any one function is non-linear then the model is called "*Non-linear Programming Problem (NLPP)*".

II. We define some basic aspects of LPP in the following :

(*a*) Convex set : A set X is said to be convex if

$$x_1, x_2 \in X, \text{ then for } 0 \le \lambda \le 1,$$
$$x_3 = \lambda x_1 + (1 - \lambda)x_2 \in X$$

Some examples of convex sets are :

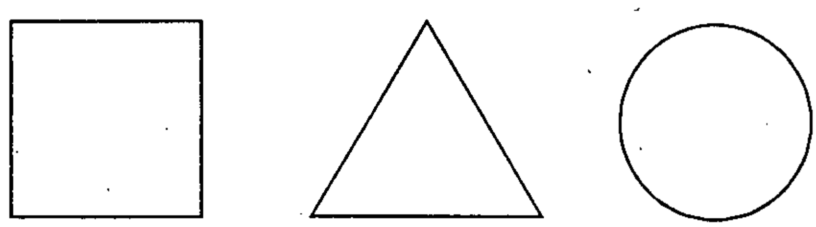

Fig. 2.1 Convex sets

Some examples of non-convex sets are :

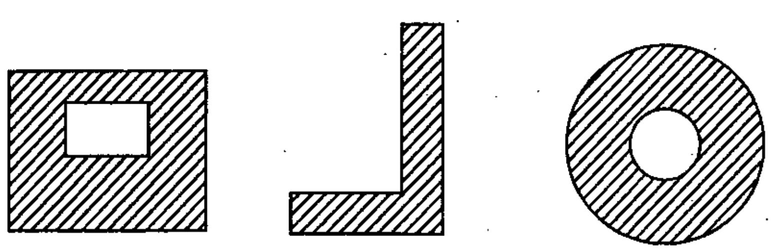

Fig. 2.2 Non-convex sets

Basically if all the points on a line segment forming by two points lies inside the set/geometric figure then it is called convex.

(*b*) Extreme point or vertex or corner point of a convex set : It is a point in the convex set which can not be expressed as $\lambda x_1 + (1 - \lambda)x_2$ where x_1 and x_2 are any two points in the convex set.

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For a triangle, there are three vertices, for a rectangle there are four vertices and for a circle there are infinite number of vertices.

(*c*) Let Ax = b be the constraints of an LPP. The set X = $\{x \mid Ax = b, x \geq 0\}$ is a convex set.

Feasible Solution : A solution which satisfies all the constraints in LPP is called feasible solution.

Basic Solution : Let m = no. of constraints and n = no. of variables and $m < n$. Then the solution from the system Ax = b is called basic solution. In this system there are ${}^{n}c_{m}$ number of basic solutions. By setting $(n - m)$ variables to zero at a time, the basic solutions are obtained. The variables which is set to zero are known as 'non-basic' variables. Other variables are called basic variables.

Basic Feasible Solution (BFS) : A solution which is basic as well as feasible is called basic feasible solution.

Degenerate BFS : If a basic variable takes the value zero in a BFS, then the solution is said to be degenerate.

Optimal BFS : The BFS which optimizes the objective function is called optimal BFS.

2.2 GRAPHICAL METHOD

Let us consider the constraint $x_1 + x_2 = 1$. The feasible region of this constraint comprises the set of points on the straight line $x_1 + x_2 = 1$.

If the constraint is $x_1 + x_2 \geq 1$, then the feasible region comprises not only the set of points on the straight line $x_1 + x_2 = 1$ but also the points above the line. Here above means away from origin.

If the constraint is $x_1 + x_2 \leq 1$, then the feasible region comprises not only the set of points on the straight line $x_1 + x_2 = 1$ but also the points below the line. Here below means towards the origin.

The above three cases depicted as follows:

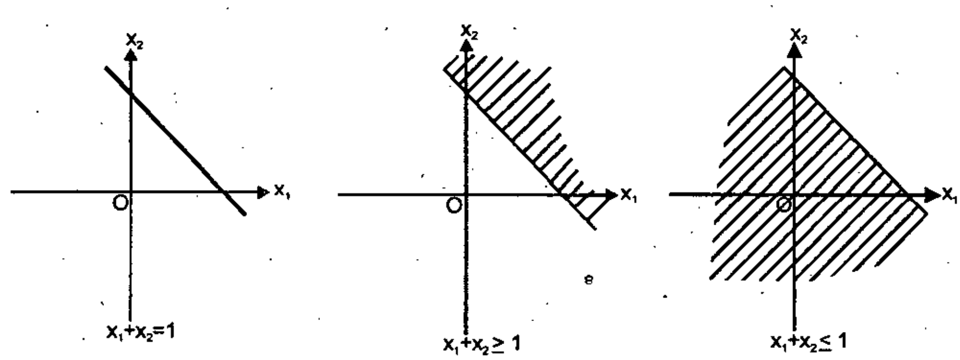


Fig. 2.3

NOTES

For the constraints $x_1 \geq 1$, $x_1 \leq 1$, $x_2 \geq 1$, $x_2 \leq 1$ the feasible regions are depicted below :

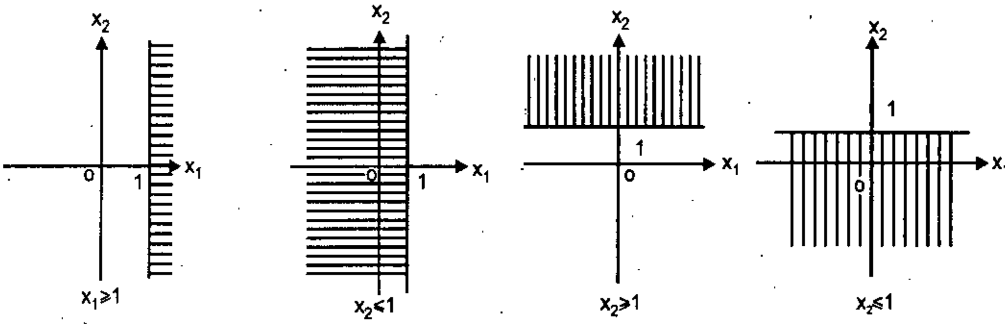


Fig. 2.4

For the constraints $x_1 - x_2 = 0$, $x_1 - x_2 \geq 0$ and $x_1 - x_2 \leq 0$ the feasible regions are depicted in Fig. 2.5.

The *steps of graphical method* can be stated as follows :

(*i*) Plot all the constraints and identify the individual feasible regions.

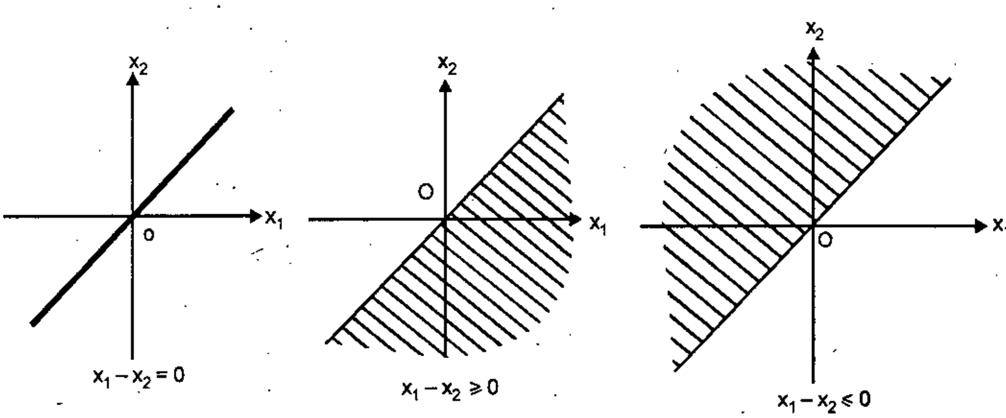


Fig. 2.5

(*ii*) Identify the common feasible region and identify the corner points *i.e.*, vertices of the common feasible region.

(*iii*) Identify the optimal solution at the corner points if exists.

Example 1. *Using graphical method solve the following LPP :*

$$\text{Maximize } z = 5x_1 + 3x_2$$

$$\begin{aligned} \text{Subject to, } 2x_1 + 5x_2 &\leq 10, \\ 5x_1 + 2x_2 &\leq 10, \\ 2x_1 + 3x_2 &\geq 6, \\ x_1 \geq 0, x_2 &\geq 0. \end{aligned}$$

Solution. Let us present all the constraints in intercept form *i.e.*,

$$\frac{x_1}{5} + \frac{x_2}{2} \leq 1 \qquad \text{...(I)}$$

$$\frac{x_1}{2} + \frac{x_2}{5} \leq 1 \qquad \text{...(II)}$$

$$\frac{x_1}{3} + \frac{x_2}{2} \geq 1 \qquad \text{...(III)}$$

NOTES

The common feasible region ABC is shown in Fig. 2.6 and the individual regions are indicated by arrows. (Due to non-negativity constraints *i.e.*, $x_1 \geq 0$, $x_2 \geq 0$, the common feasible region is obtained in the first quadrant).

The corner points are $A\left(\frac{18}{11}, \frac{10}{11}\right)$, $B\left(\frac{10}{7}, \frac{10}{7}\right)$ and C (0, 2). The value of the objective function at the corner points are $z_A = \frac{120}{11} = 10.91$, $z_B = \frac{80}{7} = 11.43$ and $z_C = 6$.

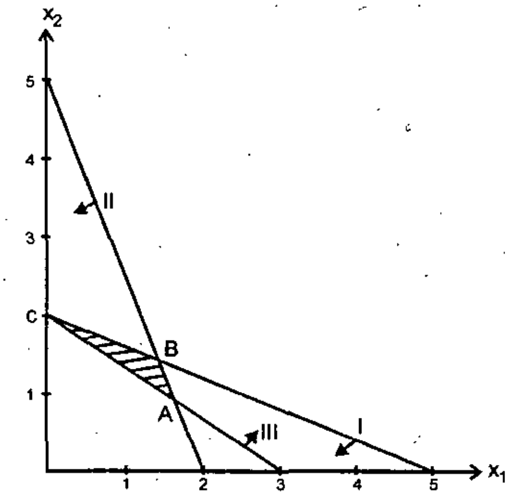


Fig. 2.6

Here the common feasible region is bounded and the maximum has occurred at the corner point B. Hence the optimal solution is

$$x_1^* = \frac{10}{7},\ x_2^* = \frac{10}{7} \text{ and } z^* = \frac{80}{7} = 11.43.$$

Example 2. *Using graphical method solve the following LPP :*

$$\text{Minimize } z = 3x_1 + 10x_2$$

$$\text{Subject to, } 3x_1 + 2x_2 \geq 6,$$
$$4x_1 + x_2 \geq 4,$$
$$2x_1 + 3x_2 \geq 6,$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

Solution. Let us present all the constraints in intercept form *i.e.*,

$$\frac{x_1}{2} + \frac{x_2}{3} \geq 1 \quad \text{...(I)}$$

$$\frac{x_1}{1} + \frac{x_2}{4} \geq 1 \quad \text{...(II)}$$

$$\frac{x_1}{3} + \frac{x_2}{2} \geq 1 \quad \text{...(III)}$$

Due to the non-negativity constraints *i.e.*, $x_1 \geq 0$ and $x_2 \geq 0$ the feasible region will be in the first quadrant.

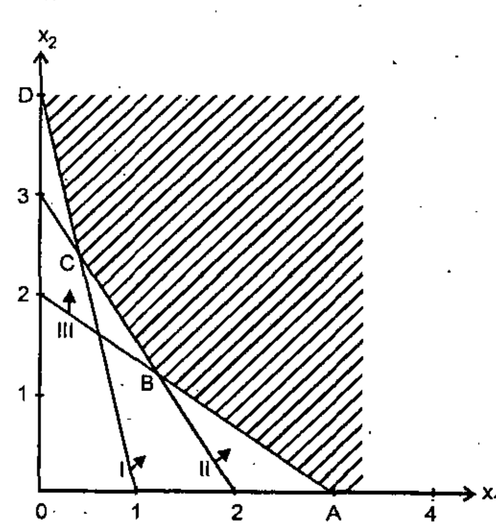


Fig. 2.7

NOTES

The common feasible region is shown in Fig. 2.7 where the individual feasible regions are shown by arrows. Here the common feasible region is unbounded. *i.e.*, open with the corner points A(3, 0), $B\left(\frac{3}{5},\frac{8}{5}\right)$, $C\left(\frac{2}{5},\frac{12}{5}\right)$ and D (0, 4). The value of the objective function at the corner points are $z_A = 9$, $z_B = \frac{89}{5} = 17.8$, $z_C = \frac{126}{5} = 25.2$, and $z_D = 40$.

Here the minimum has occurred at A and there is no other point in the feasible region at which the objective function value is lower than 9. Hence the optimal solution is

$$x_1^* = 3,\ x_2^* = 0 \text{ and } z^* = 9$$

Example 3. *Solve the following LPP by graphical method :*

$$\text{Maximize } z = 3x_1 - 15x_2$$

$$\text{Subject to, } x_1 + x_2 \le 8,$$

$$x_1 - 4x_2 \le 8,$$

$$x_1 \ge 0,\ x_2 \text{ unrestricted in sign.}$$

Solution. Since x_2 is unrestricted in sign this means x_2 may be ≥ 0 or ≤ 0. Also $x_1 \ge 0$. Then the common feasible region will be in the first and fourth quadrant. Let us present all the constraints in intercept forms *i.e.*,

$$\frac{x_1}{8} + \frac{x_2}{8} \le 1 \qquad \text{...(I)}$$

$$\frac{x_1}{8} - \frac{x_2}{2} \le 1 \qquad \text{...(II)}$$

The common feasible region is shown in Fig. 2.8 where the individual feasible regions are shown by arrows.

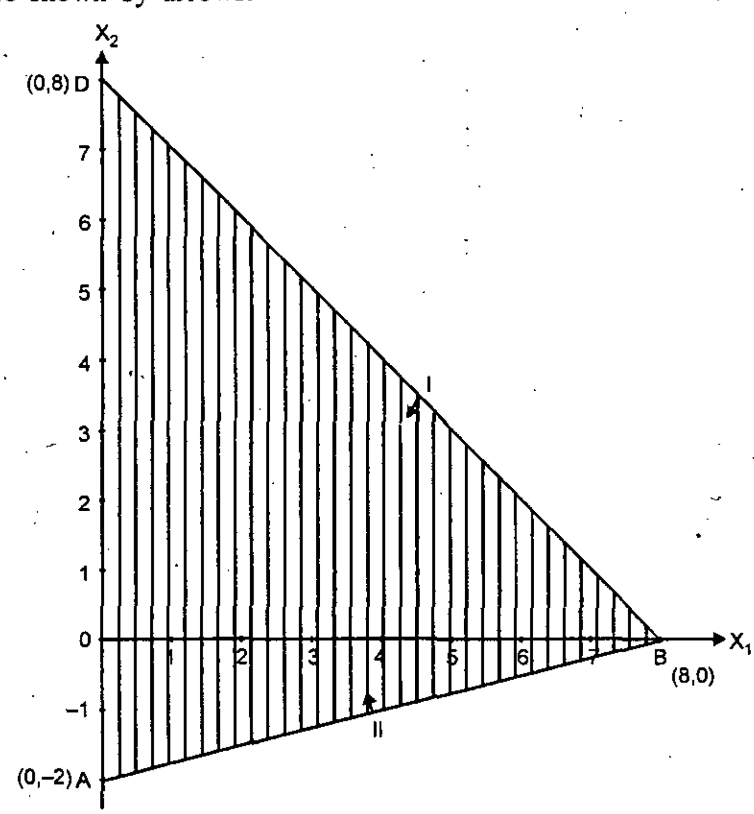


Fig. 2.8

The value of the objective function at the corner points are $z_A = 30$, $z_B = 24$ and $z_C = -120$. Since the common feasible region is bounded and the maximum has occurred at A, the optimal solution is

$$x_1^* = 0,\ x_2^* = -2 \text{ and } z^* = 30.$$

NOTES

Exceptional Cases in Graphical Method

There are three cases may arise. When the value of the objective function is maximum/minimum at more than one corner points then '*multiple optima*' solutions are obtained.

Sometimes the optimum solution is obtained at infinity, then the solution is called '*unbounded solution*'. Generally, this type of solution is obtained when the common feasible region is unbounded and the type of the objective function leads to unbounded solution.

When there does not exist any common feasible region, then there does not exist any solution. Then the given LPP is called *infeasible i.e., having no solution*. For example, consider the LPP which is infeasible

$$\begin{aligned} \text{Maximize } z &= 5x_1 + 10x_2 \\ \text{Subject to, } x_1 + x_2 &\le 2, \\ x_1 + x_2 &\ge 3, \\ x_1, x_2 &\ge 0. \end{aligned}$$

Example 4. *Solve the following LPP using graphical method :*

Maximize $\quad z = x_1 + \frac{3}{5}x_2$

Subject to, $\quad 5x_1 + 3x_2 \le 15,$

$$3x_1 + 4x_2 \le 12,$$

$$x_1, x_2 \ge 0.$$

Solution. Let us present all the constraints in intercept forms *i.e.,*

$$\frac{x_1}{3} + \frac{x_2}{5} \le 1 \qquad \ldots\text{(I)}$$

$$\frac{x_1}{4} + \frac{x_2}{3} \le 1 \qquad \ldots\text{(II)}$$

Due to non-negativity constraints *i.e.,* $x_1 \ge 0, x_2 \ge 0$ the common feasible region is obtained in the first quadrant as shown in Fig. 2.9 and the individual feasible regions are shown by arrows.

The corner points are O(0, 0), A (3, 0), $B\left(\frac{24}{11}, \frac{15}{11}\right)$ and C(0, 3). The values of the objective function at the corner points are obtained as $z_O = 0$, $z_A = 3$, $z_B = 3$, $z_C = \frac{9}{4}$.

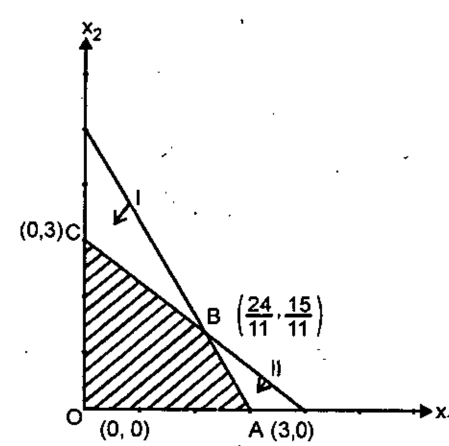


Fig. 2.9

NOTES

Since the common feasible region is bounded and the maximum has occurred at two corner points *i.e.,* at A and B respectively, these solutions are called multiple optima. So the solutions are

$$x_1^* = 3,\ x_2^* = 0 \quad \text{and} \quad x_1^* = \frac{15}{11},\ x_2^* = \frac{24}{11} \text{ and } z^* = 3.$$

Example 5. *Using graphical method show that the following LPP is unbounded.*

$$\text{Maximize } z = 10x_1 + 3x_2$$

$$\text{Subject to, } -2x_1 + 3x_2 \le 6,$$

$$x_1 + 2x_2 \ge 4,$$

$$x_1,\ x_2 \ge 0.$$

Solution. Due to the non-negativity constraints *i.e.,* $x_1 \ge 0$ and $x_2 \ge 0$ the common feasible region will be obtained in the first quadrant. Let us present the constraints in the intercept forms *i.e.,*

$$\frac{x_1}{-3} + \frac{x_2}{2} \le 1 \qquad \text{...(I)}$$

$$\frac{x_1}{4} + \frac{x_2}{2} \ge 1 \qquad \text{...(II)}$$

The common feasible region is shown in Fig. 2.10 which is unbounded *i.e.,* open region.

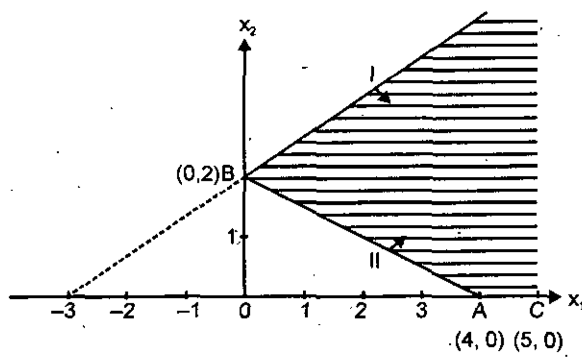


Fig. 2.10

There are two corner points A(4, 0) and B(0, 2). The objective function values are $z_A = 40$ and $z_B = 6$. Here the maximum is 40. Since the region is open, let us examine some other points.

Consider the point C(5, 0) and the value of the objective function is $z_C = 50$ which is greater than z_A. Therefore z_A is no longer optimal. If we move along x-axis, we observe that the next value is higher than the previous value and we reach to infinity for optimum value. Hence the problem is unbounded.

Note. For the same problem minimum exists which is the point B.

PROBLEMS

Using graphical method solve the following LPP :

1. $$\text{Maximize } z = 13x_1 + 117x_2$$

$$\text{Subject to, } x_1 + x_2 \le 12,$$

NOTES

$$x_1 - x_2 \geq 0,$$
$$4x_1 + 9x_2 \leq 36,$$
$$0 \leq x_1 \leq 2 \text{ and } \quad 0 \leq x_2 \leq 10.$$

2. Maximize $z = 3x_1 + 15x_2$

Subject to, $4x_1 + 5x_2 \leq 20,$
$$x_2 - x_1 \leq 1,$$
$$0 \leq x_1 \leq 4 \text{ and } \quad 0 \leq x_2 \leq 3.$$

3. Maximize $z = 5x_1 + 7x_2$

Subject to, $3x_1 + 8x_2 \leq 12,$
$$x_1 + x_2 \leq 2,$$
$$2x_1 \leq 3,$$
$$x_1, x_2 \geq 0.$$

4. Minimize $z = 2x_1 + 3x_2$

Subject to, $x_2 - x_1 \geq 2,$
$$5x_1 + 3x_2 \leq 15,$$
$$2x_1 \geq 1,$$
$$x_2 \leq 4,$$
$$x_1, x_2 \geq 0.$$

5. Minimize $z = 10x_1 + 9x_2$

Subject to, $x_1 + 2x_2 \leq 10,$
$$x_1 - x_2 \leq 0,$$
$$x_1 \leq 0, x_2 \geq 0.$$

6. Minimize $z = 4x_1 + 3x_2$

Subject to, $2x_1 + 3x_2 \leq 12,$
$$3x_1 - 2x_2 \leq 12,$$
x_1 unrestricted in sign, $x_2 \geq 0.$

7. Maximize $z = 10x_1 + 11x_2$

Subject to, $x_1 + x_2 \geq 4,$
$$0 \leq x_2 \leq 3,$$
$$x_1 \geq 2,$$
$$x_1 \geq 0.$$

8. Minimize $z = -x_1 + 2x_2$

Subject to, $x_1 - x_2 \geq 1,$
$$x_1 + x_2 \geq 5,$$
$$x_1, x_2 \geq 0.$$

9. Maximize $z = 4x_1 + 5x_2$

Subject to, $4x_1 - 5x_2 \leq 20,$
$$x_2 - x_1 \leq 1,$$
$$0 \leq x_2 \leq 3$$
$$0 \leq x_1 \leq 4.$$

10. Maximize $z = -3x_1 + 4x_2$

Subject to, $-3x_1 + 4x_2 \leq 12,$

$$2x_1 - x_2 \geq -2,$$
$$x_1 \leq 4,$$
$$x_1 \geq 0, x_2 \geq 0.$$

ANSWERS

NOTES

1. $x_1 = 2, x_2 = 2, z^* = 260$
2. $x_1 = \frac{5}{3}, x_2 = \frac{8}{3}, z^* = 45$
3. $x_1 = \frac{4}{5}, x_2 = \frac{6}{5}, z^* = \frac{62}{5}$
4. $x_1 = \frac{1}{2}, x_2 = \frac{5}{2}, z^* = \frac{17}{2}$
5. $x_1 = 0, x_2 = 0, z^* = 0$
6. $x_1 = 4, x_2 = 0, z^* = 16$
7. Unbounded solution
8. Unbounded solution
9. Multiple optima :

   $x_1 = \frac{5}{3}, x_2 = \frac{8}{3}$, and $x_1 = 4, x_2 = \frac{4}{5}$ and $z^* = 20$
10. Multiple optima :

    $x_1 = \frac{4}{5}, x_2 = \frac{18}{5}$ and $x_1 = 4, x_2 = 6$ and $z^* = 12$

2.3 SIMPLEX METHOD

The algorithm is discussed below with the help of a numerical example *i.e.*, consider

$$\text{Maximize } z = 4x_1 + 8x_2 + 5x_3$$
$$\text{Subject to, } x_1 + 2x_2 + 3x_3 \leq 18,$$
$$2x_1 + 6x_2 + 4x_3 \leq 15,$$
$$x_1 + 4x_2 + x_3 \leq 6,$$
$$x_1, x_2, x_3 \geq 0.$$

Step 1. If the problem is in minimization, then convert it to maximization as Min $z = -$ Max $(-z)$.

Step 2. All the right side constants must be positive. Multiply by -1 both sides for negative constants. All the variables must be non-negative.

Step 3. Make standard form by adding slack variables for '$\leq$' type constraints, surplus variables for '$\geq$' type constraints and incorporate these variables in the objective function with zero coefficients.

For example, $\quad$ Maximum $z = 4x_1 + 8x_2 + 5x_3 + 0, s_1 + 0.s_2 + 0.s_3$

$$\text{Subject to, } x_1 + 2x_2 + 3x_3 + s_1 = 18,$$
$$2x_1 + 6x_2 + 4x_3 + s_2 = 15,$$
$$x_1 + 4x_2 + x_3 + s_3 = 6,$$
$$x_1, x_2, x_3 \geq 0, s_1, s_2, s_3 \geq 0$$

Note that an unit matrix due to s_1, s_2 and s_3 variables is present in the coefficient matrix which is the key requirement for simplex method.

Step 4. Simplex method is an iterative method. Calculations are done in a table which is called simplex table. For each constraint there will be a row and for each

NOTES

variable there will be a column. Objective function coefficients c_j are kept on the top of the table. x_B stands for basis column in which the variables are called 'basic variables'. Solution column gives the solution, but in iteration 1, the right side constants are kept. At the bottom $z_j - c_j$ row is called 'net evaluation' row.

In each iteration one variable departs from the basis and is called departing variable and in that place one variable enter which is called entering variable to improve the value of the objective function.

Minimum ratio column determines the departing variable.

Iteration 1.

		c_j	4	8	5	0	0	0	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio
0	s_1	18	1	2	3	1	0	0	
0	s_2	15	2	6	4	0	1	0	
0	s_3	6	1	4	1	0	0	1	
	$z_j - c_j$								

Note. Variables which are forming the columns of the unit matrix enter into the basis column. In this table the solution is $s_1 = 18$, $s_2 = 15$, $s_3 = 6$, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ and $z = 0$.

To test optimality we have to calculate $z_j - c_j$ for each column as follows :

$$z_j - c_j = c_B^T.[x_j] - c_j$$

For first column, $$(0,0,0)\begin{pmatrix}1\\2\\1\end{pmatrix} - 4 = -4$$

For second column, $$(0,0,0)\begin{pmatrix}2\\6\\4\end{pmatrix} - 8 = -8$$ and so on.

These are displayed in the following table :

		c_j	4	8	5	0	0	0	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio
0	s_1	18	1	2	3	1	0	0	
0	s_2	15	2	6	4	0	1	0	
0	s_3	6	1	4	1	0	0	1	
	$z_j - c_j$		– 4	– 8 ↑	– 5	0	0	0	

NOTES

Decisions : If all $z_j - c_j \geq 0$. Then the current solution is optimal and stop. Else, Select the negative most value from $z_j - c_j$ and the variable corresponding to this value will be the entering variable and that column is called '*key column*'. Indicate this column with an upward arrow symbol.

In the given problem '– 8' is the most negative and variable x_2 is the entering variable. If there is a tie in the most negative, break it arbitrarily.

To determine the *departing variable*, we have to use minimum ratio. Each ratio is calculated as $\frac{[\text{soln.}]}{[\text{key column}]}$, componentwise division only for positive elements (*i.e.*, > 0) of the key column. In this example,

$$\min.\left\{\frac{18}{2}, \frac{15}{6}, \frac{6}{4}\right\} = \min.\{9, 2.5, 1.5\} = 1.5$$

The element corresponding to the min. ratio *i.e.*, here s_3 will be the departing variable and the corresponding row is called 'key row' and indicate this row by an outward arrow symbol. The intersection element of the key row and key column is called key element. In the present example, 4 is the key element which is highlighted. This is the end of this iteration, the final table is displayed as follow:

Iteration 1:

		c_j	4	8	5	0	0	0	Min.	
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio	
0	s_1	18	1	2	3	1	0	0	$\frac{18}{2} = 9$	
0	s_2	15	2	6	4	0	1	0	$\frac{15}{6} = 2.5$	→
0	s_3	6	1	$\boxed{4}$	1	0	0	1	$\frac{6}{4} = 1.5$	
	$z_j - c_j$		– 4	– 8	– 5	0	0	0		
				↑						

Step 5. For the construction of the next iteration (new) table the following calculations are to be made :

(*a*) Update the x_B column and the c_B column.

(*b*) Divide the key row by the key element.

(*c*) Other elements are obtained by the following formula :

$$\begin{pmatrix}\text{new}\\\text{element}\end{pmatrix} = \begin{pmatrix}\text{old}\\\text{element}\end{pmatrix} - \frac{\begin{pmatrix}\text{element}\\\text{corresponding to}\\\text{key row}\end{pmatrix} \cdot \begin{pmatrix}\text{element}\\\text{corresponding to}\\\text{key column}\end{pmatrix}}{\text{key element}}$$

(*d*) Then go to step 4.

NOTES

Iteration 2.

		c_j	4	8	5	0	0	0	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio
0	s_1	15	$\frac{1}{2}$	0	$\frac{5}{2}$	1	0	$-\frac{1}{2}$	$15\times\frac{3}{5}=6$
0	s_2	6	$\frac{1}{2}$	0	$\boxed{\frac{5}{2}}$	0	1	$-\frac{3}{2}$	$6\times\frac{2}{5}=2.4 \rightarrow$
8	x_2	$\frac{3}{2}$	$\frac{1}{4}$	1	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{3}{4}\times 4=3$
	$z_j - c_j$		-2	0	-3	0	0	2	
					$\uparrow$				

Iteration 3.

		c_j	4	8	5	0	0	0	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio
0	s_1	9	0	0	0	1	-1	1	-
5	x_3	$\frac{12}{5}$	$\frac{1}{5}$	0	1	0	$\frac{2}{5}$	$-\frac{3}{5}$	$\frac{12}{5}\times\frac{5}{1}=12$
8	x_2	$\frac{9}{10}$	$\boxed{\frac{1}{5}}$	1	0	0	$-\frac{1}{10}$	$\frac{2}{5}$	$\frac{9}{10}\times\frac{5}{1}=4.5 \rightarrow$
	$z_j - c_j$		$-\frac{7}{5}$	0	0	0	$\frac{6}{5}$	$\frac{1}{5}$	
			$\uparrow$						

Iteration 4.

		c_j	4	8	5	0	0	0	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio
0	s_1	9	0	0	0	1	-1	1	
5	x_3	$\frac{3}{2}$	0	-1	1	0	$\frac{1}{2}$	-1	
4	x_1	$\frac{9}{2}$	1	5	0	0	$-\frac{1}{2}$	2	
	$z_j - c_j$		0	7	0	0	$\frac{1}{2}$	3	

Since all $z_j - c_j \geq 0$, the current solution is optimal.

NOTES

$$\therefore \qquad x_1^* = \frac{9}{2},\ x_2^* = 0,\ x_3^* = \frac{3}{2},\ z^* = \frac{51}{2}.$$

Note (exceptional cases).

(*a*) If in the key column, all the elements are non-positive *i.e.*, zero or negative, then min. ratio cannot be calculated and the problem is said to be unbounded.

(*b*) In the net evaluation of the optimal table all the basic variables will give the value zero. If any non-basic variable give zero net evaluation then it indicates that there is an alternative optimal solution. To obtain that solution, consider the corresponding column as key column and apply one simplex iteration.

(*c*) For negative variables, $x \leq 0$, set $x = -x'$, $x' \geq 0$.

For unrestricted variables set $x = x' - x''$ where $x', x'' \geq 0$.

Example 6. *Solve the following by simplex method :*

$$\text{Maximize } z = x_1 + 3x_2$$

$$\text{Subject to, } -x_1 + 2x_2 \leq 2,\ x_1 - 2x_2 \leq 2,\ x_1, x_2 \geq 0.$$

Solution. Standard form of the given LPP can be written as follows :

$$\text{Maximum } z = x_1 + 3x_2 + 0.s_1 + 0.s_2$$

$$\text{Subject to, } -x_1 + 2x_2 + s_1 = 2,\ x_1 - 2x_2 + s_2 = 2,$$

$$x_1, x_2 \geq 0,\ s_1, s_2 \text{ slacks} \geq 0.$$

Iteration 1.

		c_j	1	3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
0	s_1	2	-1	$\boxed{2}$	1	0	$\frac{2}{2} = 1$ →
0	s_2	2	1	-2	0	1	–
	$z_j - c_j$		-1	-3 ↑	0	0	

Iteration 2.

		c_j	1	3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
3	x_2	1	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	
0	s_2	4	0	0	1	1	
	$z_j - c_j$		$-\frac{5}{2}$ ↑	0	$\frac{3}{2}$	0	

Since all the elements in the key column are non-positive, we cannot calculate min. ratio. Hence the given LPP is said to be unbounded.

PROBLEMS

NOTES

Solve the following LPP by simplex method:

1. Maximize $z = 3x_1 + 2x_2$

   S/t, $5x_1 + x_2 \le 10$, $4x_1 + 5x_2 \le 60$; $x_1, x_2 \ge 0$

2. Maximize $z = 5x_1 + 4x_2 + x_3$

   S/t, $6x_1 + x_2 + 2x_3 \le 12$, $8x_1 + 2x_2 + x_3 \le 30$,

   $4x_1 + x_2 - 2x_3 \le 16$, $x_1, x_2, x_3 \ge 0$

3. Maximize $z = 3x_1 + 2x_2$

   S/t, $3x_1 + 4x_2 \le 12$, $2x_1 + 5x_2 \le 10$, $x_1, x_2 \ge 0$

4. Maximize $z = 3x_1 + 2x_2 + x_3$

   S/t, $3x_1 + x_2 + 2x_3 \le 20$, $x_1 + 3x_2 + 4x_3 \le 16$, $x_1, x_2, x_3 \ge 0$

5. Maximize $z = 4x_1 - 2x_2 - x_3$

   S/t, $x_1 + x_2 + x_3 \le 3$, $2x_1 + 2x_2 + x_3 \le 4$, $x_1 - x_2 \le 0$,

   $x_1, x_2, x_3 \ge 0$

6. Maximize $z = 5x_1 + 3x_2 + 3x_3$

   S/t, $4x_1 + 4x_2 + 3x_3 \le 12000$, $0.4x_1 + 0.5x_2 + 0.3x_3 \le 1800$,

   $0.2x_1 + 0.2x_2 + 0.1x_3 \le 960$, $x_1, x_2, x_3 \ge 0$

7. Maximize $z = 3x_1 + 2x_2 + 2x_3$

   S/t, $2x_1 - x_2 + 3x_3 \le 18$, $x_1 + x_2 + 2x_3 \le 12$, $x_1, x_2, x_3 \ge 0$

8. Maximize $z = 3x_1 + x_2 + x_3 + x_4$

   S/t, $-2x_1 + 2x_2 + x_3 = 4$, $3x_1 + x_2 + x_4 = 6$, $x_i \ge 0$ for all i

9. Maximize $z = x_1 + x_2$

   S/t, $x_1 - 2x_2 \le 2$, $-x_1 + 2x_2 \le 2$, $x_1, x_2 \ge 0$

10. Find all the optimal BFS to the following :

    Maximize $z = x_1 + x_2 + x_3 + x_4$

    S/t, $x_1 + x_2 \le 2$, $x_3 + x_4 \le 5$, $x_1, x_2, x_3, x_4 \ge 0$

ANSWERS

1. $x_1 = 0, x_2 = 10, z^* = 20$ (It 3)
2. $x_1 = 0, x_2 = 12, x_3 = 0, z^* = 48$ (It 3)
3. $x_1 = 4, x_2 = 0, z^* = 12$ (It 2)
4. $x_1 = \frac{11}{2}, x_2 = \frac{7}{2}, x_3 = 0, z^* = \frac{47}{2}$ (It 3)
5. $x_1 = 1, x_2 = 1, x_3 = 0, z^* = 2$ (It 3)
6. $x_1 = 3000, x_2 = 0, x_3 = 0, z^* = 15000$ (It 2)
7. $x_1 = 10, x_2 = 2, x_3 = 0, z^* = 34$ (It 3)
8. Solution 1 : $x_1 = 1, x_2 = 3, x_3 = x_4 = 0$ (It 2)

   Solution 2 : $x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 4, z^* = 6$
9. Unbounded solution (It 2)
10. (2, 0, 5, 0), (0, 2, 5, 0), (0, 2, 0, 5), (2, 0, 0, 5).

NOTES

2.4 BIG M METHOD

The method is also known as 'penalty method' due to Charnes. If there is '≥' type constraint, we add surplus variable and if there is '=' type, then the constraint is in equilibrium. Generally, in these cases there may not be any unit matrix in the standard form of the coefficient matrix.

To bring unit matrix we take help of another type of variable, known as 'artificial variable'. The addition of artificial variable creates infeasibility in the system which was already in equilibrium. To overcome this, we give a very large number denoted as M to the coefficient of the artificial variable in the objective function. For maximization problem, we add "– M. (artificial variable)" in the objective function so that the profit comes down. For minimization problem we add "M.(artificial variable)" in the objective function so that the cost goes up. Therefore the simplex method tries to reduce the artificial variable to the zero level so that the feasibility is restored and the objective function is optimized.

The only **drawback** of the big M method is that the value of M is not known but it is a very large number. Therefore we cannot develop computer program for this method.

Note. (*a*) Once the artificial variable departs from the basis, it will never again enter in the subsequent iterations due to big M. Due to this we drop the artificial variable column in the subsequent iterations once the variable departs from the basis.

(*b*) If in the optimal table, the artificial variable remains with non-zero value, then the problem is said to be '*infeasible*'.

If the artificial variable remains in the optimal table with zero value, then the solution is said to be 'pseudo optimal'.

(*c*) The rule for 'multiple solution' and 'unbounded solution' are same as given by simplex method. The big-M method is a simple variation of simplex method.

Example 7. *Using Big-M method solve the following LPP :*

$$\text{Minimize } z = 10x_1 + 3x_2$$

$$\text{S/t, } x_1 + 2x_2 \geq 3,\ x_1 + 4x_2 \geq 4;\ x_1, x_2 \geq 0$$

Solution. Standard form of the given LPP is

$$\text{Min. } z = -\text{Max. } (-2 \;= -10x_1 - 3x_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$

$$\text{S/t, } x_1 + 2x_2 - s_1 + a_1 = 3$$

$$x_1 + 4x_2 - s_2 + a_2 = 4$$

$$x_1, x_2 \geq 0,\ s_1, s_2 \text{ surplus} \geq 0,\ a_1, a_2 \text{ artificial} \geq 0$$

Iteration 1.

		c_j	– 10	– 3	0	0	– M	– M	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2	ratio
– M	a_1	3	1	2	– 1	0	1	0	$\frac{3}{2} = 1.5$

NOTES

$-M$	a_2	4	1	$\boxed{4}$	0	-1	0	1	$\frac{4}{4}=1$ →
	$z_j - c_j$	8	$-2M+10$	$-6M+3$	M	M	0	0	
				↑					

Iteration 2.

		c_j	-10	-3	0	0	$-M$	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	ratio
$-M$	a_1	1	$\boxed{\frac{1}{2}}$	0	-1	$\frac{1}{2}$	1	$\frac{1}{1/2}=2$ →
-3	x_2	1	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{1}{1/4}=4$
$z_j - c_j$			$-\frac{M}{2}+\frac{37}{4}$	0	M	$-\frac{M}{2}+\frac{3}{4}$	0	
			↑					

Iteration 3.

		c_j	-10	-3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
-10	x_1	2	1	0	-2	$\boxed{1}$	$\frac{2}{1}=2$ →
-3	x_2	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	–
$z_j - c_j$			0	0	$\frac{37}{2}$	$-\frac{17}{2}$	
						↑	

Iteration 4. (Optimal)

		c_j	-10	-3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
0	s_2	2	1	0	-2	1	
-3	x_2	$\frac{3}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	
$z_j - c_j$			$\frac{17}{2}$	0	$\frac{3}{2}$	0	

Since all $z_j - c_j \geq 0$, the current solution is optimal.

$$\therefore \quad x_1^* = 0,\ x_2^* = \frac{3}{2},\ z^* = \frac{9}{2}.$$

Example 8. *Solve the following LPP by Big-M method :*

$$\text{Minimize } z = 2x_1 + x_2 + 3x_3$$

$$S/t, \; -3x_1 + x_2 - 2x_3 \geq 1, \; x_1 - 2x_2 + x_3 \geq 2; \; x_1, x_2, x_3 \geq 0.$$

Solution. The standard form of the given problem can be written as follows :

$$\text{Min. } z = -\text{Max. } (-z = -2x_1 - x_2 - 3x_3 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$

$$\text{S/t, } -3x_1 + x_2 - 2x_3 - s_1 + a_1 = 1,$$

$$x_1 - 2x_2 + x_3 - s_2 + a_2 = 2,$$

$x_1, x_2, x_3 \geq 0$, s_1, s_2 surplus $s \geq 0$, a_1, a_2 artificial variables ≥ 0.

Iteration 1.

		c_j	-2	-1	-3	0	0	$-M$	$-M$	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	a_1	a_2	ratio
$-M$	a_1	1	-3	1	-2	-1	0	1	0	
$-M$	a_2	2	1	-2	1	0	-1	0	1	
$z_j - c_j$			$2M + 2$	$M + 1$	$M + 3$	M	M	0	0	

Since all $z_j - c_j \geq 0$, the first iteration itself give optimal solution. But in solution *i.e.*, $a_1 = 1$, $a_2 = 2$ present with non-zero value. Hence the given problem does not possess any feasible solution.

2.5 TWO PHASE METHOD

To overcome the drawback of Big-M method, two phase method has been framed. In the first phase an auxiliary LP Problem is formulated as follows :

Minimize T = Sum of artificial variables

S/t, original constraints

which is solved by simplex method. Here artificial variables act as decision variables. So Big-M is not required in the objective function. If $T^* = 0$, then go to phase two calculations, else ($T^* \neq 0$) write the problem is infeasible. In phase two, the optimal table of phase one is considered with the following modifications :

Delete the artificial variable's columns and incorporate the original objective function and also update the c_B values. Calculate $z_j - c_j$ values. If all $z_j - c_j \geq 0$, the current solution is optimal else go to the next iteration.

Note. (*a*) Multiple solutions, if it exists, can be detected from the optimal table of phase two.

(*b*) In phase I, the problem is always minimization type irrespective of the type of the original given objective function.

Example 9. *Using two phase method solve the following LPP :*

$$\text{Minimize } z = 10x_1 + 3x_2$$

$$S/t, \; x_1 + 2x_2 \geq 3, \; x_1 + 4x_2 \geq 4; \; x_1, x_2 \geq 0$$

Solution. Standard form of the given LPP is

$$\text{Min. } z = -\text{Max. } (-z = -10x_1 - 3x_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$

NOTES

$$\text{S/t, } x_1 + 2x_2 - s_1 + a_1 = 3,$$
$$x_1 + 4x_2 - s_2 + a_2 = 4,$$
$$x_1, x_2 \geq s_1, s_2, \text{ surplus} \geq a_1, a_2 \text{ artificial} \geq 0$$

Phase I $\quad$ Min $T = a_1 + a_2 = -\text{Max}$

$$(-T = 0.\ x_1 + 0.x_2 + 0.s_1 + 0.s_2 - a_1 - a_2)$$

$$\text{S/t., } x_1 + 2x_2 - s_1 + a_1 = 3;\ x_1 + 4x_2 - s_2 + a_2 = 4,$$
$$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$$

Iteration 1.

		c_j	0	0	0	0	-1	-1	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2	ratio
-1	a_1	3	1	2	-1	0	1	0	$\frac{3}{2} = 1.5$
-1	a_2	4	1	$\boxed{4}$	0	-1	0	1	$\frac{4}{4} = 1$ →
$z_j - c_j$			-2	-6 ↑	1	1	0	0	

Iteration 2.

		c_j	0	0	0	0	-1	-1	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2	ratio
-1	a_1	1	$\boxed{\frac{1}{2}}$	0	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{1/2} = 2$ →
0	x_2	1	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{1/4} = 4$
$z_j - c_j$			$-\frac{1}{2}$ ↑	0	1	$-\frac{1}{2}$	0	$\frac{3}{2}$	

Iteration 3.

		c_j	0	0	0	0	-1	-1	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2	ratio
0	x_1	2	1	0	-2	1	2	-1	
0	x_2	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	
$z_j - c_j$			0	0	0	0	1	1	

Since all $z_j - c_j \geq 0$, the solution is optimal $a_1^* = 0$, $a_2^* = 0$ and $T^* = 0$. Therefore we go to phase II calculations.

Phase II

Iteration 1.

		c_j	-10	-3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
-10	x_1	2	1	0	-2	$\boxed{1}$	$\frac{2}{1}=2 \rightarrow$
-3	x_2	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	
	$z_j - c_j$		0	0	$\frac{37}{2}$	$-\frac{17}{2}$	

Iteration 2.

		c_j	-10	-3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
0	s_2	2	1	0	-2	1	
-3	x_2	$\frac{3}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	
	$z_j - c_j$		$\frac{17}{2}$	0	$\frac{3}{2}$	0	

Since all $z_j - c_j \geq 0$, the current solution is optimal.

$$\therefore \quad x_1^* = 0, x_2^* = \frac{3}{2}, \; z^* = \frac{9}{2}.$$

Example 10. *Solve the following LPP by two phase method.*

$$\text{Maximize } z = 2x_1 + x_2 - 3x_3$$

$$\text{S/t, } x_1 + 2x_2 + 2x_3 \geq 12, \; 3x_1 - 2x_2 + 4x_3 \leq 10$$

$$x_1 \geq 0, \; x_2 \leq 0, \; x_3 \geq 0.$$

Solution. Set $x_2 = -x'_2, x'_2 \geq 0$.

The standard form of the given LPP is

$$\text{Maximize } z = 2x_1 - x'_2 - 3x_3 + 0.s_1 + 0.s_2 - Ma_1$$

$$\text{S/t, } x_1 - 2x'_2 + 2x_3 - s_1 + a_1 = 12,$$

$$3x_1 + 2x'_2 + 4x_3 + s_2 = 10,$$

$$x_1, x'_2\, x_3 \geq 0, \; s_1 \text{ (surplus)} \geq 0, s_2 \text{ (slack)} \geq 0 \text{ and } a_1 \text{ (artificial)} \geq 0.$$

NOTES

Phase I.

$$\text{Minimize } T = a_1 = -\text{Max.}\ (-T = 0x_1 + 0x'_2 + 0x_3 + 0.s_1 + 0.s_2 - a_1)$$

$$\text{S/t, } x_1 - 2x'_2 + 2x_3 - s_1 + a_1 = 12$$

$$3x_1 + 2x'_2 + 4x_3 + s_2 = 10$$

$$x_1, x'_2, x_3, s_1, s_2, a_1 \geq 0.$$

Iteration 1.

		c_j	0	0	0	0	0	-1	Min.
c_B	x_B	soln.	x_1	x'_2	x_3	s_1	s_2	a_1	ratio
-1	a_1	12	1	-2	2	-1	0	1	$\frac{12}{2}=6$
0	s_1	10	3	2	$\boxed{4}$	0	1	0	$\frac{10}{4}=2.5$ →
	$z_j - c_j$		-1	2	-2 ↑	1	0	0	

Iteration 2.

		c_j	0	0	0	0	0	-1	Min.
c_B	x_B	soln.	x_1	x'_2	x_3	s_1	s_2	a_1	ratio
-1	a_1	7	$-\frac{1}{2}$	-3	0	-1	$-\frac{1}{2}$	1	
0	x_3	$\frac{5}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	1	0	$\frac{1}{4}$	0	
	$z_j - c_j$		$\frac{1}{2}$	3	0	1	$\frac{1}{2}$	0	

Since all $z_j - c_j \geq 0$, the current solution is optimal and $T^* = 7 \neq 0$.

This implies that there does not exist any feasible solution to the given LPP.

Note. In simplex, Big-M and two phase methods, if there is a tie in min. ratio or in negative most value of net evaluation, the optimal feasible solution will lead to degenerate solution.

PROBLEMS

Solve the following LPP using *Big-M* method and *Two phase* method :

1. Minimize $z = 2x_1 + 3x_2$

   S/t, $2x_1 + x_2 \geq 1$, $x_1 + 2x_2 \geq 1$; $x_1, x_2 \geq 0$.

2. Maximize $z = 5x_1 + 3x_2$

   S/t, $2x_1 - 4x_2 \leq 16$, $3x_1 + 4x_2 \geq 12$; $x_1, x_2 \geq 0$.

3. Maximize $z = 2x_1 + 3x_2 + 2x_3$

$$\text{S/t, } 3x_1 + 2x_2 + 2x_3 = 16,\ 2x_1 + 4x_2 + x_3 = 20,$$
$x_1 \geq 0$, x_2 unrestricted in sign, $x_3 \geq 0$.

NOTES

4. $$\text{Maximize } z = 2x_1 + 3x_2 + x_3$$
$$\text{S/t, } 3x_1 + 2x_2 + x_3 = 15,\ x_1 + 4x_2 = 10,$$
x_1 unrestricted in sign, $x_2, x_3 \geq 0$.

5. $$\text{Maximize } z = 2x_1 + 2x_2 + 3x_3$$
$$\text{S/t, } x_1 - 2x_2 + x_3 \leq 8,\ 3x_1 + 4x_2 + 2x_3 \geq 2,$$
$$x_1 \geq 0,\ x_2 \leq 0,\ x_3 \geq 0.$$

6. $$\text{Maximize } z = 3x_1 + 2x_2 + x_3 - x_4$$
$$\text{S/t, } x_1 + 2x_2 + 3x_3 = 16,\ 3x_1 + x_2 + 2x_3 = 20,$$
$$2x_1 + x_2 + x_3 + x_4 = 12;\ x_1, x_2, x_3, x_4 \geq 0.$$

7. $$\text{Minimize } z = x_1 + 2x_2$$
$$\text{S/t, } 2x_1 + x_2 = 4,\ 3x_1 + 4x_2 \geq 5,\ x_1 + x_2 \leq 4,$$
$$x_1, x_2 \geq 0.$$

8. $$\text{Minimize } z = x_1 + 3x_2 + 5x_3$$
$$\text{S/t, } 2x_1 + 5x_2 + x_3 \geq 12,\ x_1 + 2x_2 + 3x_3 \geq 10,$$
$$x_1, x_2, x_3 \geq 0.$$

9. $$\text{Minimize } z = 5x_1 + x_2$$
$$\text{S/t, } 2x_1 + x_2 \leq 2,\ 3x_1 + 4x_2 \geq 12;\ x_1, x_2 \geq 0.$$

10. $$\text{Maximize } z = x_1 - 3x_2$$
$$\text{S/t, } -x_1 + 2x_2 \leq 15,\ x_1 + 3x_2 = 10;\ x_1 \geq 0,\ x_2 \leq 0.$$

11. Find a BFS of the following system :
$$x_1 + x_2 \geq 1,\ -2x_1 + x_2 \geq 2,\ 2x_1 + 3x_2 \leq 6;\ x_1, x_2 \geq 0.$$

ANSWERS

1. $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{3}$, $z^* = \frac{5}{3}$ (Big-M 3 It)
2. Unbounded solution. (Big-M 4 It)
3. $x_1 = 0, x_2 = 4, x_3 = 4, z^* = 20$ (Big-M 4 It)
4. Unbounded solution (Big-M 2 It)
5. $x_1 = 0, x_2 = 0, x_3 = 8, z^* = 24$. (Big-M 5 It)
6. $x_1 = 4, x_2 = 0, x_3 = 4, x_4 = 0, z^* = 16$. (Big-M 4 It)
7. $x_1 = 2, x_2 = 0, z^* = 2$. (Big-M 3 It)
8. $x_1 = 10, x_2 = 0, x_3 = 0, z^* = 10$. (Big-M 5 It)
9. Infeasible solution.
10. Unbounded solution.
11. $x_1 = 0, x_2 = 2$ (use Phase-I) (3 It).

2.6 FORMULATION PROBLEMS

Example 11. *A manufacturer produces two types of machines M_1 and M_2. Each M_1 requires 4 hrs. of grinding and 2 hrs. of polishing whereas each M_2*

NOTES

model requires 2 hrs. of grinding and 4 hrs. of polishing. Manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hrs a week and each polisher works 40 hrs a week. Profit on an M_1 model is Rs. 3 and on an M_2 model is Rs. 4. Whatever is produced is sold in the market. How should the manufacturer allocate his production capacity to two types of models so that he may make the maximum profit in a week. Formulate the LPP and solve graphically.

Solution. Let x_1 = No. of M_1 machines, and

x_2 = No. of M_2 machines to be produced in a week.

The above data is summarized as follows :

	M_1	M_2
Grinding	4 hrs.	2 hrs.
Polishing	2 hrs.	4 hrs.
Profit	Rs. 3	Rs. 4

	Time available per week
2 Grinding	80 hrs.
3 Polishing	120 hrs.

Therefore the LPP can be formulated as follows :

$$\text{Maximize profit} = 3x_1 + 4x_2$$
$$\text{S/t, } 4x_1 + 2x_2 \leq 80 \quad \text{(grinding)}$$
$$2x_1 + 4x_2 \leq 120 \quad \text{(polishing)}$$
$$x_1, x_2 \geq 0$$

The graphical region is shown below.

Profit at A = 60

Profit at B = 126.67

Profit at C = 120

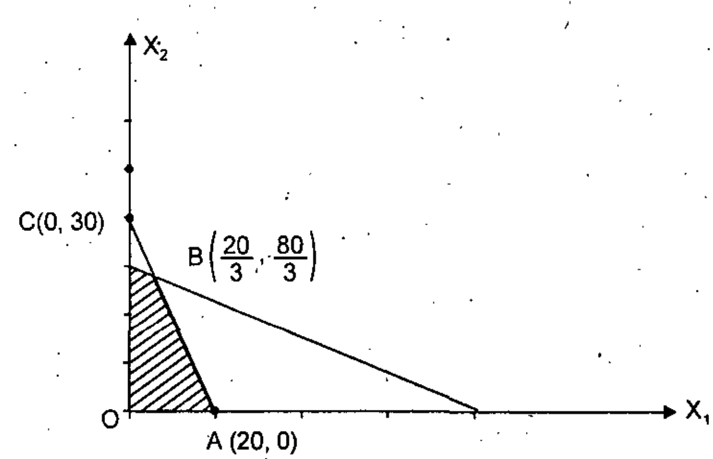


Fig. 2.11

$\therefore$ The optimal solution is $x_1 = \dfrac{20}{3}$, $x_2 = \dfrac{80}{3}$, and max. profit = Rs. 126.67.

NOTES

Example 12. *A firm can produce three types of woolen clothes, say, A, B and C using three kinds of wool, say red wool, green wool and blue wool. One unit of length of type A cloth needs 2 yards of red wool and 3 yards of blue wool; one unit length of type B cloth needs 3 yards of red wool, 2 yards of green wool and 2 yards of blue wool; and one unit length of type C cloth needs 5 yards of green wool and 4 yards of blue wool. The firm has only a stock of 8 yards of red wool, 10 yards of green wool and 15 yards of blue wool. It is assumed that income obtained from one unit length of type A cloth is Rs. 3, of type B cloth is Rs. 5 and that of type C cloth is Rs. 4. Formulate the above problem as a LP problem.*

Solution. The given data is summarized below :

Wool	*Garment type* *A*	*B*	*C*	*Stock available*
Red	2	3	-	8
Green	-	2	5	10
Blue	3	2	4	15
Income (Rs.)	3	5	4	

Suppose that he produces x_1, x_2 and x_3 unit lengths of A, B and C clothes respectively. Then the LPP is :

$$\text{Maximize income} = 3x_1 + 5x_2 + 4x_3$$

$$\text{S/t, } 2x_1 + 3x_2 \leq 8 \quad \text{(Red wool)}$$

$$2x_2 + 5x_3 \leq 10 \quad \text{(Green wool)}$$

$$3x_1 + 2x_2 + 5x_3 \leq 15 \quad \text{(Blue wool)}$$

$$x_1, x_2, x_3 \geq 0$$

The solution is obtained as $x_1^* = 1.67$, $x_2^* = 1.56$, $x_3^* = 1.38$, $z^* = 18.29$ (It 4, Simplex).

PROBLEMS

1. A manufacturer of furniture makes only chair and tables. A chair requires two hours on m/c A and six hours on m/c B. A table requires five hours on m/c A and two hours on m/c B. 16 hours are available on m/c A and 22 hours on m/c B per day. Profits for a chair and table be Rs. 1 and Rs. 5 respectively. Formulate the LPP of finding daily production of these items for maximum profit and solve graphically.

2. A tailor has 80 sq. m. of cotton material and 120 sq. m. of woolen material. A suit requires 1 sq. m. of cotton and 3 sq. m. of woolen material and a dress requires 2 sq. m. of each. A suit sells for Rs. 500 and a dress for Rs. 400. Pose a LPP in terms of maximizing the income.

3. A company owns two mines : mine A produces 1 tonne of high grade ore, 3 tonnes of medium grade ore and 5 tonnes of low grade ore each day; and mine B produces 2 tonnes of each of the three grades of ore each day. The company needs 80 tonnes

NOTES

of high grade ore, 160 tonnes of medium grade ore and 200 tonnes of low grade ore. If it costs Rs. 200 per day to work each mine, find the number of days each mine has to be operated for producing the required output with minimum total cost.

4. A company manufactures two products A and B. The profit per unit sale of A and B is Rs. 10 and Rs. 15 respectively. The company can manufacture at most 40 units of A and 20 units of B in a month. The total sale must not be below Rs. 400 per month. If the market demand of the two items be 40 units in all, write the problem of finding the optimum number of items to be manufactured for maximum profit, as a problem of LP. Solve the problem graphically or otherwise.

5. A company is considering two types of buses—ordinary and semideluxe for transportation. Ordinary bus can carry 40 passengers and requires 2 mechanics for servicing. Semideluxe bus can carry 60 passengers and requires 3 mechanics for servicing. The company can transport at least 300 persons daily and not more than 12 mechanics can be employed. The cost of purchasing buses is to be minimized, given that the ordinary bus costs Rs. 1,20,000 and semideluxe bus costs Rs. 1,50,000. Formulate this problem as a LPP.

6. A pharmaceutical company has 100 kg. of ingredient A, 180 kg. of ingredient B and 120 kg. of ingredient C available per month. They can use these ingredients to make three basic pharmaceutical products namely 5-10-5; 5-5-10 and 20-5-10; where the numbers in each case represent the percentage by weight of A, B and C respectively in each of the products. The cost of these ingredients are given below :

Ingredient	*Cost per kg. (Rs.)*
A	80
B	20
C	50
Inert ingredients	20

Selling price of these products are Rs. 40.5, Rs. 43 and Rs. 45 per kg. respectively. There is a capacity restriction of the company for the product 5-10-5, so they cannot produce more than 30 kg. per month. Determine how much of each of the products they should produce in order to maximize their monthly profit.

7. A fruit squash manufacturing company manufactures three types of squashes. The basic formula are :

5 litre lemonade	:	2 oz. lemons, 2 kg. of sugar, 2 oz. citric acid and water.
5 litre grape fruits	:	11/2 kg of grape fruit, 11/2 kg. of sugar, 11/2 oz. citric acid and water.
5 litre orangeade	:	11/2 dozen oranges, 11/2 kg. of sugar, 1 oz. citric acid and water.

The squashes sell at

Lemonade	:	Rs. 37.50 per 5 litre;
Grape fruit	:	Rs. 40.00 per 5 litre;
Orangeade	:	Rs. 42.50 per 5 litre.

In the last week of the season they have in stock 2500 dozen lemons, 2000 kgs. grape fruit, 750 dozen oranges, 5000 kgs. of sugar and 3000 ozs. citric acid. What should be their manufacturing quantities in the week to maximize the turnover ?

8. A farmer is raising cows in his farm. He wishes to determine the qualities of the available types of feed that should be given to each cow to meet certain nutritional requirements at a minimum cost. The numbers of each type of basic nutritional ingredient contained within a kg. of each feed type is given in the following table, along with the daily nutritional requirements and feed costs.

NOTES

Nutritional ingredient	*kg. of corn*	*kg. of tankage*	*kg. of green grass*	*Min. daily requirement*
Carbohydrates	9	2	4	20
Proteins	3	8	6	18
Vitamins	1	2	6	15
Cost	7	6	5	

Formulate a linear programming model for this problem so as to determine the optimal mix of feeds.

ANSWERS

1. No chairs and 3.2 tables to be produced for max. profit of Rs. 16.
2. Max. sells = $500x_1 + 400x_2$
   S/t, $x_1 + 2x_2 \le 80$, $3x_1 + 2x_2 \le 120$, x_1 = no. of suits ≥ 0 and x_2 = no. of dresses ≥ 0.
3. Mine A to be operated for 40 days and mine B to be operated for 20 days and min. cost = Rs. 12000.
4. Max. profit = $10x_1 + 15x_2$
   S/t, $x_1 \le 40$, $x_2 \le 20$, $x_1 + x_2 \ge 40$, $10x_1 + 15x_2 \ge 400$, $x_1, x_2 \ge 0$ and $x_1^* = 40$, $x_2^* = 20$, max. profit = Rs. 700.
5. Min. cost = $1{,}20{,}000x_1 + 1{,}50{,}000x_2$
   S/t, $40x_1 + 60x_2 \ge 300$, $2x_1 + 3x_2 \le 12$; $x_1, x_2 \ge 0$.
6. Let x_1, x_2, x_3 be three products in kg. to be manufactured.
   Max. profit = $16x_1 + 17x_2 + 10x_3$
   S/t, $x_1 + x_2 + 4x_3 \le 2000$, $2x_1 + x_2 + x_3 \le 3600$,
   $x_1 + 2x_2 + 2x_3 \ge 2400$, $x_1 \le 30$, $x_1 + x_2 + x_3 \ge 0$
   Solution. $x_1 = 30$, $x_2 = 1185$, $x_3 = 0$, Profit = Rs. 20,625. (Simplex 3It)
7. Let $5x_1$, $5x_2$, $5x_3$ litre be the lemonade, grape fruit and orangeade to be manufactured per week.
   Max. profit = $37.5x_1 + 40x_2 + 42.5x_3$
   S/t, $2x_1 \le 2500$, $3x_2 \le 4000$, $3x_3 \le 1500$,
   $4x_1 + 3x_2 + 3x_3 \le 10{,}000$, $4x_1 + 3x_2 + 2x_3 \le 6000$, $x_1, x_2, x_3 \ge 0$.
8. Min. cost = $7x_1 + 6x_2 + 5x_3$
   S/t, $9x_1 + 2x_2 + 4x_3 \ge 20$,
   $3x_1 + 8x_2 + 6x_3 \ge 18$
   $x_1 + 2x_2 + 6x_3 \ge 15$
   $x_1, x_2, x_3 \ge 0$.

2.7 REVISED SIMPLEX METHOD (RSM)

NOTES

I. Algorithm

Step 1. Write the standard form of the given LPP and convert it into maximization type if it is in minimization type *i.e.,*

$$\text{Max. } z = cx$$
$$\text{S/t, } Ax = b, x \geq 0.$$

Use the following notations :

$c^T = [c_1, c_2, \ldots, c_n]$ Profit coefficients.

Columns of A as $A_1, A_2, \ldots, A_m$.

$\pi = (\pi_1, \pi_2, \ldots)$ Simplex multipliers

x_B = Basis vector

c_B^T = Profit coefficient in the basis

B = Basis matrix, B^{-1} = Basis inverse

$\bar{c}_j$ = Net evaluations,

j = Index of non-basic variables

$\bar{b}$ = Current BFS

Step 2. For Iteration 1

$$B = I, B^{-1} = I$$

else for other iterations

Find $B = \left[x_{B_i}\right] = \left[A_{x_{B_i}}\right]$ and hence find B^{-1}.

Step 3. Calculate

$$\pi = c_B^T.B^{-1} \text{ and } \bar{b} = B^{-1}.b \text{ (current solution)}$$
$$\bar{c}_j = \pi A_j - c_j$$

Decisions : If all $\bar{c}_j \geq 0$ then the current BFS is optimal, else select the negative most of $\bar{c}_j$, say $\bar{c}_k$. Then x_k will be the '*Entering Variable*' and $\bar{A}_k$ = key column $= B^{-1}.A_k$.

Step 4. Produce the following revised simplex table :

x_B	B^{-1}	$\bar{b}$	*Entering*	*Key*
			variable	*column*

Encircle the key element obtained from the min. ratio $\left\{\left[\bar{b}\right]/\left[\text{Key column}\right]\right\}$.

Element corresponding to the key element will depart from $[x_B]$.

Step 5. Go to step 2.

Repeat the procedure until optimal BFS is obtained.

Note. (*a*) If, in step 4, all the elements in the key column are non-positive, then the given problem is unbounded.

(*b*) If, in the optimal BFS, artificial variables (if any) take zero value then the solution is degenerate else, for non-zero value, the given problem is said to be infeasible.

II. Advantages

NOTES

In computational point of view, the Revised Simplex Method is superior than ordinary simplex method. Due to selected column calculations in revised simplex method, less memory is required in computer. Whereas the ordinary simplex method requires more memory space in computer.

Example 13. *Using revised simplex method solve the following LPP :*

$$\text{Maximize } z = 5x_1 + 2x_2 + 3x_3$$
$$\text{S/t, } x_1 + 2x_2 + 2x_3 \le 8$$
$$3x_1 + 4x_2 + x_3 \le 7$$
$$x_1, x_2, x_3 \ge 0.$$

Solution. Standard form of the given LPP is

$$\text{Max. } z = 5x_1 + 2x_2 + 3x_3 + 0.s_1 + 0.s_2$$
$$\text{S/t, } x_1 + 2x_2 + 2x_3 + s_1 = 8$$
$$3x_1 + 4x_2 + x_3 + s_2 = 7$$
$$x_1, x_2, x_3 \ge 0,\ s_1, s_2 \text{ are slacks and } \ge 0$$

Then, $$A_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},\ A_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, A_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ A_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ A_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ b = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

Let us consider the index of the variables x_1 be 1, x_2 be 2, x_3 be 3, s_1 be 4, s_2 be 5.

Iteration 1.

$$x_B = (s_1, s_2),\ B = [A_4, A_5] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,\ B^{-1} = I.$$

$$c_B^T = (0, 0),\ \bar{b} = B^{-1}.b = b,\ J = (1, 2, 3).$$

$$\pi = c_B^T.B^{-1} = (0, 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (0, 0) = (\pi_1, \pi_2).$$

Net evaluations :

$\bar{c}_1 = \pi A_1 - c_1 = -5 \leftarrow$ negative most and entering variable is x_1

$\bar{c}_2 = \pi A_2 - c_2 = -2$

$\bar{c}_3 = \pi A_3 - c_3 = -3.$

Key column : $$B^{-1}.A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Table 1

x_B	B^{-1}	$\bar{b}$	*Entering variable*	*Key column*

s_1	1 0	8		1
s_2	0 1	7	x_1	3

This indicates the departing variable as s_2.

NOTES

Iteration 2.

$$x_B = (s_1, x_1),\ B = [A_4, A_1] = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix},\ B^{-1} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix}.$$

$$\bar{b} = B^{-1}.b = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 17/3 \\ 7/3 \end{bmatrix},\ J = (2, 3, 5).$$

$$\pi = c_B^T.B^{-1} = [0, 5] \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} = \left(0, \frac{5}{3}\right).$$

Net evaluations :

$$\bar{c}_2 = \pi A_2 - c_2 = \left(0, \frac{5}{3}\right) \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 2 = \frac{14}{3}.$$

$$\bar{c}_3 = \pi A_3 - c_3 = \left(0, \frac{5}{3}\right) \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 3 = -\frac{4}{3} \leftarrow \text{Entering variable } x_3$$

$$\bar{c}_5 = \pi A_5 - c_5 = \left(0, \frac{5}{3}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 = \frac{5}{3}$$

Key column : $B^{-1}.A_3 = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 1/3 \end{pmatrix}.$

Table 2

x_B	B^{-1}	$\bar{b}$	*Entering variable*	*Key column*
s_1	1 −1/3	17/3	x_3	5/3
x_2	0 1/3	7/3		1/3

(This indicates the departing variable as s_1).

Iteration 3.

$$x_B = (x_3, x_1),\ B = [A_3, A_1] = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ B^{-1} = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix},$$

J = 2, 4, 5

$$\bar{b} = B^{-1}.b = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 17/5 \\ 6/5 \end{pmatrix}$$

$$\pi = c_B^T.B^{-1} = (3, 5) \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix} = \left(\frac{4}{5}, \frac{7}{5}\right).$$

Net evaluations :

$$\bar{c}_2 = \pi A_2 - c_2 = \left(\frac{4}{5}, \frac{7}{5}\right)\begin{pmatrix}2\\4\end{pmatrix} - 2 = \frac{26}{5}.$$

$$\bar{c}_4 = \pi A_4 - c_4 = \left(\frac{4}{5}, \frac{7}{5}\right)\begin{pmatrix}1\\0\end{pmatrix} - 0 = \frac{4}{5}$$

$$\bar{c}_5 = \pi A_5 - c_5 = \left(\frac{4}{5}, \frac{7}{5}\right)\begin{pmatrix}0\\1\end{pmatrix} - 0 = \frac{7}{5}$$

As all $\bar{c}_j > 0 \Rightarrow$ the current $\bar{b}$ is optimal.

$\therefore\ x_1^* = \frac{6}{5},\ x_2^* = 0,\ x_3^* = \frac{17}{5}$ and $z^* = \frac{81}{5}.$

Example 14. *Solve by revised simplex method.*

$$\text{Minimize } z = 12x_1 + 20x_2$$
$$\text{S/t, } 6x_1 + 8x_2 \geq 100$$
$$7x_1 + 12x_2 \geq 120$$
$$x_1, x_2 \geq 0.$$

Solution. Standard form :

$$\text{Min. } z = -\text{Max. } (-z = -12x_1 - 20x_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$
$$\text{S/t, } 6x_1 + 8x_2 - s_1 + a_1 = 100$$
$$7x_1 + 12x_2 - s_2 + a_2 = 120$$

$x_1, x_2 \geq 0$, s_1, s_2 surplus and ≥ 0, a_1, a_2 artificial and ≥ 0.

Let $A_1 = \begin{bmatrix}6\\7\end{bmatrix}$, $A_2 = \begin{bmatrix}8\\12\end{bmatrix}$, $A_3 = \begin{bmatrix}-1\\0\end{bmatrix}$, $A_4 = \begin{bmatrix}0\\-1\end{bmatrix}$, $A_5 = \begin{bmatrix}1\\0\end{bmatrix}$, $A_6 = \begin{bmatrix}0\\1\end{bmatrix}$, $b = \begin{bmatrix}100\\120\end{bmatrix}$

Let the index of the variables $x_1, x_2, s_1, s_2, a_1, a_2$ be 1, 2, 3, 4, 5 and 6 respectively.

Iteration 1.

$$x_B = (a_1, a_2),\ B = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = B^{-1},\ \bar{b} = b,\ c_B^T = [-M, -M]$$

$$\pi = c_B^T.B^{-1} = [-M, -M],\ J = (1, 2, 3, 4)$$

Net evaluations :

$\bar{c}_1 = \pi A_1 - c_1 = -13\,M + 12$

$\bar{c}_2 = \pi A_2 - c_2 = -20\,M + 20 \leftarrow$ Most negative and x_2 as entering variable

$\bar{c}_3 = \pi A_3 - c_3 = M$

$\bar{c}_4 = \pi A_4 - c_4 = M$

Key column $= B^{-1}.A_2 = \begin{bmatrix}8\\12\end{bmatrix}.$

NOTES

Table 3

x_B	B^{-1}	$\bar{b}$	*Entering variable*	*Key column*
a_1	1 0	100	x_2	8
a_2	0 1	120		12

(This table indicates a_2 as departing variable).

Iteration 2.

$$x_B = (a_1, x_2),\ B = [A_3, A_2] = \begin{bmatrix} 1 & 8 \\ 0 & 12 \end{bmatrix},\ B^{-1} = \begin{pmatrix} 1 & -2/3 \\ 0 & 1/12 \end{pmatrix}$$

$$J = (1, 3, 4, 6)$$

$$\pi = c_B^T.B^{-1} = (-M, -20)\begin{pmatrix} 1 & -2/3 \\ 0 & 1/12 \end{pmatrix} = \left(-M, \frac{2}{3}M - \frac{5}{3}\right)$$

$$\bar{b} = B^{-1}.b = \begin{pmatrix} 1 & -2/3 \\ 0 & 1/12 \end{pmatrix}\begin{pmatrix} 100 \\ 120 \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \end{pmatrix}$$

Net evaluations :

$$\bar{c}_1 = \pi A_1 - c_1 = -\frac{4}{3}M + \frac{1}{3}$$

$$\bar{c}_3 = \pi A_3 - c_3 = M$$

$$\bar{c}_4 = \pi A_4 - c_4 = -\frac{2}{3}M + \frac{5}{3}$$

$$\bar{c}_6 = \pi A_6 - c_6 = \frac{5}{3}M - \frac{5}{3}$$

$$\text{Key column} = B^{-1}.A_1 = \begin{bmatrix} 4/3 \\ 7/12 \end{bmatrix}$$

Table 4

x_B	B^{-1}	$\bar{b}$	*Entering variable*	*Key column*
a_1	1 −2/3	20	x_1	4/3
x_2	0 1/12	10		7/12

(This table indicates a_1 as departing variable).

Iteration 3.

$$x_B = (x_1, x_2),\ B = (A_1, A_2) = \begin{pmatrix} 6 & 8 \\ 7 & 12 \end{pmatrix},\ B^{-1} = \begin{pmatrix} 3/4 & -1/2 \\ -7/16 & 3/8 \end{pmatrix}$$

$$J = (3, 4, 5, 6)$$

NOTES

$$\pi = c_B^T.B^{-1} = (-12, -20)\begin{pmatrix} 3/4 & -1/2 \\ -7/16 & 3/8 \end{pmatrix} = \left(-\frac{1}{4}, -\frac{3}{2}\right)$$

$$\bar{b} = B^{-1}.b = \begin{pmatrix} 3/4 & -1/2 \\ -7/16 & 3/8 \end{pmatrix}\begin{pmatrix} 100 \\ 120 \end{pmatrix} = \begin{pmatrix} 15 \\ 5/4 \end{pmatrix}$$

Net evaluations :

$$\bar{c}_3 = \pi A_3 - c_3 = \frac{1}{4}$$

$$\bar{c}_4 = \pi A_4 - c_4 = \frac{3}{2}$$

$$\bar{c}_5 = \pi A_5 - c_5 = M - \frac{1}{4}$$

$$\bar{c}_6 = \pi A_6 - c_6 = M - \frac{3}{2}$$

Since all $\bar{c}_j > 0 \Rightarrow$ the current $\bar{b}$ is optimal.

$\therefore\ x_1^* = 15;\ x_2^* = \frac{5}{4}$ and $z^* = 205$.

PROBLEMS

Using revised simplex method solve the following LPP :

1. Maximize $z = x_1 + x_2 + 3x_3$
   S/t, $3x_1 + 2x_2 + x_3 \le 3,\ 2x_1 + x_2 + 2x_3 \le 2;\ x_1, x_2, x_3 \ge 0$.
2. Maximize $z = 3x_1 + 4x_2$
   S/t, $x_1 - x_2 \ge 0,\ -x_1 + 3x_2 \le 3;\ x_1, x_2 \ge 0$.
3. Minimize $z = x_1 + x_2$
   S/t, $2x_1 + x_2 \ge 4,\ x_1 + 7x_2 \ge 7;\ x_1, x_2 \ge 0$.
4. Minimize $z = 2x_1 - x_2 + 2x_3$
   S/t, $-x_1 + x_2 + x_3 = 4,\ -x_1 + x_2 - x_3 \le 6,$
   $x_1 \le 0,\ x_2 \ge 0,\ x_3$ unrestricted in sign.
5. Maximize $z = -x_1 + 2x_2 + 3x_3$
   S/t, $-2x_1 + x_2 + 3x_3 = 2,\ 2x_1 + 3x_2 + 4x_3 = 1;\ x_1, x_2, x_3 \ge 0$.
6. Maximize $z = 2x_1 + x_2 + 3x_3$
   S/t, $x_1 + x_2 + 2x_3 \le 5,\ 2x_1 + 3x_2 + 4x_3 = 12;\ x_1, x_2, x_3 \ge 0$.
7. Maximize $z = 5x_1 + 2x_2 + 3x_3$
   S/t, $x_1 + 2x_2 + 2x_3 \le 8,\ 3x_1 + 4x_2 + x_3 \le 7;\ x_1, x_2, x_3 \ge 0$.

ANSWERS

1. $x_1 = 0, x_2 = 0, x_3 = 1, z^* = 3$.
2. Unbounded solution.
3. $x_1 = \frac{21}{13},\ x_2 = \frac{10}{13},\ z^* = \frac{31}{13}$.
4. $x_1 = -5, x_2 = 0, x_3 = -1, z^* = -12$ (Iteration 3)
5. Infeasible solution.

6. $x_1 = 3, x_2 = 2, x_3 = 0, z^* = 8.$

7. $x_1 = \frac{6}{5}, x_2 = 0, x_3 = \frac{17}{5}, z^* = \frac{81}{5}$ (Iteration 3)

NOTES

2.8 INTRODUCTION AND FORMULATION

For every LP Problem we can construct another LP problem using the same data. These two problems try to achieve two different objectives within the same data. The original problem is called *Primal* problem and the constructed problem is called *Dual*. This is illustrated through the following example :

A company makes three products X, Y, Z using three raw materials A, B and C. The raw material requirement is given below : (for 1 unit of product).

	X	*Y*	*Z*	*Availability*
A	1	2	1	36 units
B	2	1	4	60 units
C	2	5	1	45 units
Profit	Rs. 40	Rs. 25	Rs. 50	

Let the company decide to produce x_1, x_2 and x_3 units of the products X, Y and Z respectively in order to maximize the profit. We obtain the following LP problems :

$$\text{Maximize profit} = 40x_1 + 25x_2 + 50x_3$$

$$\begin{aligned} \text{Subject to, } x_1 + 2x_2 + x_3 &\le 36, \\ 2x_1 + x_2 + 4x_3 &\le 60, \\ 2x_1 + 5x_2 + x_3 &\le 45, \\ x_1, x_2, x_3 &\ge 0. \end{aligned}$$

Adding slack variables s_1, s_2 and s_3 to the constraints, we solve the problem by simplex method. The optimal solution is

$$x_1 = 20, x_2 = 0, \quad x_3 = 5 \text{ and optimal profit} = \text{Rs. } 1050.$$

Suppose the company wishes to sell the three raw materials A, B and C instead of using them for production of the products X, Y and Z. Let the selling prices be Rs. y_1, Rs. y_2 and Rs. y_3 per unit of raw material A, B and C respectively. The cost of the purchaser due to all raw materials is

$$36y_1 + 60y_2 + 45y_3.$$

Then the purchaser forms the following LP problem :

$$\text{Minimize T} = 36y_1 + 60y_2 + 45y_3$$

$$\begin{aligned} \text{Subject to, } y_1 + 2y_2 + 2y_3 &\ge 40, \\ 2y_1 + y_2 + 5y_3 &\ge 25, \\ y_1 + 4y_2 + y_3 &\ge 50, \end{aligned}$$

$$y_1, y_2, y_3 \geq 0.$$

The solution is obtained as :

$$y_1 = 0,\ y_2 = 10,\ y_3 = 10,\ \text{Optimal cost} = \text{Rs. } 1050.$$

In the above, the company's problem is called primal problem and purchaser's problem is called dual problem. Also we can use these two terms interchangeably. In the primal problem, the company achieve a profit of Rs. 1050 by producing 20 units of X and 5 units of Z. Instead, if the company sells the raw material B with Rs. 10 per unit and C with Rs. 10 per unit then also the company achieve a sale of Rs. 1050.

(*a*) Formulation

In the above, both the problems are called **symmetric** problem since the objective function is maximization (minimization), all the constraints are '≤' type (≥ type) and non-negative decision variables.

The decision variables in the primal are called primal variables and the decision variables in the dual are called dual variables.

Let us consider the following table for formulation of the dual.

Primal (*Maximization*)	*Dual* (*Minimization*)
Right hand side constants	Cost vector
Cost vector	Right hand side constants.
Coefficient matrix	Transpose of coefficient matrix
'≤'	'≥'
Max. $z = cx$ S/t, $Ax \leq b$ $x \geq 0$	Min. $T = b^T y$ S/t $A^T y \geq c^T$ $y \geq 0$

(*b*) Asymmetric Primal-Dual Problems

Primal (*Maximization*)	*Dual* (*Minimization*)
a. Coefficient matrix	A^T
b. Right hand side constants	Cost vector
c. Cost vector	Right hand side constants
i-th constraint	*i*-th dual variable
≤ type	$y_i \geq 0$
≥ type	$y_i \leq 0$

NOTES

= type	y_i unrestricted in sign
j-th primal variable	j-th dual constraint
x_j unrestricted in sign	= type
$x_j \leq 0$	$\leq$ type
$x_j \geq 0$	$\geq$ type

Also in (*a*) and (*b*),

No. of primal constraints = No. of dual variables.

No. of primal variables = No. of dual constraints.

Note. The dual of the dual is the primal.

Example 15. *Obtain the dual of*

$$\text{Minimize } z = 8x_1 + 3x_2 + 15x_3$$
$$\text{Subject to, } 2x_1 + 4x_2 + 3x_3 \geq 28,$$
$$3x_1 + 5x_2 + 6x_3 \geq 30,$$
$$x_1, x_2, x_3 \geq 0.$$

Solution. Let y_1 and y_2 be the variables corresponding to the first and second constraints respectively. Objective function, maximize $T = 28y_1 + 30y_2$. There will be three dual constraints due to three primal variables. In primal

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 5 & 6 \end{bmatrix}, c = [8, 3, 15]$$

∴ In dual
$$A^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq c^T$$

⇒
$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq \begin{pmatrix} 8 \\ 3 \\ 15 \end{pmatrix}$$

⇒
$$2y_1 + 3y_2 \leq 8 \text{ (due to } x_1)$$
$$4y_1 + 5y_2 \leq 3 \text{ (due to } x_2)$$
$$3y_1 + 6y_2 \leq 15 \text{ (due to } x_3)$$

Hence the dual problem is

$$\text{Maximize } T = 28y_1 + 30y_2$$
$$\text{Subject to, } 2y_1 + 3y_2 \leq 8$$
$$4y_1 + 5y_2 \leq 3$$
$$3y_1 + 6y_2 \leq 15$$
$$y_1, y_2 \geq 0.$$

Example 16. *Find the dual of*

$$\text{Maximize } z = 2x_1 + x_2 + 5x_3$$
$$\text{Subject to, } x_1 + x_2 + x_3 = 10,$$

$$4x_1 - x_2 + 2x_3 \geq 12,$$
$$3x_1 + 2x_2 - 3x_3 \leq 6,$$
$$x_1, x_2, x_3 \geq 0.$$

Solution. First we have to express all the constraints in '≤' form due to maximization problem :

NOTES

The first constraint : $x_1 + x_2 + x_3 \leq 10$

and $x_1 + x_2 + x_3 \geq 10$

$\Rightarrow$ $-x_1 - x_2 - x_3 \leq -10$

The second constraint : $-4x_1 + x_2 - 2x_3 \leq -12$

Let y_1, y_2, y_3 and y_4 be four dual variables corresponding to the newly converted constraints respectively.

Objective function : Minimize $T = 10y_1 - 10y_2 - 12y_3 + 6y_4$

Again,
$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ -4 & 1 & -2 \\ 3 & 2 & -3 \end{bmatrix}, c = [2, 1, 5]$$

$\therefore$ Constraints in dual :
$$A^T \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \geq c^T.$$

Thus the dual problem is

$$\text{Minimize } T = 10y_1 - 10y_2 - 12y_3 + 6y_4.$$

Subject to, $y_1 - y_2 - 4y_3 + 3y_4 \geq 2$ (due to x_1)

$y_1 - y_2 + y_3 + 2y_4 \geq 1$ (due to x_2)

$y_1 - y_2 - 2y_3 - 3y_4 \geq 5$ (due to x_3)

$y_1, y_2, y_3, y_4 \geq 0.$

Set $w_1 = y_1 - y_2$, $w_2 = -y_3$, $w_3 = y_4 \Rightarrow w_1$ unrestricted in sign, $w_2 \leq 0$, $w_3 \geq 0$. This conversion leads to

$$\text{Minimize } T = 10w_1 + 12w_2 + 6w_3$$

Subject to, $w_1 + 4w_2 + 3w_3 \geq 2,$

$w_1 - w_2 + 2w_3 \geq 1,$

$w_1 + 2w_2 - 3w_3 \geq 5,$

w_1 unrestricted in sign, $w_2 \leq 0$, $w_3 \geq 0.$

Example 17. *Find the dual of*

Maximize $z = 5x_1 + 4x_2 - 3x_3$

Subject to, $2x_1 + 4x_2 - x_3 \leq 14,$

$x_1 - 2x_2 + x_3 = 10,$

$x_1 \geq 0$, x_2 *unrestricted in sign,* $x_3 \leq 0.$

NOTES

Solution. First we have introduce non-negative variables.

$\therefore$ Set $\quad x_2 = x'_2 - x''_2;\ x'_2, x''_2 \geq 0$ and $x_3 = -x'_3,\ x'_3 \geq 0.$

The given problem reduces to

$$\text{Maximize } z = 5x_1 + 4x'_2 - 4x''_2 + 3x'_3$$
$$\text{Subject to, } 2x_1 + 4x'_2 - 4x''_2 + x'_3 \leq 14$$
$$x_1 - 2x'_2 + 2x''_2 - x'_3 \leq 10$$
$$x_1, x'_2, x''_2, x'_3 \geq 0$$

The second constraint is expressed as

$$x_1 - 2x'_2 + 2x''_2 - x'_3 \leq 10$$

and $$-x_1 + 2x'_2 - 2x''_2 + x'_3 \leq -10$$

Let y_1, y_2, y_3 be the three dual variables corresponding to the three constraints respectively. Then the symmetric dual is

$$\text{Minimize T} = 14y_1 + 10y_2 - 10y_3$$
$$\text{Subject to, } 2y_1 + y_2 - y_3 \geq 5 \text{ (due to } x_1)$$
$$4y_1 - 2y_2 + 2y_3 \geq 4 \text{ (due to } x'_1)$$
$$-4y_1 + 2y_2 - 2y_3 \geq -4 \text{ (due to } x''_2)$$
$$y_1 - y_2 + y_3 \geq -3 \text{ (due to } x'_3)$$
$$y_1, y_2, y_3 \geq 0$$

Set $\quad w_1 = y_1,\ w_2 = y_2 - y_3 \Rightarrow w_1 \geq 0$ and w_2 unrestricted.

Also the second and third constraint reduces to

$$4y_1 - 2y_2 + 2y_3 = 4$$

Therefore the dual is

$$\text{Minimize T} = 14w_1 + 10w_2$$
$$\text{Subject to, } 2w_1 + w_2 \geq 5,$$
$$4w_1 - 2w_2 = 4,$$
$$-w_1 + w_2 \leq 3$$

$w_1 \geq 0$ and w_2 unrestricted in sign.

2.9 DUALITY THEOREMS

Theorem 1. (Weak Duality)

Consider the symmetric primal (max. type) and Dual (min. type). The value of the objective function of the (dual) minimum problem for any feasible solution is always greater than or equal to that of the maximum problem (primal) for any feasible solution.

Proof. Let x^0 be a feasible solution to the primal.

Then $Ax^0 \leq b$, $x^0 \geq 0$ and $z = cx^0$.

NOTES

Let y^0 be a feasible solution to the dual.

Then $A^T y^0 \geq c^T$, $y^0 \geq 0$ and $T = b^T y^0$.

Taking transpose on both sides, we have

$$c \leq (y^0)^T.A$$

$$\Rightarrow \quad cx^0 \leq (y^0)^T.Ax^0$$

$$\Rightarrow \quad cx^0 \leq (y^0)^T.b$$

$$\Rightarrow \quad cx^0 \leq b^T.y^0 \quad (\because (y^0)^T b = b^T y^0)$$

Hence proved.

Theorem 2.

Let x^0 and y^0 be the feasible solutions to the corresponding primal and dual problem such that $cx^0 = b^T y^0$, then x^0 and y^0 are optimal solutions to the respective problems.

Proof. Let x^* be any other feasible solution to the primal problem.

Then by Theorem 1, $cx^* \leq b^T y^0$

$$\Rightarrow \quad cx^* \leq cx^0$$

Hence x^0 is an optimal solution to the primal problem because the primal problem is a maximization problem.

Similarly, we can prove that y^0 is an optimal solution for the dual problem.

Theorem 3. (Fundamental Theorem of Duality)

If both the primal and dual problems are feasible and both have optimal solutions then the optimal values of the objective functions of both the problems are equal.

Theorem 4. (Complementary Slackness Conditions (CSC))

Let x^0 and y^0 be the feasible solutions for the primal and dual problems respectively. Let u be the slack variables of the primal and v be the surplus variables of the dual. Then x^0 and y^0 are optimal solutions to the respective primal and dual problems respectively iff

$$(x^0)^T.v = 0 \text{ and } (y^0)^T.u = 0$$

Results on Feasibility

		Primal (Max. z)	
		Feasible Solution	*Infeasible solution*
Dual (Min. T)	Feasible solution	Max. z = Min T	Dual unbounded (Min. T $\to -\infty$)
	Infeasible solution	Primal unbounded (Max. $z \to \infty$)	May occur.

NOTES

Let the primal as : Minimize $z = -x_1 - x_2$

$$\text{S/t, } x_1 - x_2 = 3,$$
$$x_1 - x_2 = -3,$$
$$x_1 \geq 0, x_2 \geq 0.$$

Then the dual can be written as

$$\text{Maximize T} = 3y_1 - 3y_2$$
$$\text{S/t, } y_1 + y_2 \leq -1,$$
$$-y_1 - y_2 \leq -1,$$
$$y_1, y_2 \text{ unrestricted in sign.}$$

Here both the primal and the dual are inconsistent and hence no feasible solutions.

Example 18. *Using the C.S.C. find the optimal solution of the following primal.*

$$\text{Minimize } z = 2x_1 + 3x_2 + 5x_3 + 3x_4 + 2x_5$$
$$\text{S/t, } x_1 + x_2 + 2x_3 + 3x_4 + x_5 \geq 4,$$
$$2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geq 3,$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

Solution. The dual is

$$\text{Maximize T} = 4y_1 + 3y_2$$
$$\text{S/t, } y_1 + 2y_2 \leq 2$$
$$y_1 - 2y_2 \leq 3$$
$$2y_1 + 3y_2 \leq 5.$$
$$3y_1 + y_2 \leq 3$$
$$y_1 + y_2 \leq 2$$
$$y_1, y_2 \geq 0.$$

The solution of this dual, by graphically is $y_1^* = \frac{4}{5}$, $y_2^* = \frac{3}{5}$, $T^* = 5$. Let u_1, u_2, u_3, u_4 and u_5 be the slack variables of the dual and v_1, v_2 be the surplus variables of the primal. Then by C.S.C., we have

$$x_1u_1 = 0, x_2u_2 = 0, x_3u_3 = 0,$$
$$x_4u_4 = 0, x_5u_5 = 0, y_1v_1 = 0, y_2v_2 = 0.$$

Since y_1^* and y_2^* are non-zero $\Rightarrow$ $v_1 = v_2 = 0.$

It is also seen that at optimality, the two constraints $y_1 + 2y_2 \leq 2$ and $3y_1 + y_2 \leq 3$ are satisfying in equality sense which mean $u_1^* = 0$ and $u_4^* = 0$.

For the remaining constraints, u_2^*, u_3^* and u_5^* are non-zero *i.e.*, by C.S.C., $x_2^* = 0$, $x_3^* = 0$ and $x_5^* = 0$.

Then the primal constraints reduces to

$$x_1^* + 3x_4^* = 4$$
$$2x_1^* + x_4^* = 3.$$

Solving we get $x_1^* = 1$ and $x_4^* = 1.$

Hence the optimal solution of the primal is

$$x_1^* = 1, x_2^* = 0, x_3^* = 0, x_4^* = 1, x_5^* = 0 \text{ and } z^* = 5.$$

2.10 DUALITY OF SIMPLEX METHOD

NOTES

The fundamental theorem of duality helps to obtain the optimal solution of the dual from optimal table of the primal and vice-versa. Using C.S.C., the correspondence between the primal (dual) variables and slack and/or surplus variables of the dual (primal) to be identified. Then the optimal solution of the dual (primal) can be read off from the net evaluation row of the primal (dual) of the simplex table.

For example, if the primal variable corresponds to a slack variable of the dual, then the net evaluation of the slack variable in the optimal table will give the optimal solution of the primal variable.

Example 19. *Using the principle of duality solve the following problem :*

$$\text{Minimize } z = 4x_1 + 14x_2 + 3x_3$$

$$\text{S/t, } -x_1 + 3x_2 + x_3 \geq 3,$$

$$2x_1 + 2x_2 - x_3 \geq 2,$$

$$x_1, x_2, x_3 \geq 0.$$

Solution. The dual problem is

$$\text{Maximize T} = 3y_1 + 2y_2$$

$$\text{S/t, } -y_1 + 2y_2 \leq 4$$

$$3y_1 + 2y_2 \leq 14$$

$$y_1 - y_2 \leq 3$$

$$y_1, y_2 \geq 0$$

Standard form :

$$\text{Maximize T} = 3y_1 + 2y_2 + 0.u_1 + 0.u_2 + 0.u_3$$

$$\text{S/t, } -y_1 + 2y_2 + u_1 = 4$$

$$3y_1 + 2y_2 + u_2 = 14$$

$$y_1 - y_2 + u_3 = 3$$

$y_1, y_2 \geq 0$, u_1, u_2, u_3 are slacks and ≥ 0.

Let the surplus variables of the dual v_1 and v_2.

Then by C.S.C., $y_1v_1 = 0, y_2v_2 = 0,$

$$x_1u_1 = 0, x_2u_2 = 0, x_3u_3 = 0.$$

Let us solve the dual by simplex method and the optimal table is given below (Iteration 3) :

		c_j	3	2	0	0	0
c_B	x_B	Soln.	y_1	y_2	u_1	u_2	u_3
0	u_1	6	0	0	1	$-\frac{1}{5}$	$\frac{3}{5}$
2	y_2	1	0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$

3	y_1	4	1	0	0	$\frac{1}{5}$	$\frac{2}{5}$
$z_j - c_j$			0	0	0	1	0

NOTES

The optimal solution of the dual is $y_1^* = 4, y_2^* = 1, T^* = 14$.

The optimal solution of the primal can be read off from the $(z_j - c_j)$-row. Since x_1, x_2, x_3 corresponds to u_1, u_2, u_3 respectively, then

$$x_1^* = 0, x_2^* = 1, x_3^* = 0, \text{and } z^* = 14.$$

2.11 THE DUAL SIMPLEX METHOD

Step 1. Convert the minimization LP problem into an symmetric maximization LP problem (*i.e.*, all constraints are $\leq$ type) if it is in the minimization form.

Step 2. Introduce the slack variables and obtain the first iteration dual simplex table.

		c_j	
c_B	x_B	Soln.	(x)
	$[x_{B_i}]$		
	$z_j - c_j$		
	Max. ratio		

Step 3.(*a*) If all $z_j - c_j$ and x_{B_i} are non-negative, then an optimal basic feasible solution has been attained.

(*b*) If all $z_j - c_j \geq 0$ and at least one of x_{B_i} is negative then go to step 4.

(*c*) If at least one $(z_j - c_j)$ is negative, the method is not applicable.

Step 4. Select the most negative of x_{B_i}'s and that basic variable will leave the basis and the corresponding row is called 'key-row'.

Step 5.(*a*) If all the elements of the key row is positive, then the problem is infeasible.

(*b*) If at least one element is negative then calculate the maximum ratios as follows :

$$\text{Max}\left\{\frac{(z_j - c_j)\text{ value}}{\text{Negative element of the key row}}\right\}$$

The maximum ratio column is called 'key column' and the intersection element of key row and key column is called 'key element'.

Step 6. Obtain the next table which is the same procedure as of simplex method.

Step 7. Go to step 3.

Note. 1. Difference between simplex method and dual-simplex method : In simplex method, we move from a feasible non-optimal solution to feasible optimal solution. Whereas in dual simplex method, we move from an infeasible optimal solution to feasible optimal solution.
2. The term 'dual' is used in dual simplex method because the rules for leaving and entering variables are derived from the dual problem but are used in the primal problem.

Example 20. *Using dual simplex method solve the following LP problem.*

$$\text{Minimize } z = 4x_1 + 2x_2$$
$$\text{S/t, } x_1 + 2x_2 \geq 2,\ 3x_1 + x_2 \geq 3,$$
$$4x_1 + 3x_2 \geq 6,\ x_1,\ x_2 \geq 0.$$

Solution. Min. $z = -$ Max. $(-z) = -$ Max. $(-z = -4x_1 - 2x_2)$.

Multiply -1 to all the $\geq$ constraints to make $\leq$ type.

Then the standard form is obtained as follows :

$$\text{Max } -z = -4x_1 - 2x_2 + 0.s_1 + 0.s_2 + 0.s_3$$
$$\text{S/t, } -x_1 - 2x_2 + s_1 = -2$$
$$-3x_1 - x_2 + s_2 = -3$$
$$-4x_1 - 3x_2 + s_3 = -6$$

$x_1, x_2 \geq 0$, s_1, s_2, s_3 are slacks and ≥ 0.

Iteration 1.

		c_j	-4	-2	0	0	0	
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	-2	-1	-2	1	0	0	
0	s_2	-3	-3	-1	0	1	0	
0	s_3	-6	-4	-3	0	0	1	→ Key row
$z_j - c_j$			4	2	0	0	0	
Max. ratio			$\frac{4}{-4}$	$\frac{2}{-3}$ ↑ Key column				

Iteration 2.

		c_j	-4	-2	0	0	0	
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	2	5/3	0	1	0	$-2/3$	
0	s_2	-1	$-5/3$	0	0	1	$-1/3$	→ Key row
-2	x_2	2	4/3	1	0	0	$-1/3$	
$z_j - c_j$			4/3	0	0	0	2/3	
Max. ratio			$-\frac{4}{5}$ ↑ Key column	–	–	–	-2	

NOTES

Iteration 3.

		c_j	-4	-2	0	0	0
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3
0	s_1	1	0	0	1	1	-1
-4	x_1	3/5	1	0	0	$-3/5$	1/5
-2	x_2	6/5	0	1	0	4/5	$-3/5$
$z_j - c_j$			0	0	0	4/5	2/5

Hence optimal feasible solution is $x_1^* = \frac{3}{5}$, $x_2^* = \frac{6}{5}$ and $z^* = \frac{24}{5}$.

Example 21. *Use dual simplex method to solve the following LP problem.*

$$\text{Maximize } z = -4x_1 - 3x_2$$

$$\text{Subject to, } x_1 + x_2 \le 1,\ x_2 \ge 1,\ -x_1 + 2x_2 \le 1,\ x_1, x_2 \ge 0.$$

Solution. The constraint $x_2 \ge 1$ is rewritten as $-x_2 \le -1$. Adding slack variables, the standard form is

$$\text{Maximize } z = -4x_1 - 3x_2 + 0.s_1 + 0.s_2 + 0.s_3$$

$$\text{S/t, } x_1 + x_2 + s_1 = 1,\ -x_2 + s_2 = -1,\ -x_1 + 2x_2 + s_3 = 1$$

$x_1, x_2 \ge 0$, s_1, s_2, s_3 are slacks and ≥ 0.

Iteration 1.

		c_j	-4	-3	0	0	0	
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	1	1	1	1	0	0	
0	s_2	-1	0	$\boxed{-1}$	0	1	0	→ Key row
0	s_3	1	-1	2	0	0	1	
$z_j - c_j$			4	3	0	0	0	
Max. ratio			–	-3	–	–	–	
				↑ Key column				

NOTES

Iteration 2.

		c_j	-4	-3	0	0	0	
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	0	1	0	1	1	0	
-3	x_2	1	0	1	0	-1	0	
0	s_3	-1	$\boxed{-1}$	0	0	2	1	→ Key row
$z_j - c_j$			4	0	0	3	0	
Max. ratio			-4	–	–	–	–	
			↑ Key column					

Iteration 3.

		c_j	-4	-3	0	0	0	
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	-1	0	0	1	3	1	→ Key row
-3	x_2	1	0	1	0	-1	0	
-4	x_1	1	1	0	0	-2	-1	
$z_j - c_j$			0	0	0	11	4	
Max. ratio			–	–	–	–	–	

Since all the elements in the key row are positive, the given problem is infeasible.

2.12 ECONOMIC INTERPRETATION OF DUAL VARIABLE

Let x^* and y^* be the optimal solutions of the respective primal and dual problems respectively and objective function values are same *i.e.*, $z^* = T^*$. In the primal, the small change in the resources (*i.e.*, right hand side constants) gives the small change *in* z^*. *Consequently, the* y^* *value for each primal constraint gives the net* change in the optimal value of the objective function for unit increase in right hand side constants. Hence the dual variables are called '**shadow prices**'.

SUMMARY

- "Operational Research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence.

NOTES

- In decision-making all the decisions are taken through some variables which are known as decision variables. In engineering design, these variables are known as design vectors.
- A solution which satisfies all the constraints in LPP is called feasible solution.
- A solution which is basic as well as feasible is called basic feasible solution.
- If a basic variable takes the value zero in a BFS, then the solution is said to be degenerate.
- The BFS which optimizes the objective function is called optimal BFS.

REVIEW QUESTIONS

1. Obtain the dual of the following LP problems :

(a)
$$\text{Maximize } z = 4x_1 + 2x_2 + x_3 + 6x_4$$
$$\text{Subject to, } 6x_1 - 3x_2 + x_3 + 5x_4 \leq 15,$$
$$x_1 - x_2 + 6x_3 + 2x_4 \geq 8,$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

(b)
$$\text{Maximize } z = 2x_1 + x_2$$
$$\text{Subject to, } 2x_1 + 3x_2 \geq 4,$$
$$3x_1 + 4x_2 \leq 10,$$
$$x_1 + 5x_2 = 9,$$
$$x_1 \geq 0, x_2 \geq 0.$$

(c)
$$\text{Minimize } z = 3x_1 + 4x_2 - x_3$$
$$\text{Subject to, } 2x_1 + 3x_2 + 5x_3 \geq 10,$$
$$3x_1 + 10x_3 \leq 14,$$
$$x_1 \geq 0, x_2 \leq 0, x_3 \geq 0.$$

(d)
$$\text{Minimize } z = 10x_1 + 15x_2$$
$$\text{Subject to, } 3x_1 + 2x_2 = 15,$$
$$5x_1 + 4x_2 = 20,$$
$$x_1, x_2 \text{ unrestricted in sign.}$$

(e)
$$\text{Maximize } z = x_1 - 2x_2 + 3x_3$$
$$\text{Subject to, } 2x_1 + 5x_3 \leq 16,$$
$$5x_2 + 4x_3 \geq 8,$$
$$x_1 + x_2 + x_3 = 10,$$
$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ unrestricted in sign.}$$

2. Use principle of duality to solve the following LP problems :

(a)
$$\text{Minimize } z = 4x_1 + 3x_2$$
$$\text{S/t, } 2x_1 + x_2 \geq 40, x_1 + 2x_2 \geq 50, x_1 + x_2 \geq 35$$
$$x_1, x_2 \geq 0$$

(b)
$$\text{Maximize } z = 2x_1 + x_2$$
$$\text{S/t, } x_1 + 2x_2 \leq 10, x_1 + x_2 \leq 6, x_1 - x_2 \leq 2, x_1 - 2x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

(c)
$$\text{Minimize } z = 6x_1 + x_2$$
$$\text{S/t, } 2x_1 + x_2 \geq 3, x_1 - x_2 \geq 0, x_1, x_2 \geq 0$$

(*d*) Minimize $z = 30x_1 + 30x_2 + 10x_3$

S/t, $2x_1 + x_2 + x_3 \geq 6,\ x_1 + x_2 + 2x_3 \leq 8,\ x_1, x_2, x_3 \geq 0$

(*e*) Maximize $z = 5x_1 + 2x_2$

S/t, $x_1 - x_2 \leq 1,\ x_1 + x_2 \geq 4,\ x_1 - 3x_2 \leq 3,\ x_1, x_2 \geq 0$

3. Using the complementary slackness condition solve the following LP problem :

$$\text{Maximize } z = 2x_1 + 3x_2 + 6x_3$$

S/t, $x_1 + 3x_2 + 4x_3 \leq 4,\ 2x_1 + x_2 + 3x_3 \leq 2,\ x_1, x_2, x_3 \geq 0.$

4. With the help of the following example, verify that the dual of the dual is the primal.

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

$$\text{S/t, } 4x_1 + 3x_2 - x_3 \leq 20,\ 3x_1 + 2x_2 + 5x_3 = 18,$$

$$0 \leq x_1 \leq 4,\ x_2 \geq 0,\ x_3 \leq 0.$$

5. Verify the fundamental theorem of duality using the following LP problems :

(*a*) Maximize $z = 2x_1 + 10x_2$

S/t, $2x_1 + 5x_2 \leq 16,\ 6x_1 \leq 30,\ x_1, x_2 \geq 0.$

(*b*) Minimize $z = 2x_1 - x_2$

S/t, $x_1 + x_2 \leq 5,\ x_1 + 2x_2 \geq 8,\ x_1, x_2 \geq 0.$

6. Use dual-simplex method to solve the following LP problems :

(*a*) Minimize $z = x_1 + 3x_2$

S/t, $2x_1 + x_2 \geq 4,\ 3x_1 + 2x_2 \geq 5,\ x_1, x_2 \geq 0.$

(*b*) Minimize $z = 2x_1 + x_2$

S/t, $x_1 + x_2 \geq 2,\ 3x_1 + 2x_2 \geq 4,\ x_1, x_2 \geq 0.$

(*c*) Minimize $z = 2x_1 + 3x_2 + 10x_3$

$$\text{S/t, } 2x_1 - 5x_2 + 4x_3 \geq 30,$$
$$3x_1 + 2x_2 - 5x_3 \geq 25,$$
$$x_1 + 3x_2 + x_3 \leq 30,$$
$$x_1, x_2, x_3 \geq 0.$$

(*d*) Maximize $z = -2x_1 - x_2 - 3x_3$

S/t, $-3x_1 + x_2 - 2x_3 - x_4 = 1,\ x_1 - 2x_2 + x_3 - x_5 = 2,\ x_i \geq 0\ \forall i$

(*e*) Minimize $z = 2x_1 + 3x_2 + 4x_3$

S/t, $3x_1 + 10x_2 + 5x_3 \geq 3,\ 3x_1 - 10x_2 + 9x_3 \leq 30,$

$x_1 + 2x_2 + x_3 \geq 4,\ x_1, x_2, x_3 \geq 0.$

(*f*) Minimize $z = 6x_1 + 2x_2 + 5x_3 + 3x_4$

S/t, $3x_1 + 2x_2 - 3x_3 + 5x_4 \geq 10,\ 4x_2 + 3x_3 - 5x_4 \geq 12,$

$5x_1 - 4x_2 + x_3 + x_4 \geq 10,\ x_1, x_2, x_3, x_4 \geq 0.$

(*g*) Maximize $z = -x_1 - 2x_2 - 3x_3$

S/t, $2x_1 - x_2 - x_3 \geq 4,\ x_1 - x_2 + 2x_3 \leq 8,\ x_1, x_2, x_3 \geq 0.$

7. One unit of product A requires 3 units of raw material and 2 hours of labour and contributes the profit of Rs. 7. One unit of product B requires one unit of raw material and one hour of labour and contributes the profit of Rs. 5. There are 48 units of raw material and 40 hours of labour available. The objective is to maximize the profit. Calculate the shadow prices of the raw material and labour.

NOTES

ANSWERS

1. (*a*)
$$\text{Minimize } T = 15y_1 - 8y_2$$
$$\text{Subject to, } 6y_1 - y_2 \geq 4$$
$$-3y_1 + y_2 \geq 2$$
$$y_1 - 6y_2 \geq 1$$
$$5y_1 - 2y_2 \geq 6$$
$$y_1, y_2 \geq 0, \text{ and then set } w_1 = y_1 \text{ and } w_2 = -y_2.$$
(*b*)
$$\text{Minimize } T = 4w_1 + 10w_2 + 9w_3$$
$$\text{S/t, } 2w_1 + 3w_2 + w_3 \geq 2$$
$$3w_1 + 4w_2 + 5w_3 \geq 1$$
$$w_1 \leq 0,\ w_2 \geq 0,\ w_3 \text{ unrestricted.}$$
(*c*)
$$\text{Maximize } T = -10w_1 - 14w_2$$
$$\text{S/t, } -2w_1 - 3w_2 \leq 3$$
$$-w_1 \leq 4$$
$$5w_1 + 10w_2 \geq 1$$
$$w_1 \leq 0,\ w_2 \geq 0.$$
(*d*)
$$\text{Maximize } T = 15w_1 + 20w_2$$
$$\text{S/t, } 3w_1 + 5w_2 = 10,\ 2w_1 + 4w_2 = 15,$$
$$w_1, w_2 \text{ unrestricted.}$$
(*e*)
$$\text{Minimize } T = 16w_1 + 8w_2 + 10w_3$$
$$\text{S/t, } 2w_1 + w_3 \geq 1,\ 5w_2 + w_3 \leq -2,\ 5w_1 + 4w_2 + w_3 = 3$$
$$w_1 \geq 0,\ w_2 \leq 0,\ w_3' \text{ unrestricted.}$$

2. (*a*) $x_1 = 5, x_2 = 30, z^* = 110.$
   (*b*) $x_1 = 4, x_2 = 2, z^* = 10.$
   (*c*) $x_1 = 1, x_2 = 1, z^* = 7.$
   (*d*) $x_1 = \frac{4}{3}, x_2 = 0,\ x_3 = \frac{10}{3}, z^* = \frac{220}{3}.$
   (*e*) Unbounded solution.

3. $x_1 = 0,\ x_2 = \frac{4}{5},\ x_3 = \frac{2}{5},\ z^* = 4.8.$

5. (*a*) Max. $z = 32 =$ Min. T.
   (*b*) Min. $z = -5 =$ Max. T.

6. (*a*) $x_1 = 2, x_2 = 0, z^* = 2$ (It – 3)
   (*b*) $x_1 = 0, x_2 = 2, z^* = 2$ (It – 2)
   (*c*) $x_1 = 15, x_2 = 0, x_3 = 0, z^* = 30$ (It – 2)
   (*d*) Infeasible solution (It – 3)
   (*e*) $x_1 = 0, x_2 = 2, x_3 = 0, z^* = 6$ (It – 2)
   (*f*) $x_1 = 0,\ x_2 = \frac{11}{3},\ x_3 = 0,\ x_4 = \frac{8}{15},\ z^* = \frac{134}{15}$ (It – 3)
   (*g*) $x_1 = 2, x_2 = 0, x_3 = 0, z^* = -2$ (It – 2)

7. Shadow prices for raw material is zero and for labour is five.

FURTHER READINGS

- **Operations Research:** Col. D.S. Cheema, University Science Press.
- **Introductory Operations Research:** Theory & Applications 3e: Kasana, Springer.
- **Operations Research:** N.P. Agarwal, Indus Valley Publication.
- **Operations Research:** Jaya Banerjee, Shree Niwas.
- **Operations Research for Library and Information Professionals:** Dariush Alimohammadi, Ess Ess Publications.

NOTES

UNIT III ASSIGNMENT AND TRANSPORTATION PROBLEMS

NOTES

★ STRUCTURE ★

3.0 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define mathematical formulation
- know about initial basic feasible solution
- explain UV-method/Modi method
- explain about Hungarian algorithm
- discuss about Max-type assignment problems
- define routing problems.

NOTES

3.1 INTRODUCTION AND MATHEMATICAL FORMULATION

Transportation problem (T.P.) is generally concerned with the distribution of a certain commodity/product from several origins/sources to several destinations with minimum total cost through single mode of transportation. If different modes of transportation considered then the problem is called 'solid T.P'. In this chapter we shall deal with simple T.P.

Suppose there are m factories where a certain product is produced and n markets where it is needed. Let the supply from the factories be $a_1, a_2, ..., a_m$ units and demands at the markets be $b_1, b_2, ..., b_n$ units.

Also consider

c_{ij} = Unit of cost of shipping from factory i to market j.

x_{ij} = Quantity shipped from factory i to market j.

Then the LP formulation can be started as follows :

$$\text{Minimize } z = \text{Total cost of transportation}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij}$$

$$\text{Subject to, } \sum_{j=1}^{n} x_{ij} \le a_i,\ i = 1, 2, ..., m.$$

(Total amount shipped from any factory does not exceed its capacity)

$$\sum_{i=1}^{m} x_{ij} \ge b_j,\ j = 1, 2, ..., n.$$

(Total amount shipped to a market meets the demand of the market)

$x_{ij} \ge 0$ for all i and j.

Here the market demand can be met if

$$\sum_{i=1}^{m} a_i \ge \sum_{j=1}^{n} b_j.$$

If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ *i.e.*, total supply = total demand, the problem is said to be "Balanced T.P." and all the constraints are replaced by equality sign.

$$\text{Minimize } z = \sum\sum c_{ij}\, x_{ij}$$

$$\text{Subject to, } \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, ..., m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, ..., n.$$

$x_{ij} \ge 0$ for all i and j.

(Total $m + n$ constraints and mn variables)

NOTES

The T.P. can be represented by *table form* as given below :

Factories	M_1	M_2	-------	M_n	Supply
F_1	X_{11} C_{11}	X_{12} C_{12}		X_{1n} C_{1n}	a_1
F_2	X_{21} C_{21}	X_{22} C_{22}		X_{2n} C_{2n}	a_2
⋮	-------	-------	-------	-------	⋮
F_m	X_{m1} C_{m1}	X_{m2} C_{m2}		X_{mn} C_{mn}	a_m
Demand	b_1	b_2	-------	b_n	

In the above, each cell consists of decision variable x_{ij} and per unit transportation cost c_{ij}.

Theorem 1. A necessary and sufficient condition for the existence of a feasible solution to a T.P. is that the T.P. is balanced.

Proof. (Necessary part)

Total supply from an origin $\quad \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, ..., m.$

Overall supply, $\quad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i$

Total demand met of a destination

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, ..., n.$$

Overall demand, $\quad \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j\ .$

Since overall supply exactly met the overall demand.

$$\sum_{i}\sum_{j} x_{ij} = \sum_{j}\sum_{i} x_{ij}$$

$$\Rightarrow \qquad \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

(Sufficient part) $\quad$ Let $\sum_i a_i = \sum_j b_i = l$ and $x_{ij} = a_i b_j / l$ for all i and j.

$$\text{Then } \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} (a_i b_j)/l = a_i\left(\sum_{j=1}^{n} b_j\right)\bigg/ l = a_i = 1, 2, ..., m.$$

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} (a_i b_j)/l = b_j\left(\sum_{i=1}^{m} a_i\right)\bigg/ l = b_j,\ j = 1, 2, ..., n.$$

$x_{ij} \geq 0$ since a_i and b_j are non-negative.

Therefore x_{ij} satisfies all the constraints and hence x_{ij} is a feasible solution.

Theorem 2. The number of basic variables in the basic feasible solution of an $m \times n$ T.P. is $m + n - 1$.

NOTES

Proof. This is due to the fact that the one of the constraints is redundant in balanced T.P.

We have overall supply, $$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i$$

and overall demand $$\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$$

Since $\sum_i a_i = \sum_j b_i$, the above two equations are identical and we have only $m + n - 1$ independent constraints. Hence the theorem is proved.

Note. 1. If any basic variable takes the value zero then the basic feasible solution (BFS) is said to be degenerate. Like LPP, all non-basic variables take the value zero.

2. If a basic variable takes either positive value or zero, then the corresponding cell is called 'Basic cell' or 'Occupied cell'. For non-basic variable the corresponding cell is called 'Non-basic cell' or 'Non-occupied cell' or 'Non-allocated cell'.

Loop. This means a closed circuit in a transportation table connecting the occupied (or allocated) cells satisfying the following :

(*i*) It consists of vertical and horizontal lines connecting the occupied (or allocated) cells.

(*ii*) Each line connects only two occupied (or allocated) cells.

(*iii*) Number of connected cells is even.

(*iv*) Lines can skip the middle cell of three adjacent cells to satisfy the condition (*ii*).

The following are the examples of loops.

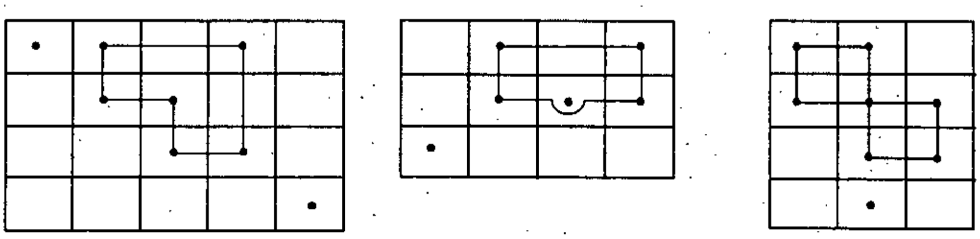


Fig. 3.1

Note. A solution of a T.P. is said to be basic if it does not consist of any loop.

3.2 FINDING INITIAL BASIC FEASIBLE SOLUTION

In this section three methods are to be discussed to find initial BFS of a T.P. In advance, it can be noted that the above three methods may give different initial BFS to the same T.P. Also allocation = minimum (supply, demand).

NOTES

(a) North-West Corner Rule (NWC)

(i) Select the north west corner cell of the transportation table.

(ii) Allocate the min (supply, demand) in that cell as the value of the variable.

If supply happens to be minimum, cross-off the row for further consideration and adjust the demand.

If demand happens to be minimum, cross-off the column for further consideration and adjust the supply.

(iii) The table is reduced and go to step (i) and continue the allocation until all the supplies are exhausted and the demands are met.

Example 1. *Find the initial BFS of the following T.P. using NWC rule.*

To

From	M_1	M_2	M_3	M_4	Supply
F_1	3	2	4	1	20
F_2	2	4	5	3	15
F_3	3	5	2	6	25
F_4	4	3	1	4	40
Demand	30	20	25	25	

Solution. Here, total supply = 100 = total demand. So the problem is balanced T.P.

The north-west corner cell is (1, 1) cell. So allocate min. (20, 30) = 20 in that cell. Supply exhausted. So cross-off the first row and demand is reduced to 10. The reduced table is

	M_1	M_2	M_3	M_4	
F_2	2	4	5	3	15
F_3	3	5	2	6	25
F_4	4	3	1	4	40
	10	20	25	25	

Here the north-west corner cell is (2, 1) cell. So allocate min. (15, 10) = 10 in that cell. Demand met. So cross-off the first column and supply is reduced to 5. The reduced table is

	M_2	M_3	M_4	
F_2	4	5	3	5
F_3	5	2	6	25
F_4	3	1	4	40
	20	25	25	

Here the north-west corner cell is (2, 2) cell. So allocate min. (5, 20) = 5 in that cell. Supply exhausted. So cross-off the second row (due to F2) and demand is reduced to 15. The reduced table is

NOTES

	M_2	M_3	M_4	
F_3	5	2	6	25
F_4	3	1	4	40
	15	25	25	

Here the north-west corner cell is (3, 2) cell. So allocate min. (25, 15) = 15 in that cell. Demand met. So cross-off the second column (due to M_2) and supply is reduced to 10. The reduced table is

	M_3	M_4	
F_3	2	6	10
F_4	1	4	40
	25	25	

Here the north-west corner cell is (3, 3) cell. So allocate min. (10, 25) = 10 in that cell. Supply exhausted. So cross-off the third row (due to F_3) and demand is reduced to 15. The reduced table is

	M_3	M_4	
F_4	1	4	40
	25	25	

continuing we obtain the allocation 15 to (4, 3) cell and 25 to (4, 4) cell so that supply exhausted and demand met. The **complete allocation** is shown below:

	M_1	M_2	M_3	M_4
F_1	[20] 3	2	4	1
F_2	[10] 2	[5] 4	5	3
F_3	3	[15] 5	[10] 2	6
F_4	4	3	[15] 1	[25] 4

Thus, the initial BFS is

$$x_{11} = 20,\ x_{21} = 10,\ x_{22} = 5,\ x_{32} = 15,\ x_{33} = 10,\ x_{43} = 15,\ x_{44} = 25.$$

The transportation cost

$$= 20 \times 3 + 10 \times 2 + 5 \times 4 + 15 \times 5 + 10 \times 2 + 5 \times 1 + 25 \times 4$$

$$= \text{Rs. } 310.$$

(b) Least Cost Entry Method (LCM) (or Matrix Minimum Method)

(i) Find the least cost from transportation table. If the least value is unique, then go for allocation.

NOTES

If the least value is not unique then select the cell for allocation for which the contributed cost is minimum.

(*ii*) If the supply is exhausted cross-off the row and adjust the demand. If the demand is met cross-off the column and adjust the supply. Thus the matrix is reduced.

(*iii*) Go to step (*i*) and continue until all the supplies are exhausted and all the demands are met.

Example 2. *Find the initial BFS of Example 1 using least cost entry method* :

	M_1	M_2	M_3	M_4	
F_1	3	2	4	1	20
F_2	2	4	5	3	15
F_3	3	5	2	6	25
F_4	4	3	1	4	40
	30	20	25	25	

Solution. Here the least value is 1 and occurs in two cells (1, 4) and (4, 3). But the contributed cost due to cell (1, 4) is 1 × min (20, 25) *i.e.*, 20 and due to cell (4, 3) is 1 × min. (40, 25) *i.e.*, 25. So we selected the cell (1, 4) and allocate 20. Cross-off the first row since supply exhausted and adjust the demand to 5. The reduced table is given below :

2	4	5	3	15
3	5	2	6	25
4	3	1	4	40
30	20	25	5	

The least value is 1 and unique. So allocate min. (40, 25) = 25 in that cell. Cross-off the third column (due to M_3) since the demand is met and adjust the supply to 15. The reduced table is given below :

2	4	3	15
3	5	6	25
4	3	4	15
30	20	5	

The least value is 2 and unique. So allocate min. (15, 30) = 15 in that cell. Cross-off the second row (due to F_2) since the supply exhausted and adjust the demand to 15. The reduced table is given below :

3	5	6	25
4	3	4	15
15	20	5	

The least value is 3 and occurs in two cells (3, 1) and (4, 2). The contributed cost due to cell (3, 1) is 3 × min. (25, 15) = 45 and due to cell (4, 2) is 3 × min. (15, 20) = 45. Let us select the (3, 1) cell for allocation and allocate 15. Cross-off the first column (due to M_1) since demand is met and adjust the supply to 10. The reduced table is given below :

5	6	10
3	4	15
20	5	

Continuing the above method and we obtain the allocations in the cell (4, 2) as 15, in the cell (3, 2) as 5 and in the cell (3, 4) as 5. The complete allocation is shown below :

	M_1	M_2	M_3	M_4
F_1	3	2	4	[20] 1
F_2	[15] 2	4	5	3
F_3	[15] 3	[5] 5	2	[5] 6
F_4	4	[15] 3	[25] 1	4

The initial BFS is

$$x_{14} = 20,\ x_{21} = 15,\ x_{31} = 15,\ x_{32} = 5,\ x_{34} = 5,\ x_{42} = 15,\ x_{43} = 25.$$

The transportation cost

$$= 20 \times 1 + 15 \times 2 + 15 \times 3 + 5 \times 5 + 5 \times 6 + 15 \times 3 + 25 \times 1$$
$$= \text{Rs. } 220.$$

Note. If the least cost is only selected columnwise then it is called 'column minima' method. If the least cost is only selected row wise then it is called 'row minima' method.

(c) Vogel's Approximation Method (VAM)

(i) Calculate the row penalties and column penalties by taking the difference between the lowest and the next lowest costs of every row and of every column respectively.

(ii) Select the largest penalty by encircling it. For tie cases, it can be broken arbitrarily or by analyzing the contributed costs.

(iii) Allocate in the least cost cell of the row/column due to largest penalty.

(iv) If the demand is met, cross off the corresponding column and adjust the supply.

If the supply is exhausted, cross-off the corresponding row and adjust the demand.

Thus the transportation table is reduced.

NOTES

(v) Go to Step (i) and continue until all the supplies exhausted and all the demands are met.

Example 3. *Find the initial BFS of example 1 using Vogel's approximation method.*

Solution.

	M_1	M_2	M_3	M_4	Row penalties
F_1	3	2	4	$\boxed{20}$ 1	20 (1)
F_2	2	4	5	3	15 (1)
F_3	3	5	2	6	25 (1)
F_4	4	3	1	4	40 (2)
Column	30	20	25	25	
penalties	(1)	(1)	(1)	(2)	

Since there is a tie in penalties, let us break the tie by considering the contributed costs. Due to M_4, the contributed cost is $1 \times \min. (20, 25) = 20$. While due to F_4, the contributed cost is $1 \times \min. (40, 25) = 25$. So select the column due to M_4 for allocation and we allocate min. (20, 25) *i.e.,* 20 in (1, 4) cell. Then cross-off the first row as supply is exhausted and adjust the corresponding demand as 5. The reduced table is

	M_1	M_2	M_3	M_4	Row penalties
F_2	2	4	5	3	15 (1)
F_3	3	5	2	6	25 (1)
F_4	4	3	1	4	40 (2)
Column	30	20	25	5	
penalties	(1)	(1)	(1)	(1)	

Here the largest penalty is 2 which is due to F_4. Allocate in (4, 3) cell as min. (40, 25) = 25. Cross-off the third column due to M_3, since demand is met and adjust the corresponding supply to 15. The reduced table is

	M_1	M_2	M_4	Row penalties
F_2	2	4	3	15 (1)
F_3	3	5	6	25 (2)
F_4	4	3	4	15 (1)
Column	30	20	5	
penalties	(1)	(1)	(1)	

Here the largest penalty is 2 which is due to F_3. Allocate in (3, 1) cell as min. (25, 30) = 25. Cross-off the third row due to F_3 since supply is exhausted and adjust the corresponding demand to 5. The reduced table is

NOTES

	M_1	M_2	M_4	Row penalties
F_2	2	4	3	15 (1)
F_4	4	3	4	15 (1)
	5	20	5	
Column penalties	(2)	(1)	(1)	

Here the largest penalty is 2 which is due to M_1. Allocate in (2, 1) cell as min. (15, 5) = 5. Corss-off the first column due to M_1 since demand is met and adjust the supply to 10. The reduced table is

	M_2	M_4	Row penalties
F_2	4	3	10 (1)
F_4	3	4	15 (1)
	20	5	
Column penalties	(1)	(1)	

Here tie has occurred. The contributed cost is minimum due to (2, 4) cell which is 3 × min. (10, 5) = 15. So allocate min. (10, 5) = 5 in (2, 4) cell. Cross-off the fourth column which is due to M_4 since demand is met and adjust the corresponding supply to 5. On continuation we obtain the allocation of 5 in (2, 2) cell and 15 in (4, 2) cell. The complete allocation in shown below :

	M_1	M_2	M_3	M_4
F_1	3	2	4	[20] 1
F_2	[5] 2	[5] 4	5	[5] 3
F_3	[25] 3	5	2	6
F_4	4	[15] 3	[25] 1	4

The initial BFS is

$$x_{14} = 20, x_{21} = 5, x_{22} = 5, x_{24} = 5, x_{31} = 25, x_{42} = 15, x_{43} = 25.$$

The transportation cost

$$= 1 \times 20 + 2 \times 5 + 4 \times 5 + 3 \times 5 + 3 \times 25 + 3 \times 15 + 1 \times 25$$

$$= \text{Rs. } 210.$$

3.3 UV-METHOD/MODI METHOD

Taking the initial BFS by any method discussed above, this method find the optimal solution to the transportation problem. The steps are given below :

(*i*) For each row consider a variable u_i and for each column consider another variable v_j.

Find u_i and v_j such that

$u_i + v_j = c_{ij}$ for every basic cells.

(*ii*). For every non-basic cells, calculate the net evaluations as follows :

$\bar{c}_{ij} = u_i + v_j - c_{ij}$

If all $\bar{c}_{ij}$ are non-positive, the current solution is optimal.

If at least one $\bar{c}_{ij} > 0$, select the variable having the largest positive net evaluation to enter the basis.

(*iii*) Let the variable x_{rc} enter the basis. Allocate an unknown quantity θ to the cell (r, c).

Identify a loop that starts and ends in the cell (r, c).

Subtract and add θ to the corner points of the loop clockwise/anticlockwise.

(*iv*) Assign a minimum value of θ in such a way that one basic variable becomes zero and other basic variables remain non-negative. The basic cell which reduces to zero leaves the basis and the cell with θ enters into the basis.

If more than one basic variables become zero due to the minimum value of θ, then only one basic cell leaves the basis and the solution is called degenerate.

(*v*) Go to step (*i*) until an optimal BFS has been obtained.

NOTES

Note. In step (*ii*), if all $\bar{c}_{ij} < 0$, then the optimal solution is unique. If at least one $\bar{c}_{ij} < 0$, then we can obtain alternative solution. Assign θ in that cell and repeat one iteration (from step (*iii*)).

Example 4. *Consider the initial BFS by LCM of Example 2, find the optimal solution of the T.P.*

Solution. Iteration 1.

	M_1	M_2	M_3	M_4	
F_1	3	2	4	$\boxed{20}$ 1	$u_1 = -5$
F_2	$\boxed{15}$ 2	4	5	3	$u_2 = -1$
F_3	$\boxed{15}$ 3	$\boxed{5}$ 5	2	$\boxed{5}$ 6	$u_3 = 0$ (Let)
F_4	4	$\boxed{15}$ 3	$\boxed{25}$ 1	4	$u_4 = -2$
	$V_1=3$	$V_2=5$	$V_3=3$	$V_4=6$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{11} = -5,\ \bar{c}_{12} = -2,\ \bar{c}_{13} = -6,\ \bar{c}_{22} = 0,\ \bar{c}_{23} = -3,\ \bar{c}_{24} = 2,\ \bar{c}_{33} = 1,\ \bar{c}_{41} = -3,\ \bar{c}_{44} = 0.$$

Since all $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal.

Select the cell (2, 4) due to largest positive value and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is as follow:

3	2	4	$\boxed{20}$ 1
$\boxed{15} - \theta$ 2	4	5	3
$\boxed{15} + \theta$ 3	$\boxed{5}$ 5	2	$\boxed{5} - \theta$ 6
4	$\boxed{15}$ 3	$\boxed{25}$ 1	4

Select θ = min. (5, 15) = 5. The cell (3, 4) leaves the basis and the cell (2, 4) enters into the basis. Thus the current solution is updated.

Iteration 2.

3	2	4	$\boxed{20}$ 1	$u_1 = -2$
$\boxed{10}$ 2	4	5	$\boxed{5}$ 3	$u_2 = 0$ (Let)
$\boxed{20}$ 3	$\boxed{5}$ 5	2	6	$u_3 = 1$
4	$\boxed{15}$ 3	$\boxed{25}$ 1	4	$u_4 = -1$
$V_1 = 2$	$V_2 = 4$	$V_3 = 2$	$V_4 = 3$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{11} = -3,\ \bar{c}_{12} = 0,\ \bar{c}_{13} = -4,\ \bar{c}_{22} = 0,\ \bar{c}_{23} = -3,\ \bar{c}_{33} = 1,\ \bar{c}_{34} = -2,\ \bar{c}_{41} = -3,\ \bar{c}_{44} = -2.$$

Since all $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal.

Select the cell (3, 3) due to largest positive value and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown below :

3	2	4	$\boxed{20}$ 1
$\boxed{10}$ 2	4	5	$\boxed{5}$ 3
$\boxed{20}$ 3	$\boxed{5} - \theta$ 5	θ 2	6
4	$\boxed{15} + \theta$ 3	$\boxed{25} - \theta$ 1	4

Select θ = min. (5, 25) = 5. The cell (3, 2) leaves the basis and the cell (3, 3) enters into the basis. Thus the current solution is updated.

Iteration 3.

3	2	4	$\boxed{20}$ 1	$u_1 = -2$
$\boxed{10}$ 2	4	5	$\boxed{5}$ 3	$u_2 = 0$ (Let)
$\boxed{20}$ 3	5	$\boxed{5}$ 2	6	$u_3 = 1$
4	$\boxed{20}$ 3	$\boxed{20}$ 1	4	$u_4 = 0$
$V_1 = 2$	$V_2 = 3$	$V_3 = 1$	$V_4 = 3$	

NOTES

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{11} = -3,\ \bar{c}_{12} = -1,\ \bar{c}_{13} = -5,\ \bar{c}_{22} = -1,\ \bar{c}_{23} = -5,\ \bar{c}_{32} = -1,\ \bar{c}_{34} = -2,\ \bar{c}_{41} = -2,\ \bar{c}_{44} = -1.$

Since all $\bar{c}_{ij}$ are non-positive, the current solution is optimal. Thus, the optimal solution is

$x_{14} = 20,\ x_{21} = 10,\ x_{24} = 5,\ x_{31} = 20,\ x_{33} = 5,\ x_{42} = 20,\ x_{43} = 20.$

The optimal transportation cost

$= 1 \times 20 + 2 \times 10 + 3 \times 5 + 3 \times 20 + 2 \times 5 + 3 \times 20 + 1 \times 20 = \text{Rs. } 205.$

Example 5. *Consider the initial BFS by VAM of Example 3, find the optimal solution of the T.P.*

Solution. Iteration 1.

3	2	4	[20] 1	$u_1 = -2$
[5] 2	[5] 4	5	[5] 3	$u_2 = 0$ (Let)
[25] 3	5	2	6	$u_3 = 1$
4	[15] 3	[25] 1	4	$u_4 = -1$
$V_1 = 2$	$V_2 = 4$	$V_3 = 2$	$V_4 = 3$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{11} = -3,\ \bar{c}_{12} = 0,\ \bar{c}_{13} = -4,\ \bar{c}_{23} = -3,\ \bar{c}_{32} = 0,\ \bar{c}_{33} = 1,\ \bar{c}_{34} = -2,\ \bar{c}_{41} = -3,\ \bar{c}_{44} = -2.$

Since all $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal.

Select the cell (3, 3) due to largest positive value and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown below :

3	2	4	[20] 1
[5] $+\theta$ 2	[5] $-\theta$ 4	5	[5] 3
[25] $-\theta$ 3	5	θ 2	6
4	[15] $+\theta$ 3	[25] $-\theta$ 1	4

Select θ = min. (5, 25, 25) = 5. The cell (2, 2) leaves the basis and the cell (3, 3) enters into the basis. Thus the current solution is updated.

Iteration 2.

3	2	4	[20] 1	$u_1 = -2$
[10] 2	4	5	[5] 3	$u_2 = 0$ (Let)
[20] 3	5	[5] 2	6	$u_3 = 1$
4	[20] 3	[20] 1	4	$u_4 = 0$
$V_1 = 2$	$V_2 = 3$	$V_3 = 1$	$V_4 = 3$	

NOTES

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{11} = -3, \bar{c}_{12} = -1, \bar{c}_{13} = -5, \bar{c}_{22} = -1, \bar{c}_{23} = -5, \bar{c}_{32} = -1, \bar{c}_{34} = -2, \bar{c}_{41} = -2, \bar{c}_{44} = -1.$

Since all $\bar{c}_{ij}$ are non-positive, the current solution is optimal. Thus the optimal solution is

$$x_{14} = 20,\ x_{21} = 10,\ x_{24} = 5,\ x_{31} = 20,\ x_{33} = 5,\ x_{42} = 20,\ x_{43} = 20.$$

The optimal transportation cost = Rs. 205.

Note. To find optimal solution to a T.P., the number of iterations by uv-method is always more if we consider the initial BFS by NWC.

3.4 DEGENERACY IN T.P.

A BFS of a T.P. is said to be degenerate if one or more basic variables assume a zero value. This degeneracy may occur in initial BFS or in the subsequent iterations of *uv*-method. An initial BFS could become degenerate when the supply and demand in the intermediate stages of any one method (NWC/LCM/VAM) are equal corresponding to a selected cell for allocation. In *uv*-method it is identified only when more than one corner points in a loop vanishes due to minimum value of θ.

For the degeneracy in initial BFS, arbitrarily we can delete the row due to supply adjusting the demand to zero or delete the column due to demand adjusting the supply to zero whenever there is a tie in demand and supply.

For the degeneracy in *uv*-method, arbitrarily we can make one corner as non-basic cell and put zero in the other corner.

Example 6. *Find the optimal solution to the following T.P. :*

Source	Destination 1	2	3	Available
1	50	30	190	10
2	80	45	150	30
3	220	180	50	40
Requirement	40	20	20	80

Solution. Let us find the initial BFS using VAM :

	1	2	3		Row penalties
1	50	30	190	10	(20)
2	80	45	150	30	(35)
3	220	180	50	40	(130)
	40	20	20		
Column penalties	(30)	(15)	(100)		

NOTES

Select (3, 3) cell for allocation and allocate min (40, 20) = 20 in that cell. Cross-off the third column as the requirement is met and adjust the availability to 20. The reduced table is given below :

	1	2	Row penalties
1	50	30	10 (20)
2	80	45	30 (35)
3	220	180	20 (40)
Column penalties	40 (30)	20 (15)	

Select (3, 2) cell for allocation. Now there is a tie in allocation. Let us allocate 20 in (3, 2) cell and cross-off the second column and adjust the availability to zero. The reduced table is given below :

	1	
1	50	10
2	80	30
3	220	0
	40	

On continuation we obtain the remaining allocations as 0 in (3, 1) cell, 30 in (2, 1) cell and 10 in (1, 1) cell. The complete initial BFS is given below and let us apply the first iteration of *uv*-method :

Iteration 1.

$\boxed{10}$ 50	30	190	$u_1 = -170$
$\boxed{30}$ 80	45	150	$u_2 = -140$
$\boxed{0}$ 220	$\boxed{20}$ 180	$\boxed{20}$ 50	$u_3 = 0$ (Let)
$V_1 = 220$	$V_2 = 180$	$V_3 = 50$	

For non-basic cells : $\quad \bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{12} = -20,\ \bar{c}_{13} = -310,\ \bar{c}_{22} = -5,\ \bar{c}_{23} = -240.$$

Since all $\bar{c}_{ij} < 0$, the current solution is optimal. Hence, the optimal solution is

$$x_{11} = 10,\ x_{21} = 30,\ x_{31} = 0,\ x_{32} = 20,\ x_{33} = 20.$$

The transportation cost

$$= 50 \times 10 + 80 \times 30 + 0 + 180 \times 20 + 50 \times 20$$
$$= \text{Rs. } 7500.$$

3.5 MAX-TYPE T.P.

Instead of unit cost in transportation table, unit profit is considered then the objective of the T.P. changes to maximize the total profits subject to supply and demand restrictions. Then this problem is called 'max-type' T.P.

NOTES

To obtain optimal solution, we consider

$$\text{Loss} = -\ \text{Profit}$$

and convert the max type transportation matrix to a loss matrix. Then all the methods described in the previous sections can be applied. Thus the optimal BFS obtained for the loss matrix will be the optimal BFS for the max-type T.P.

Example 7. *A company has three plants at locations A, B and C, which supply to four markets D, E, F and G. Monthly plant capacities are 500, 800 and 900 units respectively. Monthly demands of the markets are 600, 700, 400 and 500 units respectively. Unit profits (in rupees) due to transportation are given below :*

	D	E	F	G
A	8	5	3	6
B	7	4	5	2
C	6	8	4	2

Determine an optimal distribution for the company in order to maximize the total transportation profits.

Solution. The given problem is balanced max type T.P. All profits are converted to losses by multiplying – 1.

	D	E	F	G	
A	–8	–5	–3	–6	500
B	–7	–4	–5	–2	800
C	–6	–8	–4	–2	900
	600	700	400	500	2200

The initial BFS by LCM is given below:

[500] –8	–5	–3	–6
[100] –7	–4	[400] –5	[300] –2
–6	[700] –8	–4	[200] –2

To find optimal solution let us apply *uv*-method.

Iteration 1.

[500] –θ –8	–5	–3	θ –6	$u_1 = -1$
[100] +θ –7	–4	[400] –5	[300] –θ –2	$u_2 = 0$
–6	[700] –8	–4	[200] –2	$u_3 = 0$ (Let)
$V_1 = -7$	$V_2 = -8$	$V_3 = -5$	$V_4 = -2$	

NOTES

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{12} = -4,\ \bar{c}_{13} = -3,\ \bar{c}_{14} = 3,\ \bar{c}_{22} = -4,\ \bar{c}_{31} = -1,\ \bar{c}_{33} = -1.$$

Since all $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal. Select the cell (1, 4) due to largest positive value and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown above.

Select θ = min. (500, 300) = 300. The cell (2, 4) leaves the basis and the cell (1, 4) enters into the basis. Thus the current solution is updated.

Iteration 2.

$\boxed{200}$ −θ; −8	−5	−3	$\boxed{300}$ +θ; −6	$u_1 = -4$
$\boxed{400}$; −7	−4	$\boxed{400}$; −5	−2	$u_2 = -3$
θ; −6	$\boxed{700}$; −8	−4	$\boxed{200}$ −θ; −2	$u_3 = 0$ (Let)
$V_1 = -4$	$V_2 = -8$	$V_3 = -2$	$V_4 = -2$	

For non-basic cells,

$$\bar{c}_{12} = -7,\ \bar{c}_{13} = -3,\ \bar{c}_{22} = -7,\ \bar{c}_{24} = -3,\ \bar{c}_{31} = 2,\ \bar{c}_{33} = 2.$$

Since all the $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal. There is a tie in largest positive values. Let us select the cell (3, 1) and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown above.

Select θ = min. (200, 200) = 200. Since only one cell will leave the basis, let the cell (3, 3) leaves the basis and assign a zero in the cell (1, 1). The cell (3, 1) enters into the basis. Thus the current solution is updated.

Iteration 3.

$\boxed{0}$; −8	−5	−3	$\boxed{500}$; −6	$u_1 = -2$
$\boxed{400}$; −7	−4	$\boxed{400}$; −5	−2	$u_2 = -1$
$\boxed{200}$; −6	$\boxed{700}$; −8	−4	−2	$u_3 = 0$ (Let)
$V_1 = -6$	$V_2 = -8$	$V_3 = -4$	$V_4 = -4$	

For non-basic cells,

$$\bar{c}_{12} = -5,\ \bar{c}_{13} = -3,\ \bar{c}_{22} = -5,\ \bar{c}_{24} = -3,\ \bar{c}_{33} = 0,\ \bar{c}_{34} = -4.$$

Since all the $\bar{c}_{ij}$ are non-positive, the current solution is optimal.

Thus the optimal solution, which is degenerate, is

$$x_{11} = 0,\ x_{14} = 500,\ x_{21} = 400,\ x_{23} = 400,\ x_{31} = 200,\ x_{32} = 700.$$

The maximum transportation profit

$$= 0 + 3000 + 2800 + 2000 + 1200 + 5600 = \text{Rs. } 14600.$$

Since $\bar{c}_{33} = 0$, this indicates that there exists an alternative optimal solution. Assign an unknown quantity θ in the cell (3, 3). Identify a loop and subtract and add θ to the corner points of the loop which is as follow:

[0] −8	−5	−3	[500] −6
[400] +θ −7	−4	[400] −θ −5	−2
[200] −θ −6	[700] −8	θ −4	−2

Select θ = min. (200, 400) = 200. The cell (3, 1) leaves the basis and the cell (3, 3) enters into the basis.

Iteration 4.

[0] −8	−5	−3	[500] −6	$u_1 = -2$
[600] −7	−4	[200] −5	−2	$u_2 = -1$
−6	[700] −8	[200] −4	−2	$u_3 = 0$ (Let)
$V_1 = -6$	$V_2 = -8$	$V_3 = -4$	$V_4 = -4$	

For non-basic cells,

$$\bar{c}_{12} = -5,\ \bar{c}_{13} = -3,\ \bar{c}_{22} = -5,\ \bar{c}_{24} = -3,\ \bar{c}_{31} = 0,\ \bar{c}_{34} = -2.$$

Since all the $\bar{c}_{ij}$ are non-positive, the current solution is optimal. Thus the alternative optimal solution is

$$x_{11} = 0,\ x_{14} = 500,\ x_{21} = 600,\ x_{23} = 200,\ x_{32} = 700,\ x_{33} = 200.$$

and the maximum transportation profit is Rs. 14,600.

3.6 UNBALANCED T.P.

If total supply ≠ total demand, the problem is called unbalanced T.P.. To obtain feasible solution, the unbalanced problem should be converted to balanced problem by introducing dummy source or dummy destination, whichever is required. Suppose, (supply =) $\sum a_i > \sum b_j$ (= demand). Then add one dummy destination with demand $= \left(\sum a_i - \sum b_j\right)$ with either zero transportation costs or some penalties, if they are given. Suppose (supply =) $\sum a_i < \sum b_j$ (= demand). Then add one dummy source with supply $= \left(\sum b_j - \sum a_i\right)$ with either zero transportation costs or some penalties, if they are given.

After making it balanced the mathematical formulation is similar to the balanced T.P.

Example 8. *A company wants to supply materials from three plants to three new projects. Project I requires 50 truck loads, project II requires 40 truck loads and project III requires 60 truck loads. Supply capacities for the plants P_1, P_2 and P_3 are 30, 55 and 45 truck loads. The table of transportation costs are given as follows:*

NOTES

	I	II	III
P_1	7	10	12
P_2	8	12	7
P_3	4	9	10

Determine the optimal distribution.

Solution. Here total supplies = 130 and total requirements = 150. The given problem is unbalanced T.P. To make it balanced consider a dummy plants with supply capacity of 20 truck loads and zero transportation costs to the three projects. Then the balanced T.P. is

	To I	II	III	
From P_1	7	10	12	30
P_2	8	12	7	55
P_3	4	9	10	45
P_4 (Dummy)	0	0	0	20
	50	40	60	

Using VAM, we obtain the initial BFS as given below :

[5] 7	[20] 10	[5] 12
8	12	[55] 7
[45] 4	9	10
0	[20] 0	0

To find optimal solution let us apply *uv*-method.

Iteration 1.

[5] 7	[20] +θ 10	[5] −θ 12	$u_1 = 0$ (Let)
8	12	[55] 7	$u_2 = -5$
[45] 4	9	10	$u_3 = -3$
0	[20] −θ 0	θ 0	$u_4 = -10$
$V_1 = 7$	$V_2 = 10$	$V_3 = 12$	

For non-basic cells, $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{21} = -6,\ \bar{c}_{22} = -7,\ \bar{c}_{32} = -2,\ \bar{c}_{33} = -1,\ \bar{c}_{41} = -3,\ \bar{c}_{43} = 2,$

Since $\bar{c}_{43}$ is only positive value assign an unknown quantity θ in (4, 3) cell. Identify

NOTES

a loop and subtract and add θ to the corner points of the loop which is shown above.

Select θ = min. (5, 20) = 5 so that the cell (1, 3) leaves the basis and the cell (4, 3) enters into the basis.

Iteration 2.

[5] 7	[25] 10	12	$u_1 = 0$ (Let)
8	12	[55] 7	$u_2 = -3$
[45] 4	9	10	$u_3 = -3$
0	[15] 0	[5] 0	$u_4 = -10$
$V_1 = 7$	$V_2 = 10$	$V_3 = 10$	

For non-basic cells, we obtain

$\bar{c}_{13} = -2,\ \bar{c}_{21} = -4,\ \bar{c}_{22} = -5,\ \bar{c}_{32} = -2,\ \bar{c}_{33} = -3,\ \bar{c}_{41} = -3$

Since $\bar{c}_{ij} < 0$, the current solution is optimal. Thus the optimal solution is

Supply 15 truck loads from P_1 to I, 25 truck loads from P_1 to II, 55 truck loads from P_2 to III, 45 truck loads from P_3 to I. Demands of 15 truck loads for II and 5 truck loads for III will remain unsatisfied.

3.7 ASSIGNMENT PROBLEMS AND MATHEMATICAL FORMULATION

Consider n machines $M_1, M_2, ..., M_n$ and n different jobs $J_1, J_2, ..., J_n$. These jobs to be processed by the machines one to one basis *i.e.,* each machine will process exactly one job and each job will be assigned to only one machine. For each job the processing cost depends on the machine to which it is assigned. Now we have to determine the assignment of the jobs to the machines one to one basis such that the total processing cost is minimum. This is called an *assignment problem.*

If the number of machines is equal to the number of jobs then the above problem is called *balanced* or *standard* assignment problem. Otherwise, the problem is called *unbalanced* or *non-standard* assignment problem. Let us consider a balanced assignment problem.

For linear programming problem formulation, let us define the decision variables as

$$x_{ij} = \begin{cases} 1, \text{if job } j \text{ is assigned to machine } i \\ 0, \text{otherwise} \end{cases}$$

and the cost of processing job j on machine i as c_{ij}. Then we can formulate the assignment problem as follows :

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \qquad ...(1)$$

subject to,
$$\sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, 2, ..., n$$

(Each machine is assigned exactly to one job)

$$\sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, 2, ..., n.$$

NOTES

(Each job is assigned exactly to one machine)

$$x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j$$

In **matrix form,** Minimize $z = Cx$

subject to, $Ax = 1$,

$$x_{ij} = 0 \text{ or } 1, \; i, j = 1, 2, ..., n.$$

where A is a $2n \times n^2$ matrix and total unimodular *i.e.,* the determinant of every sub square matrix formed from it has value 0 or 1. This property permits us to replace the constraint $x_{ij} = 0$ or 1 by the constraint $x_{ij} \geq 0$. Thus we obtain

Minimize $z = Cx$

subject to, $Ax = 1, x \geq 0$

The **dual** of (1) with the non-negativity restrictions replacing the 0-1 constraints can be written as follows :

$$\text{Maximize } W = \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j$$

subject to, $u_i + v_j \leq c_{ij}$, $\qquad i, j = 1, 2, ..., n.$

u_i, v_j unrestricted in signs $\qquad i, j = 1, 2, ..., n.$

Example 9. *A company is facing the problem of assigning four operators to four machines. The assignment cost in rupees in given below :*

Machine

$$\begin{array}{cc} & \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \text{Operator} & \left[\begin{array}{c|cccc} \text{I} & 5 & 7 & - & 4 \\ \text{II} & 7 & 5 & 3 & 2 \\ \text{III} & 9 & 4 & 6 & - \\ \text{IV} & 7 & 2 & 7 & 6 \end{array}\right] \end{array}$$

In the above, operators I and III can not be assigned to the machines M_3 and M_4 respectively. Formulate the above problem as a LP model.

Solution. Let $x_{ij} = \begin{cases} 1, \text{if the } i\text{th operator is assigned to } j\text{th machine} \\ 0, \text{otherwise} \end{cases}$

$$i, j = 1, 2, 3, 4.$$

be the decision variables.

By the problem, $x_{13} = 0$ and $x_{34} = 0$.

The LP model is given below :

$$\begin{aligned} \text{Minimize } z = \; & 5x_{11} + 7x_{12} + 4x_{14} + 7x_{21} + 5x_{22} + 3x_{23} + 2x_{24} \\ & + 9x_{31} + 4x_{32} + 6x_{33} + 7x_{41} + 2x_{42} + 7x_{43} + 6x_{44} \end{aligned}$$

NOTES

subject to,

(Operator assignment constraints)

$$x_{11} + x_{12} + x_{14} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$

(Machine assignment constraints)

$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$
$$x_{23} + x_{33} + x_{43} = 1$$
$$x_{14} + x_{24} + x_{44} = 1$$
$$x_{ij} \geq 0 \text{ for all } i \text{ and } j.$$

3.8 HUNGARIAN ALGORITHM

This is an efficient algorithm for solving the assignment problem developed by the Hungarian mathematician König. Here the optimal assignment is not affected if a constant is added or subtracted from any row or column of the balanced assignment cost matrix. The **algorithm** can be started as follows :

(*a*) Bring at least one zero to each row and column of the cost matrix by subtracting the minimum of the row and column respectively.

(*b*) Cover all the zeros in cost matrix by *minimum* number of horizontal and vertical lines.

(*c*) If number of lines = order of the matrix, then select the zeros as many as the order of the matrix in such a way that they cover all the rows and columns.

(Here $A_{n \times n}$ means nth order matrix)

(*d*) If number of lines ≠ order of the matrix, then perform the following and create a new matrix :

(*i*) Select the minimum element from the uncovered elements of the cost matrix by the lines.

(*ii*) Subtract the uncovered elements from the minimum element.

(*iii*) Add the minimum element to the junction (*i.e.*, crossing of the lines) elements.

(*iv*) Other elements on the lines remain unaltered.

(*v*) Go to Step (*b*).

Example 10. *A construction company has four engineers for designing. The general manager is facing the problem of assigning four designing projects to these engineers. It is also found that Engineer 2 is not competent to design project 4. Given the time estimate required by each engineer to design a given project, find an assignment which minimizes the total time.*

NOTES

Projects

		P1	P2	P3	P4
Engineers	E1	6	5	13	2
	E2	8	10	4	–
	E3	10	3	7	3
	E4	9	8	6	2

Solution: Let us first bring zeros rowwise by subtracting the respective minima from all the row elements respectively.

4	3	11	0
4	6	0	–
7	0	4	0
7	6	4	0

Let us bring zero columnwise by subtracting the respective minima from all the column elements respectively. Here the above operations is to be performed only on first column, since at least one zero has appeared in the remaining columns.

0	3	11	0
0	6	0	–
3	0	4	0
3	6	4	0

(This completes Step-*a*)

Now (Step-*b*) all the zeros are to be covered by minimum number of horizontal and vertical lines which is shown below. It is also to be noted that this covering is not unique.

0	3	11	0
0	6	0	–
3	0	4	0
3	6	4	0

It is seen that no. of lines = 4 = order of the matrix. Therefore by Step-*c*, we can go for assignment *i.e.*, we have to select 4 zeros such that they cover all the rows and columns which is shown below:

$\boxed{0}$	3	11	0
0	6	$\boxed{0}$	–
3	$\boxed{0}$	4	0
3	6	4	$\boxed{0}$

Therefore the optimal assignment is

$$E1 \to P1, \qquad E2 \to P3, \qquad E3 \to P2, \qquad E4 \to P4$$

and the minimum total time required = 6 + 4 + 3 + 2 = 15 units.

NOTES

Example 11. *Solve the following job machine assignment problem. Cost data are given below :*

Machines

		1	2	3	4	5	6
	A	21	35	20	20	32	28
	B	30	31	22	25	28	30
	C	28	29	25	27	27	21
Jobs	D	30	30	26	26	31	28
	E	21	31	25	20	27	30
	F	25	29	22	25	30	21

Solution. Let us first bring zeros first rowwise and then columnwise by subtracting the respective minima elements from each row and each column respectively and the cost matrix, thus obtained, is as follows :

0	11	0	0	7	8
7	5	0	3	1	8
6	4	4	6	1	0
3	0	0	0	0	2
0	7	5	0	2	10
3	4	1	4	4	0

By Step-*b*, all the zeros are covered by minimum number of horizontal and vertical lines which is shown below :

0	11	0	0	7	8
7	5	0	3	1	8
6	4	4	6	1	0
3	0	0	0	0	2
0	7	5	0	2	10
3	4	1	4	4	0

Here no. of lines $\neq$ order of the matrix. Hence, we have to apply Step-*d*. The minimum uncovered element is 1. By applying Step-*d* we obtain the following matrix :

NOTES

0	11	1	0	7	9
6	4	0	2	0	8
5	3	4	5	0	0
3	0	1	0	0	3
0	7	6	0	2	11
2	3	1	3	3	0

Now, by Step-*b*, we cover all the zeros by minimum number of horizontal and vertical straight lines.

0	11	1	0	7	9
6	4	0	2	0	8
5	3	4	5	0	0
3	0	1	0	0	3
0	7	6	0	2	11
2	3	1	3	3	0

Now, the no. of lines = order of the matrix. So we can go for assignment by Step-*c*. The assignment is shown below :

$\boxed{0}$	11	1	0	7	9
6	4	$\boxed{0}$	2	0	8
5	3	4	5	$\boxed{0}$	0
3	$\boxed{0}$	1	0	0	3
0	7	6	$\boxed{0}$	2	11
2	3	1	3	3	$\boxed{0}$

The optimal assignment is A→1, B→3, C→5, D→2, E→4, F→6. An alternative assignment is also obtained as A→4, B→3, C→5, D→2, E→1, F→6. For both the assignments, the minimum cost is 21 + 22 + 27 + 30 + 20 + 21 *i.e.*, Rs. 141.

3.9 UNBALANCED ASSIGNMENTS

For unbalanced or non-standard assignment problem no. of rows ≠ no. of columns in the assignment cost matrix *i.e.*, we deal with a rectangular cost matrix. To find an assignment for this type of problem, we have to first convert this unbalanced problem into a balanced problem by adding dummy rows or columns with zero costs so that the defective function will be unaltered. For machine-job problem, if no. of machines (say, m) > no. of jobs (say, n), then create m-n dummy jobs and the processing cost of dummy jobs as zero. When a dummy job gets assigned to a machine, that machine stays idle. Similarly the other case *i.e.*, $n > m$, is handled.

Example 12. *Find an optimal solution to an assignment problem with the following cost matrix :*

	M1	M2	M3	M4	M5
J1	13	5	20	5	6
J2	15	10	16	10	15
J3	6	12	14	10	13
J4	13	11	15	11	15
J5	15	6	16	10	6
J6	6	15	14	5	12

Solution. The above problem is unbalanced. We have to create a dummy machine M6 with zero processing time to make the problem as balanced assignment problem. Therefore we obtain the following :

	M1	M2	M3	M4	M5	M6 (dummy)
J1	13	5	20	5	6	0
J2	15	10	16	10	15	0
J3	6	12	14	10	13	0
J4	13	11	15	11	15	0
J5	15	6	16	10	6	0
J6	6	15	14	5	12	0

Let us bring zeros columnwise by subtracting the respective minima elements from each column respectively and the cost matrix, thus obtained, is as follows:

7	0	6	0	0	0
9	5	2	5	9	0
0	7	0	5	7	0
7	6	1	6	9	0
9	1	2	5	0	0
0	10	0	0	6	0

NOTES

Let us cover all the zeros by minimum number of horizontal and vertical lines and is given below :

7	0	6	0	0	0
9	5	2	5	9	0
0	7	0	5	7	0
7	6	1	6	9	0
9	1	2	5	0	0
0	10	0	0	6	0

Now, the number of lines ≠ order of the matrix. The minimum uncovered element by the lines is 1. Using Step-*d* of the Hungarian algorithm and covering all the zeros by minimum no. of lines we obtain as follows :

7	0	6	0	1	1
8	4	1	4	9	0
0	7	0	5	8	1
6	5	0	5	9	0
8	0	1	4	0	0
0	10	0	0	7	1

Now, the number of lines = order of the matrix and we have to select 6 zeros such that they cover all the rows and columns. This is done in the following :

7	$\boxed{0}$	6	0	1	1
8	4	1	4	9	$\boxed{0}$
$\boxed{0}$	7	0	5	8	1
6	5	$\boxed{0}$	5	9	0
8	0	1	4	$\boxed{0}$	0
0	10	0	$\boxed{0}$	7	1

Therefore, the optimal assignment is

J1→M2, J2→M6, J3→M1, J4→M3, J5→M5, J6→M4 and the minimum cost = Rs. (5 + 0 + 6 + 15 + 6 + 5) = Rs. 37.

In the above, the job J2 will not get processed since the machine M6 is dummy.

3.10 MAX-TYPE ASSIGNMENT PROBLEMS

When the objective of the assignment is to maximize, the problem is called 'Max-type assignment problem'. This is solved by converting the profit matrix to an opportunity loss matrix by subtracting each element from the highest element of the profit matrix. Then the minimization of the loss matrix is the same as the maximization of the profit matrix.

Example 13. *A company is faced with the problem of assigning 4 jobs to 5 persons. The expected profit in rupees for each person on each job are as follows :*

	Job			
Persons	J1	J2	J3	J4
I	86	78	62	81
II	55	79	65	60
III	72	65	63	80
IV	86	70	65	71
V	72	70	71	60

Find the assignment of persons to jobs that will result in a maximum profit.

Solution. The above problem is unbalanced max-type assignment problem. The maximum element is 86. By subtracting all the elements from it obtain the following opportunity loss matrix.

0	8	24	5
31	7	21	26
14	21	23	6
0	16	21	15
14	16	15	26

Now, a dummy job J5 is added with zero losses. Then bring zeros in each column by subtracting the respective minimum element from each column we obtain the following matrix.

0	1	9	0	0
31	0	6	21	0
14	14	8	1	0
0	9	6	10	0
14	9	0	21	0

Let us cover all the zeros by minimum number of lines and is as follows:

NOTES

0	1	9	0	0
31	0	6	21	0
14	14	8	1	0
0	9	6	10	0
14	9	0	21	0

Since, the no. of lines = order of the matrix, we have to select 5 zeros such that they cover all the rows and columns. This is done in the following :

0	1	9	$\boxed{0}$	0
31	$\boxed{0}$	6	21	0
14	14	8	1	$\boxed{0}$
$\boxed{0}$	9	6	10	0
14	9	$\boxed{0}$	21	0

The optimal assignment is

I→J4, II→J2, III→J5, IV→J1, V→J3 and maximum profit = Rs. (81 + 79 + 86 + 71) = Rs. 317. Here person III is idle.

Note. The max-type assignment problem can also be converted to a minimization problem by multiplying all the elements of the profit matrix by – 1. Then the Hungarian method can be applied directly.

PROBLEMS

1. Solve the following assignment problems:

(*a*)

	A	B	C	D	E
I	12	20	20	18	17
II	20	12	5	11	8
III	20	5	12	5	9
IV	18	11	5	12	10
V	17	8	9	10	12

(*b*)

Jobs

	J1	J2	J3	J4	J5	J6
A	18	10	25	10	11	22
B	20	15	21	–	20	18
C	11	17	19	15	18	17

Persons	D	18	16	20	16	20	21
	E	20	–	21	15	11	17
	F	11	15	19	12	15	20

NOTES

2. A machine tool decides to make six sub-assemblies through six contractors A, B, C, D, E and F. Each contractor is to receive only one sub-assembly from A1, A2, A3, A4, A5 and A6. But the contractors C and E are not competent for the A4 and A2 assembly respectively. The cost of each subassembly by the bids submitted by each contractor is shown below (in hundred rupees) :

	A1	A2	A3	A4	A5	A6
A	15	10	11	18	13	22
B	9	12	18	10	14	11
C	9	15	11	–	22	11
D	14	13	9	12	15	10
E	10	–	11	22	13	18
F	10	14	15	12	13	14

Find the optimal assignments of the assemblies to contractors so as to minimize the total cost.

3. Five programmers, in a computer centre, write five programmes which run successfully but with different times. Assign the programmers to the programmes in a such a way that the total time taken by them is minimum taking the following time matrix:

Programmes

		P1	P2	P3	P4	P5
	A	80	66	65	65	73
	B	76	75	70	70	75
Programmers	C	74	73	72	70	66
	D	75	75	71	71	73
	E	76	66	66	70	75

4. Consider the problem of assigning seven jobs to seven persons. The assignment costs are given as follows :

Jobs

		I	II	III	IV	V	VI	VII
	A	9	6	12	11	13	15	11
	B	14	13	14	14	10	20	15
	C	18	6	17	11	15	13	11
Persons	D	10	11	12	15	15	14	13
	E	15	6	18	15	10	14	12
	F	9	18	15	20	14	13	11
	G	14	15	12	13	11	17	20

Determine the optimal assignment schedule.

5. Solve the following unbalanced assignment problems :

(*a*) Machines

		M1	M2	M3	M4	M5
	J1	12	15	15	13	14
Jobs	J2	8	14	11	15	8

J3	15	13	11	11	13
J4	8	12	8	11	10

NOTES

(*b*)

Machines

		M1	M2	M3	M4
	J1	16	17	14	19
Jobs	J2	15	16	14	9
	J3	10	15	9	9
	J4	10	13	10	13
	J5	15	9	14	18

6. There are five operators and six machines in a machine shop. The assignment costs are given in the table below :

Machine

		M1	M2	M3	M4	M5	M6
	A	5	–	22	6	8	6
	B	14	9	15	9	14	15
Operator	C	8	12	12	10	8	5
	D	11	13	11	6	9	14
	E	8	9	11	13	–	12

Operator A cannot operate machine M2 and operator E cannot operate machine M5. Find the optimal assignment schedule.

7. A batch of 4 jobs can be assigned to 5 different machines. The setup time for each job on various machines is given below :

Machine

		1	2	3	4	5
	J1	3	9	6	5	6
	J2	4	5	5	7	4
Jobs	J3	5	5	3	4	4
	J4	6	8	4	5	5

Find an optimal assignment of jobs to machines which will minimize the total setup time.

8. A construction company has to move six large cranes from old construction sites to new construction sites. The distances (in miles) between the old and the new sites are given below :

New sites

		A	B	C	D	E
	I	12	9	8	11	7
	II	11	10	8	12	7
Old	III	9	12	7	6	9
sites	IV	9	8	11	10	10
	V	10	9	9	6	11
	VI	11	11	7	8	9

Determine a plan for moving the cranes such that the total distance involved in the move will be minimum.

9. A company wants to assign five salesperson to five different regions to promote a product. The expected sales (in thousand) are given below :

Regions

		I	II	III	IV	V
	S1	27	54	37	100	85
	S2	55	66	45	80	32
Salesperson	S3	72	58	74	80	85
	S4	39	88	74	59	72
	S5	72	66	45	69	85

Solve the above assignment problem to find the maximum total expected sale.

10. A company makes profit (Rs.) while processing different jobs on different machines (one machine to one job only). Now, the company is facing problem of assigning 4 machines to 5 jobs. The profits are estimated as given below :

Job

		J1	J2	J3	J4	J5
	A	21	16	35	42	16
	B	15	20	30	35	15
Machine	C	20	16	30	27	18
	D	15	18	32	27	15

Determine the optimal assignment for maximum total profits.

ANSWERS

1. (*a*) I→A, II→E, III→D, IV→C, V→B, Min. cost = 38.

   (*b*) A→J6, B→J2, C→J1, D→J3, E→J5, F→J4, Min. cost = 91.

2. A→A2, B→A4, C→A1, D→A6, E→A3, F→A5

   A→A2, B→A4, C→A6, D→A3, E→A1, F→A5

   A→A2, B→A4, C→A6, D→A3, F→A5, E→A1

   For each assignment, min. cost = Rs. 6300.

3. A→P3, B→P4, C→P5, D→P1, E→P2

   A→P4, B→P3, C→P5, D→P1, E→P2

   Min. total time = 342 units.

4. A→VII, B→V, C→IV, D→I, E→II, F→VI, G→III

   A→VII, B→V, C→IV, D→VI, E→II, F→I, G→III

   A→I, B→V, C→IV, D→VI, E→II, F→VII, G→III

   Min. total cost = 73.

5. (*a*) J1→M1, J2→M5, J3→M4, J4→M3, M2 is idle.

   Min. total cost = 39.

   (*b*) J2→M4, J3→M3, J4→M1, J5→M2, J1 is not processed.

   Min. total cost = 37.

NOTES

6. A→M5, B→M2, C→M6, D→M4, E→M1, M3 is idle.
   Min. total cost = 36.
7. J1→1, J2→5, J3→3, J4→4
   J1→1, J2→5, J3→4, J4→3
   Min. total time = 15, Machine 2 is idle.
8. I→E, III→A, IV→B, V→D, VI→C
   Min. total distance = 37 miles
   Crane II is not moved.
9. S1→IV, S2→I, S3→III, S4→II, S5→5, Max. total profit = Rs. 40200.
10. A→J4, B→J2, C→J1, D→J3, Job J5 is idle.
    Max. total profit = Rs. 114.

3.11 ROUTING PROBLEMS

There are various types of routing problems which occurs in a network. The most widely discussed problem is the 'Travelling Salesman Problem (TSP)'. Suppose there is a network of n cities and a salesman wants to make a tour *i.e.,* starting from a city 1 he will visit each of the other $(n - 1)$ cities once and will return to city 1. In this tour the objective is to minimize either the total distance travelled or the cost of travelling by the salesman.

(a) Mathematical Formulation

Let the cities be numbered as 1, 2, ..., n and the distance matrix as follows:

$$D = \begin{array}{c} \text{From} \backslash \text{To} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix} \end{array}$$

Generally an infinity symbol is placed in the principal diagonal elements where there is no travelling. So d_{ij} represents the distance from city i to city j $(i \neq j)$. If the cost of travelling is considered then D is referred as cost matrix. It is also to be noted that D may be symmetric in which case the problem is called 'Symmetric TSP' or asymmetric in which case the problem is called 'Asymmetric TSP'.

Let us define the decision variables as follows :

$$x_{ij} = \begin{cases} 1, \text{ if he travels from city } i \text{ to city } j \\ 0, \text{ otherwise} \end{cases}$$

where $i, j = 1, 2, ..., n$

Then the linear programming formulation can be stated as follows :

Minimize $$z = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}\, x_{ij}$$

Subject to, $\sum_{i=1}^{n} x_{ij} = 1, \ j = 1, 2, ..., n$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad i = 1, 2, ..., n$$

$$x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j = 1, 2, ..., n$$

and $x = (x_{ij})$ is a tour.

The above problem has been solved with various approaches *e.g.*, Graph Theoretic Approach, Dynamic Programming, Genetic algorithm etc.

The above problem looks like a special type of Assignment problem. Consider a 4 × 4 assignment problem and a solution as 1 – 4, 2 – 3, 3 – 1, 4 – 2 which can also be viewed as a tour *i.e.*, 1 – 4 – 2 – 3 – 1. If the solution is 1 – 4, 2 – 3, 3 – 2, 4 – 1 then this consists of two sub-tours 1 – 4 – 1, 2 – 3 – 2.

Here one algorithm known as 'Branch and Bound' algorithm is described below :

(b) Branch and Bound Algorithm for TSP

(*i*) Ignoring tour, solve [D] using Hungarian Algorithm. The transformed matrix is denoted as $[D_0]$. If there is a tour, stop, else goto next step while storing the solution in a node denoted by TSP.

(*ii*) Calculate the **evaluation** for the variables in $[D_0]$ whose values are zero *i.e.*, $x_{ij} = 0$ where evaluation means the sum of smallest elements of the *i*-th row and the *j*-th column excluding the (i, j)th entry.

(*iii*) Select the variable with highest evaluation, say x_{ij}. If there is a tie, break it arbitrarily. The variable x_{ij} is called the branching variable.

(*iv*) Create a left branch (TSP1) with $x_{ij} = 0$. To implement this put $d_{ij} = \infty$ in $[D_0]$ *i.e.*, travelling from city *i* to city *j* is restricted.

Set [D] = transformed $[D_0]$ and goto step (*i*).

(*v*) Create a right branch (TSP2) with $x_{ij} = 1$. This means the salesman must visit city *j* from city *i*. To implement this take $[D_0]$ of the parent node. Delete the *i*-th row and *j*-th column and put $d_{ji} = \infty$ (to prevent a subtour).

Set [D] = transformed $[D_0]$ and goto step (*i*).

Note.

(*a*) There may be a situation arises in step (*i*) where further solution is not possible then we shall stop that branch.

(*b*) There may be multiple tours. We shall select the tour with minimum distance or travelling cost.

(*c*) Calculate total distance (TD) from the given [D] which increases with the level of the tree.

Example 14. *Solve the following travelling salesman problem using branch and bound algorithm.*

NOTES

$$D = \begin{array}{c|cccc} \text{From} \backslash \text{To} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & 3 & 6 & 5 \\ 2 & 3 & \infty & 5 & 8 \\ 3 & 6 & 5 & \infty & 2 \\ 4 & 5 & 8 & 2 & \infty \end{array}$$

Solution. Let us apply the Hungarian Algorithm on [D] and obtain the following matrix :

D_0 =

	1	2	3	4
1	∞	0	3	2
2	0	∞	2	5
3	4	3	∞	0
4	3	6	0	∞

The solution is 1 – 2, 2 – 1, 3 – 4, 4 – 3. *i.e.,* there exists two subtours 1 – 2 – 1, 3 – 4 –3. The total distance (TD) = 3 + 3 + 2 + 2 = 10 units.

Then we have to calculate the evaluations for the variables having the value zero in $[D_0]$.

Variable	Evaluation
x_{12}	2 + 3 = 5
x_{21}	2 + 3 = 5
x_{34}	3 + 2 = 5
x_{43}	3 + 2 = 5

Since there are ties in the values, let us select x_{12} as branching variable.

Subproblem TSP1

Let $x_{12} = 0 \Rightarrow$ Put $d_{12} = \infty$ in $[D_0]$ and obtain

D_0 =

	1	2	3	4
1	∞	∞	3	2
2	0	∞	2	5
3	4	3	∞	0
4	3	6	0	∞

⇓

	1	2	3	4
1	∞	∞	1	0
2	0	∞	2	5
3	4	0	∞	0
4	3	1	0	∞

(Apply Hungarian Algorithm)

The solution is 1 – 4, 2 – 1, 3 – 2, 4 – 3 *i.e.,* 1 – 4 – 3 – 2 – 1 which is a tour and TD = 5 + 3 + 5 + 2 = 15 units from [D].

Subproblem TSP2

Let $x_{12} = 1 \Rightarrow$ Delete row 1 and column 2 from $[D_0]$ and put $d_{21} = \infty$ to prevent subtour. The resultant transformed matrix is obtained as follows :

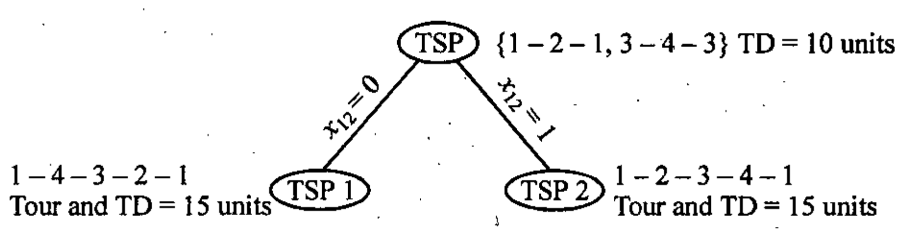
	1	3	4
2	∞	2	5
3	4	∞	0
4	3	0	∞

⇓

	1	3	4
2	∞	⓪	3
3	1	∞	⓪
4	⓪	0	∞

(Applying Hungarian Algorithm)

The solution is 1 – 2, 2 – 3, 3 – 4, 4 – 1 *i.e.,* 1 – 2 – 3 – 4 – 1 which is a tour and TD = 3 + 5 + 2 + 5 = 15 units from [D]. The above calculations is presented in the following tree diagram :



Since, there are two tours with same TD, the given problem has two solutions.

SUMMARY

- Transportation problem (T.P.) is generally concerned with the distribution of a certain commodity/product from several origins/sources to several destinations with minimum total cost through single mode of transportation.
- A BFS of a T.P. is said to be degenerate if one or more basic variables assume a zero value. This degeneracy may occur in initial BFS or in the subsequent iterations of *uv*-method.
- For the degeneracy in initial BFS, arbitrarily we can delete the row due to supply adjusting the demand to zero or delete the column due to demand adjusting the supply to zero whenever there is a tie in demand and supply.
- If total supply ≠ total demand, the problem is called unbalanced T.P.. To obtain feasible solution, the unbalanced problem should be converted to balanced problem by introducing dummy source or dummy destination, whichever is required.

NOTES

NOTES

- This is an efficient algorithm for solving the assignment problem developed by the Hungarian mathematician König. Here the optimal assignment is not affected if a constant is added or subtracted from any row or column of the balanced assignment cost matrix.
- When the objective of the assignment is to maximize, the problem is called 'Max-type assignment problem'. This is solved by converting the profit matrix to an opportunity loss matrix by subtracting each element from the highest element of the profit matrix.
- There are various types of routing problems which occurs in a network. The most widely discussed problem is the 'Travelling Salesman Problem (TSP)'.

REVIEW QUESTIONS

1. There are three sources which store a given product. The sources supply these products to four dealers. The capacities of the sources and the demands of the dealers are given. Capacities $S_1 = 150$, $S_2 = 40$, $S_3 = 80$, Demands $D_1 = 90$, $D_2 = 70$, $D_3 = 50$, $D_4 = 60$. The cost matrix is given as follows :

To

From		D_1	D_2	D_3	D_4
	S_1	27	23	31	69
	S_2	10	45	40	32
	S_3	30	54	35	57

Find the minimum cost of T.P.

2. There are three factories F_1, F_2, F_3 situated in different areas with supply capacities as 200, 400 and 350 units respectively. The items are shipped to five markets M_1, M_2, M_3, M_4 and M_5 with demands as 150, 120, 230, 200, 250 units respectively. The cost matrix is given as follows :

	M_1	M_2	M_3	M_4	M_5
F_1	2	5	6	4	7
F_2	4	3	5	8	8
F_3	4	6	2	1	5

Determine the optimal shipping cost and shipping patterns.

3. Find the initial basic feasible solution to the following T.P. using (*a*) NWC, (*b*) LCM, and (*c*) VAM :

(*i*)

To

From		D	E	F	G	H	
	A	11	7	5	8	9	50
	B	10	11	8	4	5	90
	C	9	6	12	5	5	60
		20	40	20	40	80	

NOTES

(ii)

From \ To	A	B	C	D	E	
I	9	10	0	8	9	90
II	11	12	5	8	3	20
III	4	9	1	2	0	50
IV	8	0	3	5	6	50
	80	60	20	40	10	

4. Solve the following transportation problem :

From \ To	D_1	D_2	D_3	D_4	D_5	
S_1	3	5	2	1	3	45
S_2	2	1	–	4	6	55
S_3	5	4	3	1	2	65
S_4	–	4	6	5	7	50
	27	42	51	62	33	

(Supply from S_2 to D_3 and S_4 to D_1 are restricted)

5. A transportation problem for which the costs, origin and availabilities, destinations and requirements are given below :

	D_1	D_2	D_3	
O_1	2	1	2	40
O_2	9	4	7	60
O_3	1	2	9	10
	40	50	20	

Check whether the following basic feasible solution $x_1 = 20$, $x_{13} = 20$, $x_{21} = 10$, $x_{22} = 50$, and $x_{31} = 10$ is optimal. If not, find an optimal solution.

6. Goods have to be transported from sources S_1, S_2 and S_3 to destinations D_1, D_2 and D_3. The T.P. cost per unit capacities of the sources and requirements of the destinations are given in the following table :

	D_1	D_2	D_3	Capacity
S_1	8	5	6	120
S_2	15	10	12	80
S_3	3	9	10	80
Requirement	150	80	50	

Determine a T.P. schedule so that the cost is minimized.

7. Four products are produced in four machines and their profit margins are given by the table as follows :

NOTES

	P_1	P_2	P_3	P_4	Capacity
M_1	10	7	8	6	40
M_2	5	9	6	4	55
M_3	7	4	11	5	60
M_4	4	10	7	8	45
Requirement	35	42	68	55	

Find a suitable production plan of products in machines so that the profit is maximized while the capacities and requirements are met.

8. Identical products are produced in four factories and sent to four warehouses for delivery to the customers. The costs of transportation, capacities and demands are given as below :

Warehouses

Factories	W_1	W_2	W_3	W_4	Capacities
F_1	9	6	11	5	200
F_2	4	5	8	5	150
F_3	7	8	4	6	350
F_4	3	3	10	10	250
Demands	260	100	340	200	

Find the optimal schedule of delivery for minimization of cost of transportation. Is there any alternative solution? If yes, then find it.

9. Starting with LCM initial BFS, find the optimal solution to the following T.P. problem :

To

From						Supply
	5	1	2	4	3	60
	1	4	2	3	6	55
	4	2	3	5	2	40
	3	5	6	3	7	50
Demands	42	33	41	52	27	

10. A company manufacturing air coolers has two plants located at Mumbai and Kolkata with a weekly capacity of 200 units and 100 units respectively. The company supplies air coolers to its 4 show-rooms situated at Ranchi, Delhi, Lucknow and Kanpur which have a demand of 75, 100, 100 and 30 units respectively. The cost per unit (in Rs.) is shown in the following table:

	Ranchi	Delhi	Lucknow	Kanpur
Mumbai	90	90	100	100
Kolkata	50	70	130	85

Plan the production programmes so as to minimize the total cost of transportation.

NOTES

11.

From \ To	1	2	3	4
1	∞	10	6	4
2	8	∞	5	8
3	7	5	∞	2
4	4	10	2	∞

12.

From \ To	1	2	3	4	5
1	∞	4	9	5	10
2	4	∞	7	6	8
3	10	6	∞	5	4
4	5	6	5	∞	3
5	8	7	4	3	∞

13.

From \ To	1	2	3	4	5
1	∞	5	2	7	4
2	5	∞	3	5	6
3	2	3	∞	4	1
4	7	5	4	∞	3
5	4	6	1	3	∞

ANSWERS

1. $x_{11} = 30, x_{12} = 70, x_{13} = 50, x_{24} = 40, x_{31} = 60, x_{34} = 20.$
   Minimum T.P. cost = Rs. 8190.
2. Solution 1:
   $x_{11} = 150, x_{15} = 50, x_{22} = 120, x_{23} = 80, x_{25} = 200, x_{33} = 150, x_{34} = 200.$
   Solution 2 :
   $x_{11} = 150, x_{15} = 50, x_{22} = 120, x_{23} = 230, x_{25} = 50, x_{34} = 200, x_{35} = 150.$
   Minimum shipping cost = Rs. 3510.
3. (*i*) (*a*) $x_{11} = 20, x_{12} = 30, x_{22} = 10, x_{23} = 20, x_{24} = 40, x_{25} = 20, x_{35} = 60.$
   T.P. cost = Rs. 1260.
   (*b*) $x_{11} = 20, x_{12} = 10, x_{13} = 20, x_{24} = 40, x_{25} = 50, x_{32} = 30, x_{35} = 30.$
   T.P. cost = Rs. 1130.
   (*c*) $x_{12} = 40, x_{13} = 10, x_{21} = 20, x_{23} = 10, x_{24} = 40, x_{25} = 20, x_{35} = 60.$
   T.P. cost = Rs. 1170.
   (*ii*) (*a*) $x_{11} = 80, x_{12} = 10, x_{22} = 20, x_{32} = 30, x_{33} = 20, x_{34} = 0, x_{44} = 40, x_{45} = 10.$
   T.P. cost = Rs. 1610.
   (*b*) $x_{11} = 70, x_{13} = 20, x_{21} = 10, x_{22} = 10, x_{31} = 0, x_{34} = 40, x_{35} = 10, x_{42} = 50.$
   T.P. cost = Rs. 940.
   (*c*) $x_{11} = 60, x_{12} = 10, x_{13} = 20, x_{21} = 10, x_{25} = 10, x_{31} = 10, x_{34} = 40, x_{42} = 50.$
   T.P. cost = Rs. 900.

NOTES

4. Solution 1 :

$x_{13} = 15, x_{14} = 30, x_{21} = 27, x_{22} = 28, x_{34} = 32, x_{35} = 33, x_{42} = 14, x_{43} = 36.$

Solution 2 :

$x_{13} = 45, x_{21} = 27, x_{22} = 28, x_{34} = 32, x_{35} = 33, x_{42} = 14, x_{43} = 6, x_{44} = 30.$

Minimum T.P. cost = Rs. 512.

5. The new optimal solution is $x_{11} = 30, x_{13} = 10, x_{22} = 50, x_{23} = 10, x_{31} = 10.$

Minimum T.P. cost = Rs. 360.

6. $x_{11} = 70, x_{12} = 0, x_{13} = 50, x_{22} = 80, x_{31} = 80.$

Minimum T.P. cost = Rs. 1900.

7. $x_{11} = 35, x_{14} = 5, x_{22} = 42, x_{23} = 8, x_{24} = 5, x_{33} = 60, x_{44} = 45.$

Total Profit = Rs. 1846.

8. Solution 1 :

$x_{14} = 160, x_{21} = 110, x_{24} = 40, x_{33} = 340, x_{41} = 150, x_{42} = 100.$

Solution 2 :

$x_{14} = 200, x_{21} = 110, x_{33} = 340, x_{41} = 150, x_{42} = 100.$

Minimum T.P. cost = Rs. 3550.

9. Solution 1 :

$x_{12} = 33, x_{13} = 27, x_{21} = 42, x_{23} = 11, x_{24} = 2, x_{33} = 3, x_{35} = 27, x_{44} = 50.$

Solution 2 :

$x_{12} = 30, x_{13} = 30, x_{21} = 42, x_{23} = 11, x_{24} = 2, x_{32} = 3, x_{35} = 27, x_{44} = 50.$

Minimum T.P. cost = Rs. 370.

10. $x_{12} = 75, x_{13} = 95, x_{14} = 30, x_{21} = 75, x_{22} = 25.$

Minimum T.P. cost = Rs. 24750.

11. 1 – 4 – 3 – 2 – 1, TD = 19 units.

12. 1 – 4 – 5 – 3 – 2 – 1, TD = 22 units.

13. 1 – 3 – 5 – 4 – 2 – 1 and 1 – 2 – 4 – 5 – 3 – 1, TD = 16 units.

FURTHER READINGS

- **Operations Research:** Col. D.S. Cheema, University Science Press.
- **Introductory Operations Research:** Theory and Applications 3e: Kasana, Springer
- **Operations Research:** N.P. Agarwal, Indus Valley publication
- **Operations Research:** Jaya Banerjee, Shree Niwas
- **Operations Research for Library and Information Professionals:** Dariush Alimohammadi, Ess Ess Publications

UNIT IV QUEUING THEORY AND MARKOV CHAIN

NOTES

★ STRUCTURE ★

4.0 LEARNING OBJECTIVES

After going through this chapter, you will be able to:

- discuss the basic elements of queuing model
- explain notations and kendall's notations
- explain queuing models based on birth and Death processes
- describe non-poisson queuing model
- define properties of vectors and matrices
- explain fixed points and regular stochastic matrices
- define markov chains..

4.1 INTRODUCTION

Queuing systems are prevalent throughout society. The formation of waiting lines is a common phenomenon which occurs whenever the current demand for a service exceeds the current capacity to provide that service. Commuters waiting to board a bus, cars waiting at signals, machines waiting to be serviced by a repairman, letters waiting to be typed by a typist, depositors waiting to deposit the money to a counter in a bank provide some examples of queues. There are applications of queuing theory in several disciplines. A schematic diagram of a queuing system is given below :

NOTES

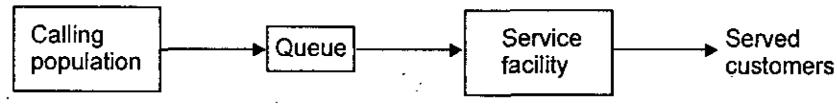


Fig. 4.1 The basic queuing system

4.2 BASIC ELEMENTS OF QUEUING MODEL

The basic elements of a queuing model depend on the following factors :

(*a*) *Arrival's distribution.* Customers arrive and join in the queue according to a probability distribution. The arrival may be single or bulk.

(*b*) *Service-time distribution.* The service offered by the server also follows a probability distribution. The server(s) may offer single or bulk services e.g., one man barber shop, a computer with parallel processing.

(*c*) *Design of service facility.* The services can be offered by the servers in a series, parallel or network stations. A facility comprise a number of series stations through which the customer many pass for service is called 'Tandem queues'. Waiting lines may or may not be allowed between the stations. Similarly parallel queue and network queue are defined.

(*d*) *Service discipline/Queue discipline.* There are three types of discipline e.g.,

FIFO – First In First Out

LIFO – Last In First Out

SIRO – Service in Random Order

'Stack' is an example of LIFO and selling tickets in a bus is an example of SIRO. Sometimes FIFO is referred as GD (*i.e.*, General Discipline).

Also there is priority service which is two types.

Preemptive. The customers of high priority are given service over the low priority customer.

Non-preemptive. A customer of low priority is served before a high priority customer.

(*e*) *Queue Size.* Generally it is referred as length of the queue or line length. Queue size may be finite or infinite (*i.e.*, a very large queue). Queue size along with the server(s) form the capacity of the system.

(*f*) *Calling Population.* It is also called calling source. Customers join in the queue from a source is known as calling population which may be finite or infinite (*i.e.*, a very large number). To reserve a ticket in a railway reservation counter, customers may come from anywhere of a city. Then the population of the city forms the calling population which can be considered as infinite.

(*g*) *Human behaviour.* In a queuing system three types of human behaviours are observed.

NOTES

Jockeying – If one queue is shorter then one join from a larger queue to it.

Balking – If the length of the queue is large, one decides not to enter into it.

Reneging – When a person becomes tired *i.e.*, loses patience on standing on a queue, the person leaves the queue.

4.3 NOTATIONS

P_n = Probability of n customers in a system (steady state)

$P_n(t)$ = Probability of n customers at time t in a system (transient state)

L_s = Expected number of customers in a system

L_q = Expected number of customers in queue

W_s = Expected waiting time in a system (in queue + in service)

W_q = Expected waiting time in queue

λ_n = Mean arrival rate when n customers are in the system.
(If λ_n = Constant for all n, this constant is denoted by λ)

μ_n = Mean service rate for overall system when n customers are in the system.
(If μ_n = Constant for all n, this constant is denoted by μ)

$\rho = \dfrac{\lambda}{\mu}$ = Traffic intensity/utilization factor

s = Number of servers

$N(t)$ = Number of customers in queuing system at time t.

It has been known that $L_s = \lambda W_s$, $L_q = \lambda W_q$. These relations are called 'Little's formula'. Also we have,

$$W_s = W_q + \frac{1}{\mu}, \text{ for } n \geq 1$$

and $$L_s = L_q + \frac{\lambda}{\mu}$$

4.4 KENDALL'S NOTATION

A convenient notation to denote queuing system is as follows :

a/*b*/*c* : *d*/*e*/*f*

where

a = Arrivals' distribution

b = Service time distribution

c = Number of servers or channels

d = Service discipline

e = System capacity

f = Calling population.

NOTES

In the above, a and b usually take one of the following distribution with its symbol:

M = Exponential distribution for inter-arrival or service time and Poisson arrival.

E_k = Erlangian or gamma distribution with parameter k.

D = Constant or deterministic inter-arrival or service time.

G = General distribution (of service time)

Generally f is taken as ∞ (infinity) and it is omitted while representing a queue.

4.5 QUEUING MODELS BASED ON BIRTH-AND-DEATH PROCESSES

The assumptions of the birth-and-death processes are the following :

(*a*) Given $N(t) = n$, ($n = 0, 1, 2, ...$), the current probability distribution of the remaining time until the next arrival is exponential with parameter λ_n.

(*b*) Given $N(t) = n$, ($n = 1, 2, ...$), the current probability distribution of the remaining time until the next service completion is exponential with parameter μ_n.

(*c*) Only one birth or death can occur at a time.

In the queuing models, arrival of a customer implies a birth and departed customer implies a death. In queuing system both arrivals and departures take place simultaneously. This makes difference from birth-and-death process. In the following queuing models mean arrival rate and mean service rate are constant.

4.6 M/M/1 : FIFO/∞ MODEL

In this queuing model, arrivals and departures are Poisson with rates λ and μ respectively. There is one server and the capacity of the system is infinity *i.e.*, very large. We shall derive the steady-state probabilities and other characteristics.

Let $\quad P_n(t)$ = Probability of n arrivals during time interval t

If $h > 0$ and small then

$P_n(t + h)$ = P (n arrivals during t and none during h)

or

P ($n - 1$ arrivals during t and one during h)

or $\quad$ P ($n + 1$ arrivals during t and one departure during h)

$$\left\{\text{For Poisson process, } P[X = n] = \frac{(\alpha t)^n . e^{-\alpha t}}{n!}\right\}$$

Now,

$$P\text{ (zero arrival in } h) = e^{-\lambda h} \approx 1 - \lambda$$

$$P\text{ (one arrival in } h) = 1 - e^{-\lambda h} \approx \lambda h$$

Similarly,

$$P\text{ (zero departure in } h) = he^{-\mu h} \approx 1 - \mu h$$

$$P\text{ (one departure in } h) = 1 - e^{-\mu h} \approx \mu h.$$

Then for $n > 0$, we can write

$$P_n(t + h) = P_n(t).(1 - \lambda h)(1 - \mu h) + P_{n-1}(t)(\lambda h)(1 - \mu h) + P_{n+1}(t)(1 - \lambda h).(\mu h)$$

$$\Rightarrow \quad \frac{P_n(t+h) - P_n(t)}{h} = \lambda P_{n-1}(t) + \mu P_{n+1}(t) - (\lambda + \mu)\, P_n(t).$$

Taking $h \to 0$, we obtain

$$P'_n(t) = \lambda P_{n-1}(t) + \mu P_{n+1}(t) - (\lambda + \mu)\, P_n(t) \qquad ...(1)$$

For $n = 0$, we can write

$$P_0(t + h) = P_0(t).(1 - \lambda h).1 + P_1(t)\,(1 - \lambda h).(\mu h).$$

$$\Rightarrow \quad \frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \mu P_1(t)$$

Taking $h \to 0$, we obtain

$$P'_0(t) = -\lambda P_0(t) + \mu P_1(t) \qquad ...(2)$$

These are called difference-differential equations.

The solution of (1) and (2) will give the transient-state probabilities, $P_n(t)$. But the solution procedure is complex. So with certain assumptions we shall obtain the steady state solution.

For steady state, let us consider

$$t \to \infty, \lambda < \mu,$$

$$P'_n(t) \to 0,\ P_n(t) \to P_n \text{ for } n = 0, 1, 2, ...$$

(Here $\lambda = \mu \Rightarrow$ No queue and $\lambda > \mu \Rightarrow$ explosive state).

From (1) and (2) we obtain,

$$\lambda P_{n-1} + \mu P_{n+1} - (\lambda + \mu)\, P_n = 0, \qquad n > 0 \qquad ...(3)$$

$$-\lambda P_0 + \mu P_1 = 0, \qquad n = 0 \qquad ...(4)$$

From (4),
$$P_1 = \frac{\lambda}{\mu}.P_0.$$

From (3), for $n = 1$,

$$P_2 = \left(\frac{\lambda + \mu}{\mu}\right) P_1 - \frac{\lambda}{\mu} P_0$$

$$= \left(\frac{\lambda + \mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right) P_0 - \left(\frac{\lambda}{\mu}\right) P_0$$

$$= \left(\frac{\lambda}{\mu}\right)^2 P_0$$

Similarly, for $n = 2, 3, ...$

$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

....

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

....

NOTES

Also, we have $$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow \quad P_0 \sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n = 1$$

$$\Rightarrow \quad P_0\left[1-\frac{\lambda}{\mu}\right]^{-1} = 1, \; \frac{\lambda}{\mu}<1$$

$$\Rightarrow \quad P_0 = \frac{1}{1-\frac{\lambda}{\mu}} = \frac{1}{1-\rho}, \; \rho < 1.$$

Therefore, $$P_n = P[X=n] = \left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

$$= P^n\,(1-\rho),\; \rho < 1,\; n \geq 0.$$

Now L_s = Expected number of customers in the system.

$$= \sum_{n=0}^{\infty} n.P_n \qquad = \sum_{n=0}^{\infty} n.\rho^n(1-\rho)$$

$$= (1-\rho)\rho\sum_{n=0}^{\infty} n.\rho^{n-1} = (1-\rho)\rho\frac{d}{d\rho}\left(\sum_{n=0}^{\infty}\rho^n\right)$$

$$= (1-\rho).\rho.\frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) (\because \rho<1)$$

$$= (1-\rho).\rho.\frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$$

W_s = Expected waiting time in the system

$$= \frac{L_s}{\lambda} \text{ (By Little's formula)} = \frac{1}{\mu-\lambda}$$

L_q = Average queue length

$$= L_s - \frac{\lambda}{\mu} = \frac{\rho^2}{1-\rho}$$

W_q = Expected waiting time in queue

$$= \frac{L_q}{\lambda} \text{ (By Little's formula)} = \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}.$$

P (at least n customers in the system)

$$= P \text{ (queue size} \geq n)$$

$$= \sum_{j=n}^{\infty} P_j = \sum_{j=n}^{\infty} \rho^j (1-\rho)$$

$$= (1-\rho)\rho^n \sum_{j=n}^{\infty} \rho^{j-n} = (1-\rho)\rho^n \sum_{k=0}^{\infty} \rho^k \text{ (let } k = j-n)$$

$$= (1-\rho)\rho^n . \frac{1}{1-\rho} = \rho^n.$$

Let m = No. of customers in the queue

$$P[m>0] = P[n>1] = 1 - P[n \le 1]$$

$$= 1 - \{P[n=0] + P[n=1]\}$$

$$= 1 - \{(1-\rho) + \rho(1-\rho)\} = \rho^2.$$

Therefore, **Average length of non-empty queue**

$$= E[m|m>0]$$

$$= \frac{E(m)}{P[m>0]} = \frac{L_q}{P[m>0]} = \frac{\rho^2}{1-\rho} \cdot \frac{1}{\rho^2} = \frac{1}{1-\rho}.$$

Variance of system length/Fluctuation of queue

$$= \sum_{n=0}^{\infty} [n - L_s]^2 P_n = \sum_{n=0}^{\infty} n^2 . P_n - [L_s]^2$$

$$= \frac{\rho}{(1-\rho)^2}, \text{ after simplification.}$$

(a) Waiting Time Distributions

Let the time spent by a customer in the system be given as follows

$$T_s = t'_1 + t_2 + ... + t_n + t_{n+1}$$

where t'_1 is the additional time taken by the customer in service, t_2, ..., t_n are the service times of other customers ahead of him and t_{n+1} is the service time of arriving customer. Here T_s is the sum of $(n+1)$ independently identically exponentially distributed random variables and follows a gamma distribution with parameters μ and $n + 1$. The conditional *pdf* $w(t \mid n+1)$ of T_s is given by

$$w(t \mid n+1) = \frac{\mu}{n!} (\mu t)^n . e^{-\mu t}, \; t > 0.$$

Then the *pdf* of T_s is obtained by first multiplying the expression $w(t \mid n+1)$ with the probability that there are n customers in the system and then summing over all values of n from 0 to ∞ and is given below :

$$pdf \text{ of } T_s = (u - \lambda)\, e^{-(\mu - \lambda)t}, \; t > 0$$

NOTES

which is an exponential distribution with parameter $(\mu - \lambda)$. We can also compute the *pdf* T_q of waiting time of an incoming customer before he receives the service following a similar line of argument. Thus

$$pdf \text{ of } T_q = \begin{cases} \rho(\mu - \lambda)\, e^{-(\mu-\lambda)t}, & t > 0 \\ 1 - \rho & t = 0 \end{cases}$$

The second component means the customer starts receiving service immediately after the arrival if there is no customer in the system.

Also we can obtain $\quad W_s = \int_0^\infty t.T_s\, dt = \dfrac{1}{\mu - \lambda}, \; W_q = \int_0^\infty t.T_q\, dt = \dfrac{\lambda}{\mu(\mu - \lambda)}.$

Example 1. *At a public telephone booth arrivals are considered to be Poisson with an average inter-arrival time of 10 minutes. The length of a phone call may be treated as service, assumed to be distributed exponentially with mean = 2.5 minutes. Calculate the following :*

(*a*) *Average number of customers in the booth*

(*b*) *Probability that a fresh arrival will have to wait for a phone call.*

(*c*) *Probability that a customer completes the phone call in less than 10 minutes and leave.*

(*d*) *Probability that queue size exceeds at least 5.*

Solution. Here $\quad \lambda = \dfrac{1}{10}$ customers per minute.

$\mu = \dfrac{1}{2.5}$ customers per minute.

and $\quad \rho = \dfrac{\lambda}{\mu} = \dfrac{2.5}{10} = 0.25 < 1$

(*a*) Average number of customers in the booth

$$= L_s = \frac{\rho}{1-\rho} = \frac{0.25}{1-0.25} = 0.33$$

(*b*) P (a fresh arrival will have to wait)

= 1 – P (a fresh arrival will not have to wait)

= 1 – P (no customers in the booth)

$= 1 - P[X = 0]$

$= 1 - (1 - \rho) = \rho = 0.25$

(*c*) P (phone call completes in less than 10 min.)

$= P[T_s < 10]$

$$= \int_0^{10} (\mu - \lambda)\, e^{-(\mu-\lambda)t}\, dt = 1 - e^{-(\mu-\lambda).10} = 0.95$$

Example 2. *At a one-man barber shop, customers arrive according to the Poisson distribution with a mean arrival rate of 4 per hour and his hair cutting time was exponentially distributed with an average hair cut taking 12 minutes. There is no restriction in queue length. Calculate the following :*

(*a*) *Expected time in minutes that a customer has to spend in the queue.*

(*b*) *Fluctuations of the queue length.*

(*c*) *Probability that there is at least 5 customers in the system.*

(*d*) *Percentage of time the barber is idle in 8-hr. day.*

Solution.
$$\lambda = 4 \text{ per hour} = \frac{1}{15} \text{ per minute.}$$

$$\mu = \frac{1}{12} \text{ per minute.}$$

$$\rho = \frac{\lambda}{\mu} = \frac{12}{15} = 0.8 < 1$$

(*a*)
$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{1/15}{\frac{1}{12}\left(\frac{1}{12} - \frac{1}{15}\right)} = 48 \text{ minutes}$$

(*b*) Fluctuations of queue length $= \dfrac{\rho}{(1-\rho)^2} = \dfrac{0.8}{(0.2)^2} = 20$

(*c*) P (at least 5 customers in the system) $= \rho^5 = (0.8)^5 = 0.33$

(*d*) P (barber is idle) $=$ P (no customers in the shop)

$= 1 - \rho = 1 - 0.8 = 0.2$

Percentage of time barber is idle $= 8 \times .2 = 1.6$.

Example 3. *In the Central Railway station 15 computerised reservation counters are available. A customer can book the ticket in any train on any day in any one of these counters. The average time spent per customer by each clerk is 5 minutes. Average arrivals per hour during three types of activity periods have been calculated and customers have been surveyed to determine how long they are willing to wait during each type of period.*

Type of period	*Arrivals per hr.*	*Customer's acceptable waiting time*
Peak	*110*	*15 Minutes*
Normal	*60*	*10 Minutes*
Low	*30*	*5 Minutes*

Making suitable assumptions on this queuing process, determine how many counters should be kept open during each type of period.

NOTES

Solution. Assumptions are as follows :

(*i*) Arrivals follow Poisson distribution with average arrival rate λ.

(*ii*) Service time follows exponential distribution with average service rate μ.

(*iii*) The given system can be considered as m different single server queuing system ($m \leq 15$) and there is no jockeying and balking.

Then the arrival rate λ (per hour) = $100/m$ (peak period)

$= 60/m$ (normal period)

$= 30/m$ (low period)

$$\text{The service rate at all the periods} = \frac{60}{5} = 12 \text{ per hour}$$

Peak period :
$$W_q = \frac{15}{60} = \frac{100/m}{12(12-100/m)} \Rightarrow m = 12.22$$

Hence, 13 counters must be kept open to ensure that the average waiting time does not exceed 15 minutes.

Normal period :
$$W_q = \frac{10}{60} = \frac{60/m}{12(12-60/m)} \Rightarrow m = 7.5$$

Hence 8 counters must be kept open during the normal period so that the waiting time does not exceed 10 minutes.

Low period :
$$W_q = \frac{5}{60} = \frac{30/m}{12(12-30/m)} \Rightarrow m = 5$$

Hence, 5 counters must be kept open during the low period so that the waiting time does not exceed 5 minutes.

4.7 M/M/1 : FIFO/N MODEL

In this model the system capacity is restricted to N. Therefore, (N + 1)th customer will not join and the difference-differential equations of the previous model are valid if $n < N$. Then for $n = N$, we have

$$P_N(t+h) = P_N(t)(1-\mu h) + P_{N-1}(t)(\lambda h)(1-\mu h)$$

On simplification, the additional difference-differential equation is obtained as

$$P'_N(t) = -\mu P_N(t) + \lambda P_{N-1}(t).$$

Under steady-state this equation reduces to

$$0 = -\mu P_N + \lambda P_{N-1}.$$

Hence, we have three difference equations

$$\lambda P_{n+1} = (\lambda+\mu)P_n - \lambda P_{n-1},\ 1 \leq n \leq N-1$$

$$\mu P_1 = \lambda P_0,\ n = 0$$

and
$$\mu P_N = \lambda P_{N-1},\ n = N.$$

NOTES

As before, the first two equations give

$$P_n = \left(\frac{\lambda}{\mu}\right)^n . P_0, \; n \le N-1$$

$$\Rightarrow \qquad P_n = \rho^n . P_0.$$

This equation satisfies the third difference equation for $n = N$.

To determine P_0, we use, $\sum_{n=0}^{N} P_n = 1$.

$$\Rightarrow \qquad 1 = P_0 . \sum_{n=0}^{N} \rho^n = \begin{cases} P_0 . \left(\dfrac{1-\rho^{n+1}}{1-\rho}\right), & \rho \ne 1 \\ P_0 (N+1), & \rho = 1 \end{cases}$$

$$\Rightarrow \qquad P_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^{N+1}}, & \rho \ne 1 \\ \dfrac{1}{N+1}, & \rho = 1 \end{cases}$$

Hence

$$P_n = \begin{cases} \dfrac{(1-\rho)\rho^n}{1-\rho^{N+1}}, & \rho \ne 1 \\ \dfrac{1}{N+1}, & \rho = 1. \end{cases}$$

So in this model, ρ can be > 1 or < 1.

$$L_s = \sum_{n=0}^{N} n . P_n$$

For $\rho \neq 1$,

$$L_s = P_0 . \sum_{n=0}^{N} n . \rho^n$$

$$= P_0 . \rho \sum_{n=0}^{N} n . \rho^{n-1} = P_0 . \rho \frac{d}{d\rho}\left(\sum_{n=0}^{N} \rho^n\right)$$

$$= P_0 . \rho \frac{d}{d\rho}\left(\frac{1-\rho^{N+1}}{1-\rho}\right)$$

$$= P_0 . \rho . \frac{(1-\rho)\left(-(N+1)\rho^N\right) - (1-\rho^{N+1})(-1)}{(1-\rho)^2}$$

$$= P_0 . \rho . \frac{1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)^2}$$

$$= \left(\frac{1-\rho}{1-\rho^{N+1}}\right) . \rho . \frac{1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)^2}$$

$$= \frac{\rho\left[1-(N+1)\rho^N + N\rho^{N+1}\right]}{(1-\rho)(1-\rho^{N+1})}$$

NOTES

For $\rho = 1$,

$$L_s = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} n.\frac{1}{(N+1)} = \frac{1}{N+1}.\sum_{n=0}^{N} n$$

$$= \frac{1}{N+1}.\frac{N(N+1)}{2} = \frac{N}{2}.$$

Thus

$$L_s = \begin{cases} \dfrac{\rho(1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)\,(1-\rho^{N+1})}, & \rho \neq 1 \\ \dfrac{N}{2}, & \rho = 1 \end{cases}$$

Let $\bar{\lambda}$ = Effective arrival rate.

Here $\lambda_n = 0$ for $n \geq N$

and

$$\sum_{n=0}^{N} P_n = 1$$

$$\Rightarrow \quad P_N + \sum_{n=0}^{N-1} P_n = 1$$

$$\Rightarrow \quad \sum_{n=0}^{N-1} P_n = 1 - P_N.$$

Therefore,

$$\bar{\lambda} = \sum_{n=0}^{N-1} \lambda_n.P_n = \lambda.\sum_{n=0}^{N-1} P_n \qquad (\because \lambda_n = \lambda)$$

$$= \lambda(1 - P_N).$$

The **other measures** are obtained as follows :

$$L_s = L_q + \frac{\bar{\lambda}}{\mu} \Rightarrow L_q = L_s - \frac{\bar{\lambda}}{\mu}$$

$$W_q = \frac{L_q}{\bar{\lambda}}$$

$$W_s = W_q + \frac{1}{\mu}.$$

All will give two values *i.e.*, one for $\rho \neq 1$ and the other for $\rho = 1$, *e.g.*,

$$L_q = \begin{cases} \dfrac{\rho^2\left[1 - N\rho^{N-1} + (N-1)\,\rho^N\right]}{(1-\rho)\,(1-\rho^{N+1})}, & \rho \neq 1 \\ \dfrac{N(N-1)}{2(N+1)}, & \rho = 1. \end{cases}$$

Example 4. *Assume that the trucks with goods are coming in a market yard at the rate of 30 trucks per day and suppose that the inter-arrival times follows an exponential distribution. The time to unload the trucks is assumed to be exponential with an average of 42 minutes. If the market yard can admit 10 trucks at a time, calculate P (the yard is empty) and find the average queue length.*

If the unload time increases to 48 minutes, then again calculate the above two questions.

Solution. Here
$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \text{ trucks per minute.}$$

and
$$\mu = \frac{1}{42} \text{ trucks per minute.}$$

$$N = 10,\ \rho = \frac{\lambda}{\mu} = \frac{42}{48} = 0.875$$

$$\text{P (yard is empty)} = \text{P (no trucks in the yard)}$$
$$= P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.875}{1-(0.875)^{11}}$$
$$= \frac{0.125}{0.7698} = 0.16$$

$$\text{Average queue length} = \frac{\rho^2\left[1-N\rho^{N-1}+(N-1)\rho^N\right]}{(1-\rho)(1-\rho^{N+1})}$$
$$= \frac{(0.875)^2\left[1-10.(0.875)^9+9.(0.875)^{10}\right]}{(1-0.875)\left(1-(0.875)^{11}\right)}$$
$$= \frac{0.2765}{0.0962} = 2.87.$$

Next part :
$$\mu = \frac{1}{48} \text{ trucks per minute}$$

and
$$\rho = \frac{\lambda}{\mu} = 1$$

$$\text{P (yard is empty)} = P_0 = \frac{1}{N+1} = \frac{1}{11} = 0.09.$$

$$\text{Average queue length} = \frac{N(N-1)}{2(N+1)} = \frac{10 \times 9}{2 \times 11} = 4.09.$$

Example 5. *Cars arrive in a pollution testing centre according to poisson distribution at an average rate of 15 cars per hour. The testing centre can accommodate at maximum 15 cars. The service time (i.e., testing time) per car is an exponential distribution with mean rate 10 per hour.*

NOTES

(a) Find the effective arrival rate at the pollution testing centre.

(b) What is the probability that an arriving car has not to wait for testing.

(c) What is the probability that an arriving car will find a vaccant place in the testing centre.

(d) What is the expected waiting time until a car is left from the testing centre.

Solution.

$$\lambda = 15 \text{ cars/hour}$$

$$\mu = 10 \text{ cars/hour}$$

$$\rho = \frac{\lambda}{\mu} = 1.5, \ N = 15$$

(*a*) $\bar{\lambda} = \lambda\,(1 - P_N) = 15(1 - P_{15}) = 15(1 - 0.333) = 10$ cars/hour.

(*b*) P (arriving car has not to wait for testing)

$$= P_0 = \frac{1 - 1.5}{1 - (1.5)^{16}} = 0.00076.$$

(*c*) $P_0 + P_1 + ... + P_{14} = 1 - P_{15} = 1 - 0.333 = 0.667.$

(*d*)
$$L_s = \frac{\rho\left[1 - (N+1)\,\rho^N + N\rho^{N+1}\right]}{(1-\rho)\,(1-\rho^{N+1})} = 13.016$$

$$\therefore \quad W_s = \frac{L_s}{\lambda} = 1.301 \text{ hours}.$$

4.8 M/M/S : FIFO/∞ MODEL

In this model, the arrival rate of the customers is λ, but maximum of s customers can be served simultaneously.

If μ be the average number of services per unit time per server, then we have

$$\lambda_n = \lambda, \ n = 0, 1, 2, ...$$

$$\mu_n = \begin{cases} n\mu, & 0 \le n \le s \\ s\mu, & n \ge s \end{cases}$$

When $n < s$, there is no queue $\rho = \dfrac{\lambda}{\mu}$.

In this model the condition of existence of steady state solution is $\dfrac{\rho}{s} < 1$.

The **steady-state probabilities** are obtained as

$$P_n = \begin{cases} \dfrac{\rho^n}{n!}.P_0, & 0 \le n \le s \\ \dfrac{\rho^n}{s^{n-s}\,s!}.P_0, & n \ge s \end{cases}$$

where $$P_0 = \left[\sum_{n=0}^{s-1}\frac{\rho^n}{n!} + \frac{\rho^s}{s!(1-\rho/s)}\right]^{-1}$$

$$L_q = \sum_{n=s}^{\infty}(n-s)P_n$$

$$= \sum_{n=s}^{\infty}(n-s)\frac{\rho^n}{s^{n-s}.s!}.P_0 = \frac{\rho^s}{s!}.P_0.\sum_{i=0}^{\infty} i.\left(\frac{\rho}{s}\right)^i \qquad (\text{let } i = n-s)$$

$$= \frac{\rho^{s+1}}{s.s!}.P_0.\frac{d}{du}\left(\sum_{i=0}^{\infty}u^i\right) \qquad (\text{let } u = \rho/s)$$

$$= \frac{\rho^{s+1}}{s.s!}.P_0.\frac{d}{du}\left(\frac{1}{1-u}\right) = \frac{\rho^{s+1}}{s.s!}P_0.\frac{1}{(1-u)^2}$$

$$= \frac{\rho^{s+1}}{s.s!}.\frac{1}{\left(1-\frac{\rho}{s}\right)^2}P_0 = \frac{\rho^{s+1}}{(s-1)!.(s-\rho)^2}.P_0.$$

The **other measures** are obtained as follows :

$$L_s = L_q + \rho,\ W_q = L_q/\lambda,\ W_s = W_q + \frac{1}{\mu}.$$

Expected number of customers in the service = ρ

Expected time for which a server is busy = $\frac{\rho}{s}$

Expected time for which a server is idle = $1-\frac{\rho}{s}$

$$P\ (\text{all servers are busy}) = P_s + P_{s+1} + \ldots$$

$$= \sum_{n=s}^{\infty}\frac{\rho^n}{s!.s^{n-s}}.P_0 = \frac{\rho^s}{s!}\left(1-\frac{\rho}{s}\right)^{-1}.P_0.$$

P (an arrival has to wait) = P (all servers are busy).

Example 6. *A post office has two counters, which handles the business of money orders, registration letters etc. It has been found that the service time distributions for both the counters are exponential with mean service time of 4 minutes per customer. The customers are found to come in each counter in a Poisson fashion with mean arrival rate of 11 per hour. Calculate*

(a) Probability of having to wait for service of a customer.

(b) Average waiting time in the queue.

(c) Expected number of idle counters.

Solution. Here λ = 11 customers/hour.

$\mu = \frac{60}{4} = 15$ customers/hour.

$s = 2.$

NOTES

$$\rho = \frac{\lambda}{\mu} = \frac{11}{15},\ \frac{\rho}{s} = \frac{11}{30} < 1.$$

$$P_0 = \left[\sum_{n=0}^{1} \frac{\rho^n}{n!} + \frac{\rho^2}{2!.(1-\rho/2)}\right]^{-1};\quad P_1 = \rho.P_0 = 0.34$$

$$= \left[1 + \frac{11}{15} + \frac{(11/15)^2}{2.(1-11/30)}\right]^{-1} = 0.463.$$

(*a*) P (an arrival has to wait) = P (all servers are busy)

$$= \left(\frac{11}{15}\right)^2 . \frac{1}{2!}\left(1 - \frac{11}{30}\right)^{-1} . (0.463) = 0.197.$$

(*b*)

$$L_q = \frac{\rho^{s+1}}{(s-1)!.(s-\rho)^2}.P_0$$

$$= \frac{(11/15)^3}{(2-11/15)^2}.(0.463) = 0.114.$$

$\therefore$ Average waiting time in queue $(W_q) = \frac{L_q}{\lambda} = \frac{0.114}{11} = 0.01$ hour $= 0.62$ min.

(*c*) Expected number of idle counters

$$= 2.P_0 + 1.P_1 = 1.266.$$

4.9 M/M/S : FIFO/N, S ≤ N MODEL

This is called *s*-server model with finite system capacity.

$$\text{Arrival rate } \lambda_n = \begin{cases} \lambda, & 0 \le n < N \\ 0, & n \ge N \end{cases}$$

$$\text{Service rate } \mu_n = \begin{cases} n\mu, & 0 \le n \le s \\ s\mu, & s \le n \le N \end{cases}$$

$$\rho = \frac{\lambda}{\mu}.$$

The **steady-state probability** are given as

$$P_n = \begin{cases} \dfrac{\rho^n}{n!}.P_0 & 0 \le n \le s \\ \dfrac{\rho^n}{s!.s^{n-s}}.P_0 & s \le n \le N \end{cases}$$

where

$$P_0 = \left[\sum_{i=0}^{s-1} \frac{\rho^i}{i!} + \sum_{i=s}^{N} \frac{\rho^i}{s!.s^{i-s}}\right]^{-1}$$

The **other characteristics** of this model are given below :

$$L_q = \sum_{n=s}^{N}(n-s).P_n = P_0.\frac{\rho^{s+1}}{(s-1)!.(s-\rho)^2}\left\{1 - x^{N-s} - (N-s)x^{N-s}(1-x)\right\}$$

where $$x = \frac{\rho}{s}$$

Expected number of idle servers $\bar{s} = \sum_{k=0}^{s}(s-k).P_k, k = \text{arrival}$

$$\bar{\lambda} = \lambda(1 - P_N) = \mu(s - \bar{s})$$

$$W_q = \frac{L_q}{\bar{\lambda}}$$

$$W_s = W_q + \frac{1}{\mu}$$

$$L_s = L_q + (s - \bar{s}).$$

Expected number of busy servers = Expected number of customers in service

$$= s - \bar{s} = \frac{\bar{\lambda}}{\mu}$$

Proportion of busy time for a server $= \dfrac{s - \bar{s}}{s}$.

Special Cases : (I) M/M/*s* : FIFO/*s*

In this model $s = N$, the steady state probabilities are given by

$$P_n = \frac{\rho^n}{n!}.P_0, \quad 0 \le n \le s$$

$$= 0, \quad n > s$$

where $$P_0 = \left[\sum_{n=0}^{s}\left(\frac{\rho^n}{n!}\right)\right]^{-1}$$

$$L_s = \sum_{n=1}^{s} n.P_n = P_0 \sum_{n=1}^{s}\frac{\rho^n}{n!-1}, \quad L_q = W_q = 0.$$

(II) M/M/∞ : FIFO/∞ (Self-service queuing model)

In this model a customer joining the system becomes a server. So this is called self-service system. The steady state probabilities are given by

$$P_n = \frac{e^{-\rho}.\rho^n}{n!}, n = 0, 1, 2, ... \text{(Poisson distribution)}$$

$$\bar{\lambda} = \lambda,\ L_s = \rho,\ W_s = \frac{L_s}{\lambda} = \frac{1}{\mu}$$

$$L_q = 0,\ W_q = 0.$$

NOTES

Example 7. *A barber shop has two barbers and four chairs for customers. Assume that customers arrive in a Poisson fashion at a rate of 4 per hour and that each barber services customers according to an exponential distribution with mean of 18 minutes. Further, if a customer arrives and there are no empty chairs in the shop he will leave. Calculate the following :*

(a) Probability that the shop is empty.

(b) Effective arrival rate.

(c) Expected number of busy servers.

(d) Expected number of customers in queue.

Solution. Here, $s = 2$, $N = 4$, $\lambda = \frac{4}{60} = \frac{1}{15}$ customers per minute

$$\mu = \frac{1}{18} \text{ customers per minute.}$$

$$\rho = \frac{\lambda}{\mu} = \frac{18}{15} = 1.2,\ \frac{\rho}{s} = 0.6$$

$$P_0 = \left[1 + \rho + \sum_{i=2}^{4} \frac{\rho^i}{2!.(2)^{i-2}}\right]^{-1} = [3.6112]^{-1} = 0.28$$

$$P_1 = \rho.P_0 = 0.34,\ P_4 = \frac{\rho^4.P_0}{2!.2^2} = 0.07, P_3 = 0.12$$

(*a*) P (shop is empty) $= P_0 = 0.28$

(*b*) $$\bar{\lambda} = \lambda(1 - P_4) = \frac{1}{15}(1 - 0.07) = 0.062$$

(*c*) Expected no. of busy servers $= \frac{\bar{\lambda}}{\mu} = 18 \times 0.062 = 1.116$

(*d*) $$L_q = \sum_{n=2}^{4}(n-2).P_n = 1.P_3 + 2.P_4$$

$$= 0.12 + 2.(0.07) = 0.26.$$

4.10 NON-POISSON QUEUING MODELS

In these queuing models, either arriving time distribution or service time distribution or both does not follow Poisson distribution. The results of two such models are summarize as follows:

10(*a*) M/E$_k$/1 : FIFO/∞ Model

This queuing model has single server and Poisson input process with mean arrival rate λ. However the service time distribution is Erlang distribution with *k*-phases. The density function of Erlang distribution is given as

$$f(t) = \frac{(\mu k)^k}{(k-1)!} t^{k-1} . e^{-k\mu t}, t \geq 0$$

where μ and *k* are positive parameters and *k* is integer.

In this *k* phases of service, a new customer enters the service channel if the previous customer finishes the *k*-phases of the service. The **characteristics** of this model are listed below :

If λ = mean arrival rate, $\frac{1}{k\mu}$ = Expected service time of each phase, then

(*i*) $P_0 = 1 - k\rho$

(*ii*) $W_q = \frac{k+1}{2k} . \frac{\lambda}{\mu(\mu - \lambda)}$

(*iii*) $W_s = W_q + \frac{1}{\mu}$

(*iv*) $L_s = \lambda . W_s$

(*v*) $L_q = \lambda . W_q$

10(*b*) M/G/1 : FIFO/∞ Model

This queuing model has single server and Poisson input process with mean arrival rate λ. However there is a general distribution for the service time whose mean and variance are $1/\mu$ and σ^2 respectively. For this system the steady state condition is $\rho = \frac{\lambda}{\mu} < 1$. The characteristics of this model are given below :

(*i*) $P_0 = 1 - \rho$

(*ii*) $L_q = \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)}$

(*iii*) $L_s = L_q + \rho$

(*iv*) $W_q = \frac{L_q}{\lambda}$

(*v*) $W_s = W_q + \frac{1}{\mu}$.

Example 8. *A barber shop with a one-man takes exactly 20 minutes to complete one haircut. If customers arrive in a Poisson fashion at an average rate of 2 customers per hour calculate the average waiting time in the queue and expected number of customers in the shop.*

Solution. $\lambda = 2$ customers/hour $= \frac{1}{30}$ customers/min.

$$\mu = \frac{1}{20} \text{ customers/min.}$$

$$\rho = \frac{\lambda}{\mu} = \frac{2}{3} < 1.$$

NOTES

Since service time is constant, we can take $k = \infty$.

$$\therefore \quad W_q = \underset{k \to \infty}{\text{Lt}} \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1}{2} \cdot \frac{\lambda}{\mu(\mu-\lambda)} = 20 \text{ min.}$$

and $$L_s = \lambda . W_s = \lambda\left(W_q + \frac{1}{\mu}\right) = \frac{1}{30}(20+20) = 1.33.$$

4.11 PROPERTIES OF VECTORS AND MATRICES

Properties of vectors and matrices are reviewed here because of its use in Markov chains.

The vector v means an n-tuple of numbers.

$$v = (v_1, v_2, \ldots\ldots, v_n).$$

The v_i are called the components of v. If $v_i = 0$, then v is called the zero vector. If we multiply v by a scalar real number k it becomes kv.

Thus $$kv = (kv_1, kv_2, \ldots\ldots, kv_n).$$

The conditions that two vectors are equal are that the corresponding components are equal *i.e.*,

If $$u = (u_1, u_2, \ldots\ldots, u_n)$$

and $$v = (v_1, v_2, \ldots\ldots, v_n)$$

for $u = v$, we get

$$u_1 = v_1, u_2 = v_2, \ldots\ldots, u_n = v_n.$$

A matrix A is the rectangular array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad \ldots(1)$$

The $(a_{11}, a_{12}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots\ldots, (a_{m1}, a_{m2}, \ldots, a_{mn})$ are called rows of A.

And $\begin{pmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{pmatrix}, \ldots\ldots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{pmatrix}$ are called columns of A.

a_{ij} is the ij-entry which appears in the ith row and the jth column. Thus, $A = (a_{ij})$. The equation (1) is $m \times n$ matrix *i.e.*, m rows by n columns. For $m = n$ the matrix is called n-square matrix. Note that a matrix having only one row may be viewed as a vector, and vice-versa. The multiplication of A and B is possible when the numbers of column of A is equal to the number of rows of B.

Thus A is an $m \times p$ matrix and B is an $p \times n$ matrix.

Their product AB is $m \times n$ matrix whose ij-entry can be obtained by multiplying the elements of the ith row of A by the corresponding elements of the jth column of B.

$$\begin{pmatrix} a_{11} & \dots & a_{1p} \\ \dots & \dots & \dots \\ a_{i1} & \dots & a_{ip} \\ a_{m1} & \dots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ b_{p1} & \dots & b_{pj} & \dots & b_{pn} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \dots & \dots & \dots \\ \dots & c_{ij} & \dots \\ c_{m1} & \dots & c_{mn} \end{pmatrix}$$

Where $c_{ij} = a_{i2}\, b_{1j} + a_{i2} b_{2j} + \dots\dots + a_{ip} b_{pj} = \sum_{k=1}^{p} a_{ik}\, b_{kj}$

If A is an n-square matrix, then

$$A^2 = AA,\ A^3 = AA^2,\ A^4 = AA^3, \dots$$

In addition, if v is a vector of n components, then the product $v\,A$ is also a vector of n componets. For $v \neq 0$ is a fixed vector (or fixed point) of A, if v is "left fixed", *i.e.*, is not changed, when multiplied by A *i.e.*,

$$vA = v$$

For scalar $k \neq 0$, we have

$$(kv)\,A = k(vA) = kv.$$

Example 9.

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} ra_1 + sb_1 & ra_2 + sb_2 & ra_3 + sb_3 \\ ta_1 + ub_1 & ta_2 + ub_2 & ta_3 + ub_3 \end{pmatrix}$$

Example 10.

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1\times1 + 2\times3 & 1\times2 + 2\times4 \\ 3\times1 + 4\times3 & 3\times2 + 4\times4 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

NOTES

Example 11.

$$(1, 2, 3)\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} = (1 + 8 + 21,\ 2 + 10 + 24,\ 3 + 12 + 27)$$

$$= (30, 36, 42).$$

Example 12.

Let $v = (2, -1)$ a fixed point of

$$A = \begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}$$

So
$$vA = (2, -1), \begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix} = (2 \times 2 + (-1)2;\ 2 \times 1 + (-1)3)$$

$$= (4 - 2,\ 2 - 3) = (2, -1) = v$$

Thus, $2v = (4, -2)$ is also a fixed point of A; because

$$(2v)\,A = (+4, -2)\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix} = (4 \times 2 + (-2)2,\ 4 \times 1 + (-2)3)$$

$$= (8 - 4,\ 4 - 6) = (4, -2) = 2v.$$

4.12 PROBABILITY VECTORS, STOCHASTIC MATRICES

Probability Vectors

$v = (v_1, v_2, \ldots\ldots, v_n)$ is called a probability vector if v_i are non-negative and $\sum_{i=1}^{n} v_i = 1$.

Example 13.

If $u = \left(\frac{3}{4}, 0, -\frac{1}{3}, \frac{1}{2}\right)$, $v = \left(\frac{1}{2}, \frac{1}{4}, 0, \frac{3}{2}\right)$ and $w = \left(\frac{1}{8}, \frac{1}{4}, 0, \frac{5}{8}\right)$

Then

- u is not a probability vector because third component *i.e.*, $-\frac{1}{3}$ is negative.
- v is not a probability vector because the sum of its components is greater than 1;
- w is a probability vector because $\frac{1}{8} + \frac{1}{4} + 0 + \frac{5}{8} = 1$ and all components are non-negative.

Stochastic Matrix

NOTES

$$P = (p_{ij}) \text{ is called a stochastic matrix if}$$

$$\sum_{\substack{i=1\\ j=1}}^{i=m,\, j=n} p_{ij} = 1 \text{ and all } p_{ij} \geq 0$$

p_{ij} are probability vectors.

Example 14.

$$u = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \ v = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \text{ and}$$

$$w = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$$

- u is not a stochastic matrix since the entry in the second row and third column is negative.
- v is not a stochastic matrix since the sum of the entries in the first row is less than one.
- w is a stochastic matrix since each row equals to unity and all entries are non-negative.

Theorem of Product of Stochastic Matrices

If A and B are stochastic matrices, then the product AB is also a stochastic matrices.

Thus all powers of A *i.e.*, A^n are stochastic matrices.

Regular Stochastic Matrices

P is said to be regular stochastic matrices if all the entries of some power p^m are non-zero positive.

Example 15.

The stochastic matrix $\quad A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

NOTES

Now
$$A^2 = AA$$
$$= \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
$$= \begin{pmatrix} 0\times 0+1\times\frac{1}{2} & 0\times 1+1\times\frac{1}{2} \\ \frac{1}{2}\times 0+\frac{1}{2}\times\frac{1}{2} & \frac{1}{2}\times 1+\frac{1}{2}\times\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

As power of A *i.e.*, A^2 is having all the elements as non-zero positive numbers, the matrix A is regular.

Example 16.

The stochastic matrix $A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Now $A^2 = \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix}$ and $A^4 = \begin{pmatrix} 1 & 0 \\ \frac{15}{16} & \frac{1}{16} \end{pmatrix}$

As there is a zero in the 1st row of all the powers of A, the stochastic matrix A is not regular.

4.13 FIXED POINTS AND REGULAR STOCHASTIC MATRICES

Theorem 1: *If P be a regular stochastic matrix, then*

(*i*) P has a unique fixed probability vector t, and the components of t are all positive.

(*ii*) the sequence P, P^2, P^3, ..., of powers of P approaches the matrix T whose rows are each the fixed point t;

(*iii*) if p is any probability vector, then sequence of vectors pP, pP^2, pP^3, ... approaches the fixed point t;

Thus p^n approaches T means that each entry of p^n approaches the corresponding entry of T, and pP^n approaches t means that each component of pP^n approaches the corresponding component of t.

Example 17.

The regular stochastic matrix $p = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and the probability vector with two components of $t = (x, 1 - x)$

If $t\ p = t$, we have

$$(x,\ 1-x)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (x,\ 1-x)$$

or $$\left(\frac{1}{2}-\frac{1}{2}x, \frac{1}{2}+\frac{1}{2}x\right) = (x,\ 1-x)$$

Thus $$\frac{1}{2}-\frac{1}{2}x = x \Rightarrow x=\frac{1}{3}$$

and $$\frac{1}{2}+\frac{1}{2}x = 1-x \quad \Rightarrow x=\frac{1}{3}$$

Thus $t = \left(\frac{1}{3}, \frac{2}{3}\right)$ is the unique fixed probability vector of P.

By theorem under section 3, the sequence p, p^2, p^3 approaches the matrix T whose rows are each the vector t.

$$\therefore \quad T = \begin{pmatrix} \frac{1}{3}, & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 0.33 & 0.67 \\ 0.33 & 0.67 \end{pmatrix}$$

and $$p^2 = \begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix}, p^3 = \begin{pmatrix} 0.25 & 0.75 \\ 0.37 & 0.63 \end{pmatrix}$$

$$p^4 = \begin{pmatrix} 0.37 & 0.63 \\ 0.31 & 0.66 \end{pmatrix}, p^5 = \begin{pmatrix} 0.31 & 0.69 \\ 0.34 & 0.66 \end{pmatrix}$$

$\Rightarrow$ approaches to T.

Example 18.

Find the unique fixed probability vector of the regular stochastic matrix.

$$p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Solution:

Method 1: Let the probability vector with three components be

$$t = (x,\ y,\ 1-x-y), \text{ such that } t\,p = t$$

$$\therefore (x,\ y,\ 1-x-y)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (x,\ y,\ 1-x-y)$$

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$$\frac{1}{2} - \frac{1}{2}x - \frac{1}{2}y = x \qquad 3x + y = 1 \qquad x = \frac{1}{5}$$

$$\therefore \quad x + \frac{1}{2} - \frac{1}{2}x - \frac{1}{2}y = y \quad \text{or} \quad x - 3y = -1 \quad \text{or} \quad y = \frac{2}{5}$$

$$y = 1 - x - y \qquad x + 2y = 1$$

Thus $t = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$ is the unique fixed probability vector of P.

Method 2: Let $v = (x, y, z)$ is the fixed probability vector

$$x, y, z \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (x, y, z)$$

or
$$\frac{1}{2}z = x$$

$$x + \frac{1}{2}z = y$$

$$y = z$$

We know that the system has a non-zero solution; hence we can arbitrarily assign a value to one of the unknowns.

Let $z = 2$, so $x = 1$, $y = 2$

$\therefore$ $v = (1, 2, 2)$ is a fixed point of P.

As we know any real multiply of v will also be a fixed probability vector

So $$t = \frac{v}{5} = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right).$$

4.14 MARKOV CHAINS

Let X_1, X_2, ..., be outcomes of a sequence of trials, which satisfy the following two properties:

(*i*) Each outcome belongs to a finite set of outcomes a_1, a_2, ..., a_n called the state space of the system, if the outcome on the *n*th trial is a_i, then we say that the system is in state a_i at time *n* or at the *n*th step.

(*ii*) The outcome of any trial depends at most upon the outcome of the immediately preceding trial and not upon any other previous outcome; with each pair

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of states (a_i, a_j) there is given the probability p_{ij} that a_j occurs immediately after a_i occurs.

Such a stochastic process is called a (finite) Markov chain. The numbers p_{ij}, called the transition probabilities can be arranged in a matrix.

$$p = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & P_{mn} \end{pmatrix} \text{ called the transition matrix.}$$

Thus for ith row $(p_{i1}, p_{i2}, \ldots\ldots, P_{im})$ of transition matrix p there is each state a_i; if the system is in state a_i, then this row vector represents the probabilities of all the possible outcomes of the next trial and so it is a probability vector. Thus we have the following theorem. "The transition matrix P of a Markov chain is a stochastic matrix".

Example 19.

A man either drives his car or catches a train to work each day. Suppose he never goes by train two days in a row; but if he drives to work, then the next day he is just as likely to drive again as he is to travel by train. Construct transition matrix of the Markov chain.

Solution:

Let $\{t$ (train), d (drive)$\}$ is the state space of the system. This is a stochastic process of Markov chain because the outcome on any day depends only on what happened the preceding day. The transition matrix of the Markov chain is

$$\begin{array}{cc} & \begin{array}{cc} t & d \end{array} \\ \begin{array}{c} t \\ d \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

The first row corresponds to the fact that he never goes by the train two days in a row and so he definitely will drive the day after he travels by train.

The second row of the matrix corresponds to the fact that the day after he drives he will drive or go by the train with equal probability.

Higher Transition Probabilities

p_{ij} means the system changes from state a_i to the state a_j in one step: $a_i \to a_j$. The question may arise that what the probability is that the system changes from state a_i to state a_j in exactly n step *i.e.*, $P_{ij}^{(n)}$.

$$a_i \to a_{k_1} \to a_{k_2} \ldots \to a_{k_{n-1}} \to a_j$$

The following theorem answers this question.

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Theorem 1: *Let P be the transition matrix of a Markov chain. Then the n-step transition matrix is equal to the nth power of P; that is,* $p^{(n)} = p^n$.

Now suppose that, at some arbitrary time, the probability that the system is in state a_i, is p_i; we denote these probabilities by the probability vector $p = (p_1, p_2, \ldots\ldots, p_m)$ which is called the probability distribution of the system at that time. In particular, we write

$$p^{(0)} = (p_1^{(0)}, p_2^{(0)}, \ldots\ldots\ldots, p_m^{(0)})$$

as the initial probability distribution, *i.e.*, the distribution when the process beains, and we write

$$p^{(0)} = (p_2^{(n)}, p_2^{(n)}, \ldots\ldots\ldots, p_m^{(n)})$$

as the nth step probability distribution i.e., the distribution after the first n steps and we write the following theorem.

Theorem 2: *Let p be the transition matrix of a Markov chain process. If* $p = (p_i)$ *is the probability distribution of the system at some arbitrary time, then pP is the probability 'distribution of the system one step later and* pP^n *is the probability distribution of the system n steps later'. In particular*

$$p^{(1)} = p^{(0)}p,$$
$$p^{(2)} = p^{(1)}p,$$
$$p^{(3)} = p^{(2)}p,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$p^{(n)} = p^{(0)}p^n$$

Example 20.

Let $$P = \begin{array}{c c} & \begin{array}{cc} t & d \end{array} \\ \begin{array}{c} t \\ d \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

$$P^4 = P^2 . P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{array}{c c} & \begin{array}{cc} t & d \end{array} \\ \begin{array}{c} t \\ d \end{array} & \begin{pmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{pmatrix} \end{array}$$

Thus the probability that the system changes from state t to state d in exactly 4 steps is $\frac{5}{8}$ *i.e.*, $p_{td}^{(4)} = \frac{5}{8}$, similarly $p_{tt}^{(4)} = \frac{3}{8}$, $p_{dt}^{(4)} = \frac{5}{16}$ and $p_{dd}^{(4)} = \frac{11}{16}$.

Now suppose that on the first day of work (see example 19), the man tossed a fair die and drove to work if and only if a 6 appeared. In other words, $p^{(0)} = \left(\frac{5}{6}, \frac{1}{6}\right)$ is the initial probability distribution, then

$$p^{(4)} = p^{(0)}p^{(4)} = \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{pmatrix}$$

$$= \left(\frac{35}{96}, \frac{61}{96}\right)$$

is the probability distribution after 4 days

i.e. $$p_t^{(4)} = \frac{35}{96},\ p_d^{(4)} = \frac{61}{96}.$$

Stationary Distribution of Regular Markov Chains

Let P is the transition matrix of a regular Markov chain. Thus the sequence of n-step transition matrices P^n approaches the matrix T whose rows are each the unique fixed probability vector t of P; hence the probability $p_{ij}^{(n)}$ that a_j occurs for sufficiently large n is independent of the original state a_i and it approaches the component t_j of t. We write the following theorem.

Theorem 3: *If P be the transition matrix of a regular Markov chain, then in the long run, the probability that any state a_j occurs is approximately equal to the component t_j of the unique fixed probability vector t of P.*

Example 21.

$$P = \begin{array}{c c} & \begin{array}{cc} t & d \end{array} \\ \begin{array}{c} t \\ d \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

The unique fixed probability vector of the above matrix is $\left(\frac{1}{3}, \frac{2}{3}\right)$ (see Example 17).

Thus, in the long run, the man will take the train to work $\frac{1}{3}$ of the time, and drive to work the other $\frac{2}{3}$ of the time.

Absorbing States

A state a_i is absorbing if and only if the ith row of the transition matrix P has a 1 on the main diagonal and zeros everywhere else. (The main diagonal of an n-square matrix $A = (a_{ij})$ consists of the components a_{i1}, a_{j1}, ..., a_n.

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Example 22.

$$P = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

The states a_2 and a_5 are each absorbing, since each of the second and fifth rows has a 1 on the main diagonal.

Ergodic Markov Chain

If it is possible to go from one state to another in a finite number of steps, regardless of present state, then the Markov chain is called ergodic Markov chain.

Example 23.

A manufacturing company has a certain piece of equipment that is inspected at the end of each day and classified as just overhauled, good, fair or imperative. If the item is imperative it is overhauled, the procedure that takes one day. Let us denote the four classification as states 1,2,3 and 4 respectively. Assume that the working condition of the equipment follows a Markov process with the following transition matrix.

$$p = \text{Today} \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \text{Tomorrow} \\ \begin{array}{cccc} 1 & 3 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

If costs Rs. 125 to overhaul a machine (including lost time) on the average and Rs. 75 if production is lost if a machine is found imperative, using the steady-state probabilities, compute the expected per day cost of maintenance.

$$[p_1, p_2, p_3, p_4]\,[P] = [P_1, P_2, P_3, P_4]$$

Solving, we get

$$p_1 = p_4,\ p_2 = \frac{3}{4}p_1 + \frac{1}{2}p_2,\ p_3 = \frac{1}{4}p_1 + \frac{1}{2}p_2 + \frac{1}{2}p_3$$

$$p_4 = \frac{1}{2}p_3 \text{ also } p_1 + p_2 + p_3 + p_4 = 1.$$

Simplifying we have

$$p_1 = \frac{2}{11}, p_2 = \frac{3}{11}, p_3 = \frac{4}{11}, p_4 = \frac{2}{11}$$

Hence the expected (average) cost per day of maintenance will be equal to

$$\frac{2}{11} \times 125 + \frac{2}{11} \times 75 = \text{Rs. } 36.36 \textbf{ Ans.}$$

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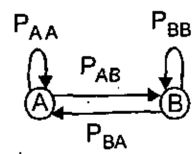
Example 24.

Brand Switching models



The figure shows a brand switching model whose transition matrix is as follows?

$$\begin{array}{cc} & \text{To} \\ & \begin{array}{cc} A & B \end{array} \\ \text{From} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} P_{AA} & P_{AB} \\ P_{BA} & P_{BB} \end{bmatrix} \end{array}$$

SUMMARY

- Queue Size is ref erred as length of the queue or line length. Queue size may be finite or infinite (*i.e.*, a very large queue). Queue size alongwith the server(s) form the capacity of the system.
- Calling Population is also called calling source. Customers join in the queue from a source is known as calling population which may be finite or infinite (*i.e.*, a very large number).
- Human Behaviour is a Queuing System. Three types of human behaviour are observed, Jockeing, Balking and Reneging.

REVIEW QUESTIONS

1. Customers at a box office window, being manned by a single individual arrive according to a Poisson process with a rate of 30 per hr. The time taken to serve a customer has an exponential distribution with a mean of 80 sec. Find the average waiting time of a customer.
2. At a certain Petrol Pump, customers arrive according to a Poisson process with an average time of 6 min. between arrivals. The service time is exponentially distributed with mean time as 3 min. Calculate the following :

(a) What would be the average number of customers in the petrol pump ?

(b) What is the average waiting time of a car before receiving petrol ?

(c) The per cent of time that the petrol pump is idle.

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3. A xerox machine in an office is operated by a person who does other jobs also. The average service time for a job is 6 minutes per customer. On an average, every 12 minutes one customer arrives for xeroxing. Find :

(a) the xerox machine utilisation.

(b) percentage of times that an arrival has not to wait.

(c) average time spent by a customer.

(d) average queue length.

(e) the arrival rate if the management is willing to deploy the person exclusively for xeroxing when the average time spent by a customer exceeds 15 minutes.

4. A repairman is to be hired to repair machines, which breakdown at an average rate of 3 per hour. Breakdowns are distributed in time in a manner that may be regarded as Poisson. Non-productive time on any one machine is considered to cost the company Rs. 5 per hour. The company has narrowed the choice to 2 repairmen—one slow but cheap, the other fast but expensive. The slow-cheap repairman asks Rs. 2 per hour, in return he will service breakdown machine exponentially at an average rate of 4 per hour. The second fast expensive repairman demands Rs. 8 per hour and will repair machines exponentially at an average rate of 5 per hour. Which repairman should be hired ? (Assume a day is 8 hours).

5. A computer manufacturing firm has a troubleshooting station that can replace a computer component in an average time of 3 minutes. Service is provided on a FIFO basis and the service rate is Poisson distributed. Arrival rates are also Poisson distributed with a mean of 18 per hour. Assuming that this is a single channel, single phase system,

(a) What is the average waiting time before a component is replaced?

(b) What is the probability that a component is replaced in less than 10 minutes?

6. A bank has two tellers working on savings accounts. The first teller handles withdrawals only while the second teller on deposits only. For both tellers the service time is exponential with mean service time 3 minutes per customer. Arrivals follow Poisson distribution with mean arrival rate of 16 per hour for depositors and 14 per hour for withdrawers respectively. What would be the effect on the average waiting time for depositors and withdrawers if each teller handles both withdrawals and deposits ?

7. Customers arrive at a bank counter manned by a single person according to a Poisson input process with a mean rate of 10 per hour. The time required to serve a customer has an exponential distribution with a mean of 4 minutes. Find :

(a) the average number in the system.

(b) the probability that there would be 3 customer in the queue.

(c) the probability that the time taken by a customer in the queue is more than 3 minutes.

8. There are two booking clerks at a railway ticket counter. Passengers arrive according to Poisson process with an average rate of 176 per 8 hour day. The mean service time is 5 minutes. Find the idle time of a booking clerk in a day and the average number of customers in the queue.

9. In a cycle repair shop, the inter-arrival times of the customers are exponential with an average time of 10 minutes. The length of service time is assumed to be exponentially distributed with mean 5 minutes. Services are offered by a mechanic. Find the following :

   (*a*) the probability that an arrival will have to wait for more than 10 minutes before getting a service.

   (*b*) the probability that an arrival will have to spent at most 15 minutes in the shop.

   (*c*) the probability that there will be two or more customers in the shop.

10. In a petrol pump vehicles are arriving to buy petrol or diesel according to Poisson distribution with an inter-arrival time of 6 minutes for petrol and 4 minutes for diesel and two different queues are maintained. The service time for both the queues are assumed to be distributed exponentially with an average of 3 minutes. Compare the (*a*) average waiting time in the two queues, (*b*) average length of queues from time to time.

11. In a self service queuing system the customers are coming in a Poisson process with an average inter-arrival time of 8 minutes. The service time is exponential with a mean of 4 minutes. Calculate

    (*a*) probability that there are more than one customer in the system.

    (*b*) average number of customers in the system.

12. Five components of a certain machine needs to be testing by a mechanic for efficiency in production. For machines the testing is done in a Poisson fashion at an average rate of 2 per hour. Mechanic tests the components in a prescribed order. The testing times of all the five components are identical exponential distributions with mean 5 minutes.

    (*a*) Find the average time spent before taking service.

    (*b*) Find the average time the machine remains with the mechanic.

13. Three boys *A*, *B* and *C* are throwing a ball to each other. *A* always throws the ball to *B* and *B* always throws the ball to *C*; but *C* is just as likely to throw the ball to *B* as to *A*. Construct transition matrix of the Markov chain.

14. Show that the following transition matrix is ergodic markov chain.

$$\text{Present states}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\overset{\displaystyle\text{Future states}}{\overset{\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}}{\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}}}$$

15. Suppose there are two market products *A* and *B*. Let each of the brands have exactly 50% of the total market in the same period and the market size is fixed. The transition matrix is as follows.

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$$
\begin{array}{cc}
 & \textit{To} \\
 & \begin{array}{cc} A & B \end{array} \\
\text{From} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}.
\end{array}
$$

If the initial market share break down is 50% for each brand, then determine their market shares in the steady state.

ANSWERS

1. 1.36 mins. per customer in queue and 2.03 mins. per customer in system.
2. (*a*) 1, (*b*) 3 min., (*c*) 50%
3. (*a*) 50% of time, (*b*) $P_0 = 0.5 \Rightarrow$ 50% of time,
   (*c*) W_s = 12 minutes (*d*) $L_q = 0.5$
   (*e*) If $\lambda > 6$ customers per hr., W_s exceeds 15 minutes.
4. Fast mechanic (Total cost = charge of mechanic + downtime cost)
5. (*a*) W_q = 27 min., (*b*) 0.99
6. Before combining, W_q = 12 min. for depositors and 7 min. for withdrawers. After combining, W_q = 3.86 min. for both.
7. (*a*) $L_s = 2$, (*b*) 0.099, (*c*) 0.52
8. Idle time = $\frac{2}{3}$ hrs., $L_q = 9.54$
9. (*a*) 0.18, (*b*) 0.78, (*c*) 0.25
10. (*a*) W_q (petrol) = 0.05, W_q = (diesel) = 0.15,
    (*b*) (petrol) 2, (diesel) 4
11. (*a*) 0.09, (*b*) 0.5
12. (*a*) $M/E_k/1$ (*i*) W_q = 75 min., (*b*) 100 min.

FURTHER READINGS

- **Operations Research:** Col. D.S. Cheema, University Science Press.
- **Introductory Operations Research:** Theory & Applications 3e: Kasana, Springer
- **Operations Research:** N.P. Agarwal, Indus Valley Publication.
- **Operations Research:** Jaya Banerjee, Shree Niwas.
- **Operations Research for Library and Information Professionals:** Dariush Alimohammadi, Ess Ess Publications.

UNIT V CONTROL TECHNIQUES

NOTES

★ STRUCTURE ★

5.0 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- explain notations
- describe purchasing and manufacturing model without shortage
- explain probabilistic models
- describe about network scheduling by CPM/PERT
- explain about time calculations in network
- define decision trees analysis.

5.1 INTRODUCTION

The term inventory may be defined as stock in hand on a given time. Inventories of physical goods are maintained to meet up demands from commercial, governmental

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and military sectors. As the inventory of an item gets depleted to fulfil demands, it needs to be replenished through procurement actions. Different organizations have different inventory problems. Let us discuss the *basic terms* related to inventories.

Demand. It can be categorized according to their size, rate and pattern. Demand size refers to the magnitude of demand and has the dimension of quantity. Demand may be constant from period to period or may be variable. When the demand size is known, then it is called deterministic demand. When the demand size is not known, it is possible in some cases to ascertain its probability distribution and the demand is called probabilistic demand. The demand rate is simply the demand size per unit of time.

Replenishments. The replenishments are usually instantaneous, uniform or batch. Its size refers to the quantity or size of the order to be received into inventory which may be constant or variable.

Lead Time. This is the period between the time an order is placed (administrative lead time) and the time when it is received (delivery lead time). When the lead time is known, it is called deterministic. When it is not known, it can govern by a random variable.

The level of inventory of an item depends upon the length of its lead time.

Costs. The main costs involved in inventory problems are described below :

(*a*) *Manufacturing or Purchase Cost.* The purchase cost of an item is the unit purchase price if it is obtained from an external source. The price breaks (quantity discounts) wherever allowed due to bulk purchasing must be taken into account. Manufacturing cost is the unit production cost when the item is produced internally.

(*b*) *Setup Cost.* This cost is incurred due to setting up of machinery before production.

(*c*) *Ordering Cost.* This cost originates from the expense of issuing a purchase order to the outside supplier.

(*d*) *Holding Cost/Inventory Carrying Cost/Storage Cost.* This cost is associated with investing in inventory and in general, it is directly proportional to the level of inventory and to the time the inventory is maintained.

(*e*) *Shortage Cost/Stockout Cost.* The shortage cost is the economic consequence of an external or internal shortage. An external shortage occurs when a consumer's order is not filled. An internal shortage occurs when an order of a group or department within the organization is not filled. Actually shortage cost is the penalty costs that are incurred as a result of running out of stock. It costs money, less business and goodwill loss.

(*f*) *Selling Price/Revenue Cost/Salvage Cost.* Actually the price and demand are not control of a company. When the company unable to meet the demand and the sale is lost then the revenue cost is included in the company's inventory policies.

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Time Horizon. This is the period over which the inventory is controlled. It can be finite or infinite.

Constraints. These are the limitations placed on the inventory system such as space constraint, capital constraint etc.

Economic Order Quantity. This is also known as 'economic lot size' or EOQ which is the optimum quantity to be purchased or produced such that the total cost of the inventory is minimized.

Re-order Level. The level between the maximum and minimum stock at which purchasing (or manufacturing) activities must start for replenishment is known as re-order level.

Order Cycle. The time period between placement of two successive orders is referred to as order cycle.

The above basic terms highlight the various activities in inventory. Now the **basic needs** of an inventory are given below :

(*a*) It helps to run the business-smooth and efficiently.

(*b*) It gives the advantage of price discounts by bulk purchasing.

(*c*) It takes the advantage of batching and longer runs.

(*d*) It economizes the transportation, clearing and forwarding charges.

In inventory control attempts are made to answer the following **two basic** problems:

(*i*) When should an order to be placed or the production to be run?

(*ii*) How much quantity to be produced or to be ordered for each time interval?

5.2 NOTATIONS

D = total number of units produced or supplied per time period to meet the demand

Q = lot size in each production run/order size

C_s = setup cost per production run/ordering cost

c_1 = holding cost per unit per unit time

c_2 = shortage cost per unit per unit time

r = demand rate

k = production rate

c_A = Average total cost per unit time.

5.3 MODEL I : PURCHASING MODEL WITHOUT SHORTAGE

Assumptions. Demand is known and uniform, purchasing at equal interval, zero lead time, no shortages and instantaneous replenishment.

The corresponding model is shown in Fig. 5.1.

NOTES

Let D = annual demand and c_1 = holding cost/unit/year and time horizon = 1 year.

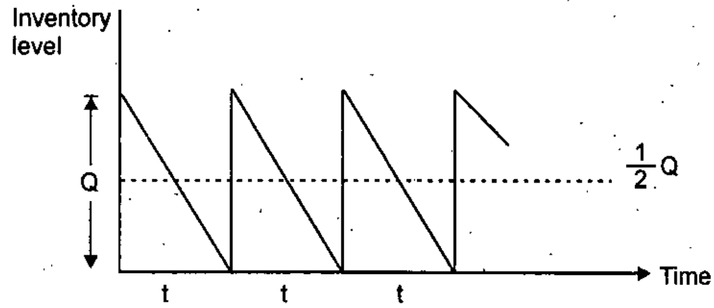


Fig. 5.1 Purchasing model without shortage

Total inventory over the time period t = Area of the first triangle

$$= \frac{1}{2}Q.t$$

$\therefore$ Average inventory at any time $= \left(\frac{1}{2}Q.t\right)\Big/t = \frac{1}{2}Q$

Total cost = Ordering cost + Holding cost + Purchase cost (constant)

$$\text{Minimize TC(Q)} = c_s.\frac{D}{Q} + c_1.\frac{1}{2}Q + \text{Constant}$$

$$\Rightarrow \quad \text{Minimize TC(Q)} = c_s.\frac{D}{Q} + c_1.\frac{1}{2}Q$$

We apply calculus method *i.e.*, $\frac{d\,TC(Q)}{dQ} = 0$ and $\frac{d^2TC(Q)}{dQ^2} > 0.$

$$\therefore \quad \frac{d\,TC(Q)}{dQ} = 0$$

$$\Rightarrow \quad -c_s.\frac{D}{Q^2} + \frac{1}{2}c_1 = 0$$

$$\Rightarrow \quad Q^2 = \frac{2c_sD}{c_1}$$

$$\Rightarrow \quad Q = \sqrt{\frac{2c_sD}{c_1}}$$

Also, $$\frac{d^2TC(Q)}{dQ} = \frac{2c_sD}{Q^3} > 0 \text{ at } Q = \sqrt{\frac{2c_sD}{c_1}}$$

Hence optimum EOQ is $Q^* = \sqrt{\frac{2c_sD}{c_1}}$

$$TC^*(Q) = c_s.D.\sqrt{\frac{c_1}{2c_sD}} + \frac{1}{2}.c_1.\sqrt{\frac{2c_sD}{c_1}}$$

$$\Rightarrow \quad c_A = \sqrt{2c_1c_sD}$$

Time between orders $t^* = \frac{Q^*}{D}$.

n^* = optimum number of orders placed per year

$= \frac{D}{Q^*}$.

Note. If the holding cost is given as a percentage of average value of inventory held, then total annual holding cost,

$c_1 = c \times I$, where c = unit cost

I = % of the value of the average inventory.

The cost functions are shown in the Fig. 5.2

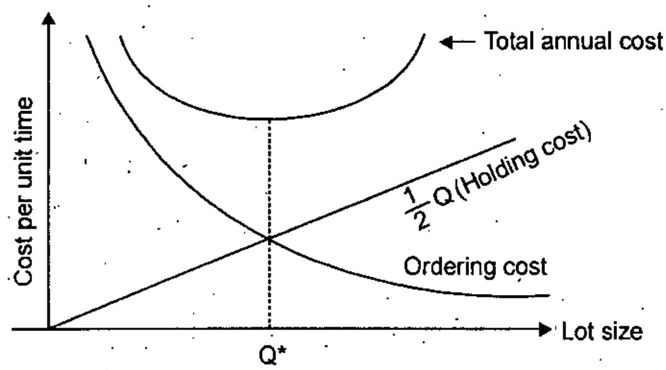


Fig. 5.2 Cost functions of Model I.

Example 1. *A medical wholesaler supplies 30 bottles cough syrup each week to various shops. Cough syrups are purchased from the manufacturer in lots of 120 each for Rs. 1200 per lot. Ordering cost is Rs. 210 per order. All orders are filled the next day. The incremental cost is Rs. 0.60 per year to store a bottle in inventory. The wholesaler finances inventory investments by paying its holding company 2% monthly for borrowed funds. Suppose multiple and fractional lots also can be ordered. How many bottles should be ordered and how frequently he should order?*

Solution. Consider 1 year = 52 weeks as working time.

Annual demand, $D = 30 \times 52 = 1560$.

Unit cost of purchases $= \frac{1200}{120} = \text{Rs.}\,10$

Ordering cost, c_s = Rs. 210.

Inventory carrying cost = 0.6 + 10 × 24/100 = Rs. 3 per unit/year.

$$\therefore \quad Q^* = \sqrt{\frac{2Dc_s}{c_1}} = \sqrt{\frac{2 \times 1560 \times 210}{3}} = 467.33$$

$$t^* = \frac{Q^*}{D} = \frac{467.33}{1560} = 0.3 \text{ year}.$$

NOTES

Example 2. *A company purchases in lots of 500 items which is a 3 month supply. The cost per item is Rs. 50 and the ordering cost is Rs. 100. The inventory carrying cost is estimated at 20% of unit value. What is the total cost of the existing inventory policy? How much money could be saved by employing the economic order quantity?*

Solution. Given c_s = Rs. 100

Number of items per order = 500

$\therefore$ Annual demand, D = 500 × 4 = 2000.

c_1 = Procurement price × inventory carrying cost per year

= 50 × 0.20 = Rs. 10

Total annual cost of the existing inventory policy

$$= \frac{D}{Q}.c_s + \frac{Q}{2}.c_1 = \frac{2000}{500} \times 100 + \frac{500}{2} \times 10 = \text{Rs. } 2900$$

Now,
$$Q^* = \sqrt{\frac{2Dc_s}{c_1}} = \sqrt{\frac{2 \times 2000 \times 100}{10}} = 200$$

Then the corresponding annual cost

$$= \frac{2000}{200} \times 100 + \frac{200}{2} \times 10 = \text{Rs. } 2000.$$

Hence, by employing the economic order quantity, the company may save Rs.(2900 – 2000) = Rs. 900.

5.4 MODEL II : PURCHASING MODEL WITH SHORTAGE

Assumptions. All the assumptions of model I except shortage occurs here. Backlogs due to shortage to be met with penalty.

One inventory cycle is given in the Fig. 5.3.

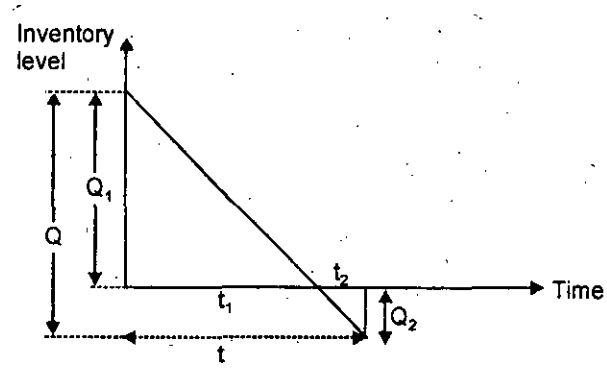


Fig. 5.3 Purchasing model with shortage

During time period t_1 inventory exhaust and during time period t_2 shortages developed.

Here Q_1 = Actual inventory in hand.

Q_2 = Shortage/Stock out

NOTES

$$Q = Q_1 + Q_2;\ t = t_1 + t_2 = \text{cycle time.}$$

$$\text{Total cost} = \text{Holding cost} + \text{Ordering cost} + \text{Shortage cost}$$

The optimum values are given as

$$Q^* = \sqrt{\frac{2c_sD}{c_1} \cdot \frac{c_1 + c_2}{c_2}}$$

$$Q_1^* = \sqrt{\frac{2c_sD}{c_1} \cdot \frac{c_2}{c_1 + c_2}}$$

$$Q_2^* = Q^* - Q_1^*$$

$$t^* = \frac{Q^*}{D},\ n^* = \frac{D}{Q^*},\ t_1^* = \frac{Q_1^*}{D},\ t_2^* = \frac{Q_2^*}{D}.$$

$$\text{Total optimum cost} = \sqrt{2Dc_sc_1 \cdot \frac{c_2}{c_1 + c_2}}.$$

Example 3. *The demand for a certain item is 50 units per year. Unsatisfied demand causes a shortage cost of Rs. 0.45 per unit per short period. The ordering cost for purchase is Rs. 20 per order and the holding cost is 15% of average inventory valuation per year. Item cost is Rs. 5 per unit. Find the EOQ, the shortage inventory and the minimum cost.*

Solution.

$$D = 50 \text{ units/year}$$

$$c_2 = \text{Rs. } 0.45\text{/unit/shortage period.}$$

$$c_s = \text{Rs. } 20$$

$$c_1 = 5 \times 0.15 = \text{Rs. } 0.75\text{/unit/year.}$$

$$Q^* = \sqrt{\frac{2c_sD}{c_1} \cdot \frac{c_1 + c_2}{c_2}} = \sqrt{\frac{2 \times 20 \times 50}{0.75} \cdot \frac{0.75 + 0.45}{0.45}}$$

$$= 84.33 \text{ units.}$$

$$Q_2^* = Q^* - \sqrt{\frac{2c_sD}{c_1} \cdot \frac{c_2}{c_1 + c_2}}$$

$$= 84.33 - 31.62 = 52.71$$

$$\text{Total minimum cost} = \sqrt{2Dc_sc_1 \cdot \frac{c_2}{c_1 + c_2}}$$

$$= \sqrt{2 \times 50 \times 20 \times 0.75 \times \frac{0.45}{1.2}} = \text{Rs. } 23.72.$$

5.5 MODEL III : MANUFACTURING MODEL WITHOUT SHORTAGE

Assumptions. Items are manufactured, shortages are not allowed, demand is uniform, lead time is zero, items are produced and used to meet demand simultaneously for a portion of an inventory cycle.

NOTES

One inventory cycle is illustrated in Fig. 5.4.

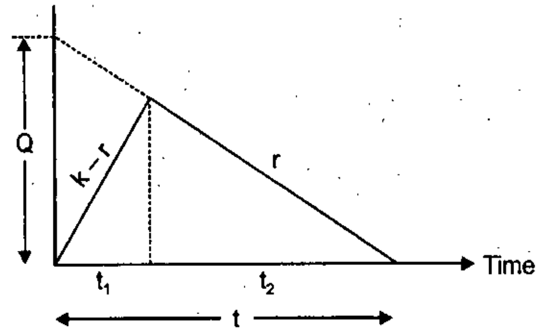


Fig. 5.4 Manufacturing model without shortage

In this model,

(*i*) Inventory is building up at a constant rate of $(k - r)$ units per unit time during t_1.

(*ii*) No production during t_2 and the demand is met at the rate of r per unit of time.

The optimal quantities are obtained as given below :

$$Q^* = \text{EOQ/Economic batch quantity}$$

$$= \sqrt{\frac{2c_s.r}{c_1(1 - r/k)}}$$

$$t_1^* = \text{period of production as well as consumption}$$

$$= \frac{Q^*}{k}$$

$$t_2^* = \text{period of consumption only} = \frac{(k - r)\,t_1^*}{r}$$

$$n^* = \text{optimum number of production runs per year}$$

$$= \frac{r}{Q^*}$$

$$t^* = t_1^* + t_2^*$$

$$\text{Total minimum cost} = \sqrt{2rc_s\,c_1(1 - r/k)}\,.$$

Example 4. *A contractor has to supply 10000 bolts per day to a customer. He finds that during a production run he can produce 20000 bolts per day. The cost of holding bolt in stock for one year is 3 paise and set up cost of a production run is Rs. 20. How frequently should production run be made?*

Solution.

$$r = 10000 \text{ bolts/day}$$

$$k = 20000 \text{ bolts/day}$$

$$c_1 = \text{Rs. } 0.03\text{/bolt/year}$$

$$= \text{Rs. } 0.000082\text{/bolt/day}$$

$$c_s = \text{Rs. } 20\text{/production run.}$$

$$Q^* = \sqrt{\frac{2r\,c_s}{c_1(1 - r/k)}} = \sqrt{\frac{2 \times 10000 \times 20}{0.000082\,(1 - 1/2)}}$$

$$= 98772.96 \approx 98773 \text{ bolts.}$$

$$t^* = \frac{Q^*}{r} = 9.88 \text{ days.}$$

$$\text{Length of production cycle} = \frac{Q^*}{k} = 4.94 \text{ days.}$$

⇒ Production cycle starts at an interval of 9.88 days and production continues for 4.94 days, so that in each cycle a batch of 98773 bolts is produced.

NOTES

5.6 MODEL IV : MANUFACTURING MODEL WITH SHORTAGE

One inventory cycle of this model is given in Fig. 5.5.

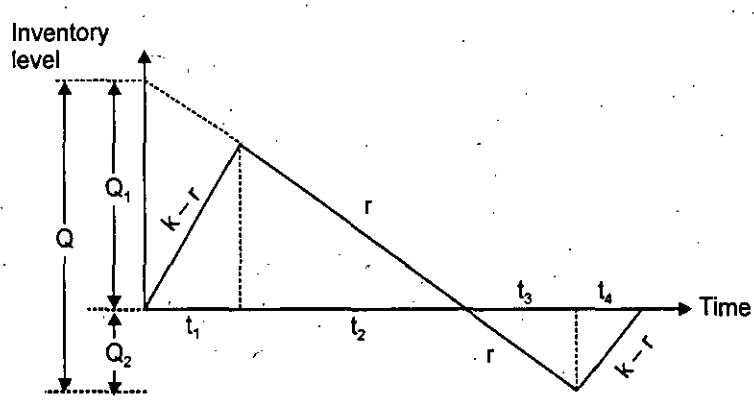


Fig. 5.5 Manufacturing model with shortage

Here during t_1, inventory is built up at the rate of $(k - r)$,

during t_2, inventory is consumed at the rate of r,

during t_3, shortage is building at the rate of r,

during t_4, shortage is being filled at the rate of $(k - r)$.

The optimum quantities are obtained as given below :

$$Q^* = \sqrt{\frac{2c_s(c_1 + c_2)}{c_1 c_2} \cdot \frac{kr}{k - r}}$$

$$Q_1 + Q_2 = \left(1 - \frac{r}{k}\right)Q$$

$$Q_2^* = \text{no. of shortages} = \sqrt{\frac{2c_s\,c_1}{(c_1 + c_2).c_2} \cdot r\left(1 - \frac{r}{k}\right)}$$

$$Q_1^* = \text{maximum inventory level} = \left(1 - \frac{r}{k}\right)Q^* - Q_2^*$$

$$t^* = \text{production cycle time} = \frac{Q^*}{r}$$

$$\text{Manufacturing time} = \frac{Q^*}{k}$$

NOTES

$$t_1^* = \frac{Q_1^*}{k-r}, t_2^* = \frac{Q_1^*}{r}, \; t_3^* = \frac{Q_2^*}{r}, \; t_4^* = \frac{Q_2^*}{k-r}.$$

$$\text{Total minimum production inventory cost} = \sqrt{\frac{2c_1c_2c_sr}{c_1+c_2}.\left(1-\frac{r}{k}\right)}.$$

Example 5. *The demand for an item is 10000 units per year. Its production rate is 1500 units per month. The holding cost is Rs. 20/unit/year and the set up cost is Rs. 800 per set up. The shortage cost is Rs. 1000 per unit per year. Find the EOQ, maximum shortage and total minimum production inventory cost.*

Solution.

$$r = 10000 \text{ units/year}$$
$$k = 1500 \text{ units/month} = 18000 \text{ units/year}$$
$$c_1 = \text{Rs. } 20\text{/unit/year}$$
$$c_s = \text{Rs. } 800$$
$$c_2 = \text{Rs. } 1000\text{/unit/year.}$$

$$Q^* = \sqrt{\frac{2c_s(c_1+c_2)}{c_1c_2}.\frac{kr}{k-r}}$$

$$= \sqrt{\frac{2\times 800\,(20+1000)}{20\times 1000}.\frac{18000\times 10000}{(18000-10000)}} = 1355 \text{ units}$$

$$Q_2^* = \sqrt{\frac{2c_sc_1}{(c_1+c_2)\,c_2}.r\left(1-\frac{r}{k}\right)}$$

$$= \sqrt{\frac{2\times 800\times 20}{(20+1000)\,1000}.10000\left(1-\frac{10000}{18000}\right)} = 11.81 \text{ units.}$$

Total minimum production inventory cost

$$= \sqrt{\frac{2c_1c_2c_sr}{c_1+c_2}.\left(1-\frac{r}{k}\right)}$$

$$= \sqrt{\frac{2\times 20\times 1000\times 800\times 10000}{(20+1000)}.\left(1-\frac{10000}{18000}\right)}$$

$$= \text{Rs. } 11808.20.$$

NOTES

5.7 PROBABILISTIC MODELS

It seldom happens that future demand is not known exactly *i.e.*, uncertain. We assume that the probability distribution of future demand is known which can be determined by some forecasting analysis. Probability distributions can be discrete or continuous. We will discuss some probabilistic models where only demand follows some probability distribution.

(a) Model I (Discrete Case)

Assumptions. Demand is instantaneous, total demand is filled at the beginning of the period, no setup cost, lead time is zero.

Let t = constant interval between orders.

Q = stock (in discrete units) for time t

d = estimated (random) demand with probability $p(d)$.

Since, the total demand is filled at the beginning of the period, the inventory position just after the demand occurs may be either positive (surplus) or negative (shortage).

When the demand does not exceed the stock Q *i.e.*, $d \leq Q$, the holding cost per unit time is

$$= (Q - d).c_1, \qquad d \leq Q$$
$$= 0.c_1, \qquad d > Q.$$

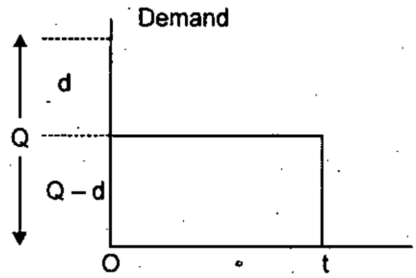


Fig. 5.6 Case d ≤ Q

When the demand d exceeds the stock Q *i.e.*, $d > Q$, the shortage cost per unit time becomes

$$= 0.c_2, \qquad d \leq Q$$
$$= (d - Q).c_2, \qquad d > Q.$$

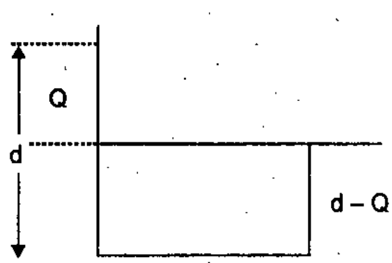


Fig. 5.7 Case d > Q

The total expected cost per unit time is

$$c(\mathrm{Q}) = \sum_{d=0}^{\mathrm{Q}} (\mathrm{Q}-d).c_1 p(d) + \sum_{d=\mathrm{Q}+1}^{\infty} (d-\mathrm{Q}).c_2 p(d)$$

NOTES

For minimum $c(\mathrm{Q})$, the condition

$$\Delta c(\mathrm{Q}-1) < c(\mathrm{Q}) < \Delta c(\mathrm{Q}).$$

must be satisfied.

Now,
$$\Delta c(\mathrm{Q}) = (c_1 + c_2).\sum_{d=0}^{\mathrm{Q}} p(d) - c_2.$$

Using the condition for minimum, we must have

$$\Delta c(\mathrm{Q}) > 0 \qquad i.e., \quad \sum_{d=0}^{\mathrm{Q}} p(d) > \frac{c_2}{c_1+c_2}.$$

Also,
$$\Delta c(\mathrm{Q}-1) < 0 \qquad i.e., \quad \sum_{d=0}^{\mathrm{Q}-1} p(d) < \frac{c_2}{c_1+c_2}$$

Thus combining the optimum value of stock level Q can be obtained from the relationship.

$$\sum_{d=0}^{\mathrm{Q}-1} p(d) < \frac{c_2}{c_1+c_2} < \sum_{d=0}^{\mathrm{Q}} p(d).$$

Example 6. *A newspaper boy buys papers for Rs. 2 and sells them for Rs. 3 each. He cannot return the unsold newspapers. Daily demand has the following distribution*

No. of customers	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*
Probability	*0.02*	*0.02*	*0.06*	*0.15*	*0.15*	*0.25*	*0.15*	*0.1*	*0.05*	*0.05*

If each day's demand is independent of the previous day's, how many papers should be ordered each day?

Solution. Here

$$c_1 = \text{Rs. } 2$$
$$c_2 = \text{Rs } (3-2) = \text{Rs. } 1$$

$$\therefore \quad \frac{c_2}{c_1+c_2} = \frac{1}{2+1} = \frac{1}{3} = 0.33.$$

Now, we have to calculate the cumulative distribution *i.e.*,

No. of customer	22	23	24	25	26	27	28	29	30	31
Cumulative probability	0.02	0.04	0.1	0.25	0.4	0.65	0.8	0.9	0.95	1

Since, the value 0.33 lies between 0.25 and 0.4, the optimal Q^* is taken as 26. *i.e.*, the newspaper boy should order 26 newspaper per day to minimize the cost.

NOTES

(b) Model II (Continuous Case)

Assumptions. The assumptions are same as in the discrete case expect that $f(x)$ will be used as probability density function of demand x.

Proceeding in the same manner as in Model I (discrete case), the cost equation can be formulated as follows :

$$c(Q) = c_1 \cdot \int_0^Q (Q - x) f(x)\, dx + c_2 \cdot \int_Q^\infty (x - Q) f(x)\, dx$$

The optimal value of Q can be obtained from $\frac{dc(Q)}{dQ} = 0$ which gives

$$(c_1 + c_2) \cdot \int_0^Q f(x)\, dx - c_2 = 0$$

$$\Rightarrow \quad \int_0^Q f(x)\, dx = \frac{c_2}{c_1 + c_2}$$

Thus, we can find out the optimum value of Q, satisfying the above equation.

Example 7. *A baking company makes a profit of Rs. 5 per kg. on each kg. cakes sold on the day it is baked. It disposes all cakes not sold on the date, it is baked at a loss of Rs. 1.20 per kg. If demand is known to be rectangular between 2000 and 3000 kg., determine the optimal daily amount to be baked if the demand is instantaneous.*

Solution. Let $c_1 = \text{Rs. } 5,\ c_2 = \text{Rs. } 1.20.$

$$f(x) = \frac{1}{1000},\ 2000 \le x \le 3000.$$

Also $$\frac{c_2}{c_1 + c_2} = \frac{1.2}{6.2} = 0.1935$$

Then $$\int_{2000}^Q f(x)\, dx = \frac{c_2}{c_1 + c_2}$$

$$\Rightarrow \quad \frac{1}{1000} \cdot \int_{2000}^Q dx = 0.1935$$

$$\Rightarrow \quad Q - 2000 = 193.5$$

$$\Rightarrow \quad Q = 2193.5.$$

$\therefore$ The company should bake 2193.5 kg. daily.

(c) Model III (Discrete Case with Uniform Demand)

Compare to Model I, the demand is uniform here rather than instantaneous and the other assumptions are the same.

The optimum stock (*i.e.*, EOQ) Q is obtained by the same relationship *i.e.*,

$$\sum_{d=0}^{Q-1} p(d) < \frac{c_2}{c_1 + c_2} < \sum_{d=0}^{Q} p(d).$$

(*d*) Model IV (Continuous Case with Uniform Demand)

NOTES

Compare to Model III, the demand is uniform here rather than instantaneous and the other assumptions are the same.

Here the optimum stock Q is obtained by the following relationship :

$$\int_0^Q f(x)\,dx + \int_Q^\infty \frac{Q}{x}.f(x)\,dx = \frac{c_2}{c_1 + c_2}.$$

5.8 NETWORK SCHEDULING BY CPM/PERT

Let us define 'Project'.

A project can be considered to be any series of activities and tasks that

(*i*) have a specific objective to be completed within certain specifications.

(*ii*) have defined start and end dates.

(*iii*) have funding limits and consume resources.

A number of techniques have been developed to assist in planning, scheduling and control of projects. The most popular methods are the Critical Path Method (CPM) and the Program Evaluation and Review Technique (PERT). These techniques decompose the project into a number of activities, represent the precedence relationships among activities through a network and then determine a critical path through the network.

The basic concepts are described below :

(*a*) *Activity*

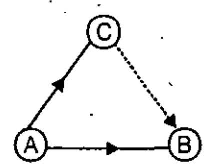
An activity is an item of work to be done that consumes time, effort, money or other resources. It is represented by an arrow. Tail represents start and head represents end of that activity.

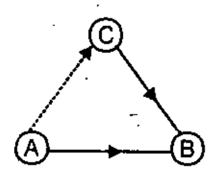
(*b*) *Event/Node*

It represents a point time signifying the completion of an activity and the beginning of another new activity. Here beginning of an activity represents tail event and end of an activity represents head event.

(*c*) *Dummy Activity*

This shows only precedence relationship and they do not represent any real activity and is represented by a dashed line arrow or dotted line arrow and does not consume any time. e.g.,





NOTES

(d) Rules for Construction of a Network

(1) Each activity is shown by one and only one arrow.

(2) There will be only one beginning node/event and only one end node/event.

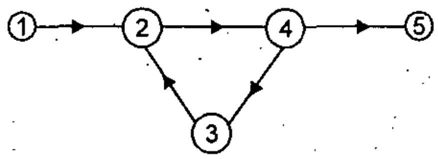
(3) No two activities can be identified by the same head and tail events.

(4) All events/node should be numbered distinctly.

(5) Time flows from left to right.

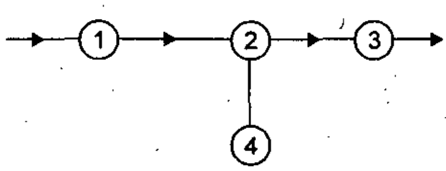
(e) Common Errors in Network

(1) Loops :



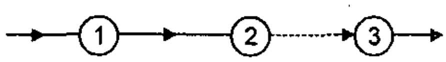
This situation can be avoided by checking the precedence relationship of the activities and by numbering them in a logical order.

(2) Dangling :



This situation can be avoided by keeping in mind that all events except the starting and ending event of the whole project must have at least one entering and one leaving activity. A dummy activity can be introduced to avoid this dangling.

(3) Redundancy :



The dummy activity is redundant and can be eliminated.

(f) Critical Path

It is the longest path in the project network. Any activity on this path is said to be critical in the sense that any delay of that activity will delay the completion time of the project.

5.9 TIME CALCULATIONS IN NETWORK

Let t_{ij} be the duration of an activity (i, j).

(*a*) *Earliest Start Time (ES).* This is the earliest occurrence time of the event from which the activity emanates.

For the beginning event, $ES_1 = 0$ and let ES_i = ES of all the activities emanating from node i. Then

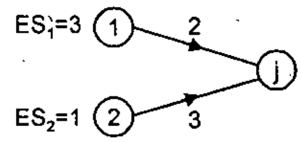
$$ES_j = \max_i \{ES_i + t_{ij}\}$$

(*b*) *Earliest Finish/Completion Time (EF).* This is the ES plus the activity duration

$$EF_i = ES_i + t_{ij}.$$

NOTES

For example, Consider a part of the network.

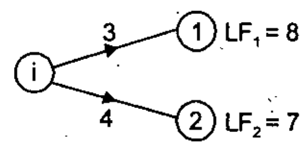


$$ES_j = \text{Max.} \{ES_1 + t_{1j}, ES_2 + t_{2j}\}$$
$$= \text{Max.} \{3 + 2, 1 + 3\} = 5$$
$$EF_1 = 3 + 2 = 5, EF_2 = 1 + 3 = 4.$$

(*c*) *Latest Finish/Completion Time (LF).* This is the latest occurrence time of the event at which the activity terminates.

$$LF_i = \min_j \{LF_j - t_{ij}\}$$

For example, consider a part of the network



Then $$LF_i = \text{Min.} \{LF_1 - t_{i1}, LF_2 - t_{i2}\}$$
$$= \text{Min.} \{8 - 3, 7 - 4\} = 3.$$

(*d*) *Latest Start Time (LS_i).* This is the last time at which the event can occur without delaying the completing of the project.

(*e*) *Total Floats (TF).* It is a time duration in which an activity can be delayed without affecting the project completion time.

$\therefore$ $$TF_{ij} = LF_j - ES_i - t_{ij}$$
$$= LF_j - (ES_i + t_{ij})$$
$$= LF_j - EF_{ij}.$$

Also $$TF_{ij} = LS_{ij} - ES_i$$
$$= (LF_j - t_{ij}) - ES_i.$$

(*f*) *Free Floats (FF).* It is a time duration in which the activity completion time can be delayed without affecting the earliest start time of immediate successor activities in the network.

$$EF_{ij} = ES_j - ES_i - t_{ij}$$
$$= ES_j - (ES_i + t_{ij})$$

$$= ES_j - EF_{ij}.$$

An activity (i, j) is said to be critical if all the following conditions are satisfied:

$$ES_i = LF_i,\ ES_j = LF_j,\ ES_j - ES_i = LF_j - LF_i = t_{ij}.$$

Thus any critical activity will have zero total float and zero free float.

NOTES

(g) *Independent Floats.* It is defined as the difference between the free float and the tail slack.

Note. Slack is with reference to an event and float is with respect to an activity. Slack is generally used with PERT and float with CPM, but they may be used interchangeably used.

5.10 CRITICAL PATH METHOD (CPM)

CPM was developed by E.I. duPont in 1957 and was first applied to construction and maintenance of chemical plants. Since then, the use of CPM has grown at a rapid rate. There are computer programs to perform the calculations.

Let the project network be drawn. Then this method consists of two phases calculations. In Phase 1, which is also called *forward pass,* Earliest start times (ES) of all the nodes are calculated.

In Phase 2, which is also called *backward pass*, Latest finish time (LF) of all the nodes are calculated.

These two calculations are displayed in the network diagram in a two chamber boxes. Upper chamber represents LF and the lower one as ES.

The critical activities (*i.e.*, ES = LF) are identified. The critical path is obtained by joining them using double arrow.

Example 8. *A project schedule has the following characteristics :*

Activity	*Time*	*Activity*	*Time*
1–2	*3*	*5–6*	*5*
1–3	*1*	*5–7*	*8*
2–4	*1*	*6–8*	*1*
2–5	*1*	*7–9*	*2*
3–5	*5*	*8–10*	*4*
4–9	*6*	*9–10*	*6*

Draw the project network and find the critical path. Also calculate the total floats and free floats.

Solution.

NOTES

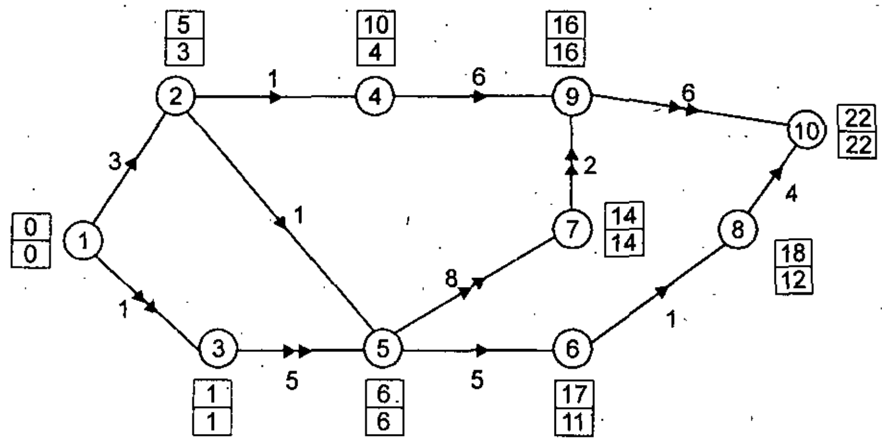


Fig. 5.8

Set $ES_1 = 0$

$ES_2 = ES_1 + t_{12} = 0 + 3 = 3$

$ES_3 = ES_1 + t_{13} = 0 + 1 = 1$

$ES_4 = ES_2 + t_{24} = 4$

$ES_5 = \text{Max. } \{ES_3 + t_{35}, ES_2 + t_{25}\} = \text{Max. } \{6, 4) = 6$

$ES_6 = ES_5 + t_{56} = 11$

$ES_7 = ES_5 + t_{57} = 14$

$ES_8 = ES_6 + t_{68} = 12$

$ES_9 = \text{Max. } \{ES_4 + t_{49}, ES_7 + t_{79}\} = \text{Max. } \{10, 16) = 16$

$ES_{10} = \text{Max. } \{ES_9 + t_{910}, ES_8 + t_{810}\} = \text{Max. } \{22, 16) = 22$

Set. $LF_{10} = ES_{10} = 22$

$LF_9 = LF_{10} - t_{910} = 22 - 6 = 16$

$LF_8 = LF_{10} - t_{810} = 22 - 4 = 18$

$LF_7 = LF_9 - t_{79} = 16 - 2 = 14$

$LF_6 = LF_8 - t_{68} = 17$

$LF_5 = \text{Min. } \{LF_7 - t_{57}, LF_6 - t_{56}\} = \text{Min. } \{6, 12\}. = 6.$

$LF_4 = LF_9 - t_{49} = 10$

$LF_3 = LF_5 - t_{35} = 1$

$LF_2 = \text{Min. } \{LF_4 - t_{24}, LF_5 - t_{25}\} = \text{Min. } \{9, 5\} = 5.$

$LF_1 = \text{Min. } \{LF_3 - t_{13}, LF_2 - t_{12}\} = \text{Min. } \{0, 2\} = 0.$

Activity (i, j)	Duration t_{ij}	Total Float TF_{ij}	Free float FF_{ij}
1 – 2	3	2	0
1 – 3	1	0	0
2 – 4	1.	6	0

NOTES

2 – 5	1	2	2
3 – 5	5	0	0
4 – 9	6	6	6
5 – 6	5	6	0
5 – 7	8	0	0
6 – 8	1	6	0
7 – 9	2	0	0
8 – 10	4	6	6
9 – 10	6	0	0

The critical path is 1—3—5—7—9—10.

Example 9. *Consider the following informations :*

Activity	*Immediate predecessors*	*Duration*
A	*None*	*2*
B	*None*	*3*
C	*A*	*1*
D	*B*	*4*
E	*C, D*	*3*
F	*D*	*1*
G	*E*	*2*
H	*F*	*3*

Draw the project network and find the critical path.

Solution. The network is drawn below :

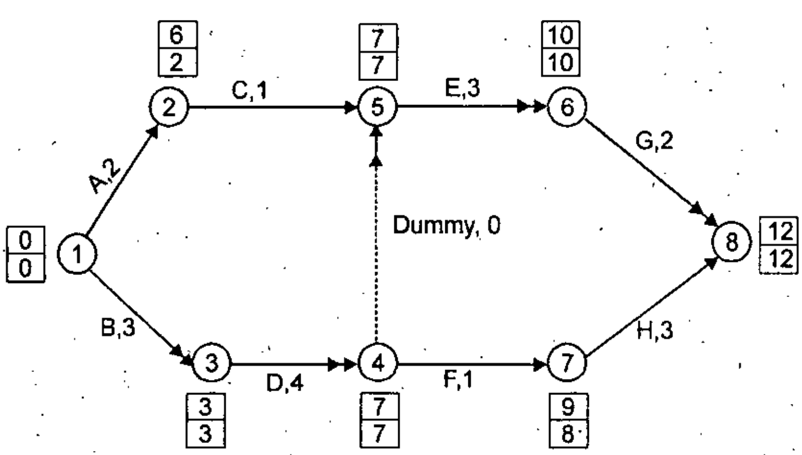


Fig. 5.9

Set $ES_1 = 0$

$ES_2 = ES_1 + t_{12} = 0 + 2 = 2$

NOTES

$$ES_3 = ES_1 + t_{13} = 0 + 3 = 3$$
$$ES_4 = ES_3 + t_{34} = 3 + 4 = 7$$
$$ES_5 = \text{Max.}\ \{ES_2 + t_{25}, ES_4 + t_{45}\}$$
$$= \text{Max}\ \{2 + 1, 7 + 0\} = 7$$
$$ES_6 = ES_5 + t_{56} = 10$$
$$ES_7 = ES_4 + t_{47} = 8$$
$$ES_8 = \text{Max.}\ \{ES_6 + t_{68}, ES_7 + t_{78}\}$$
$$= \text{Max}\ \{10 + 2, 8 + 3\} = 12$$

Set. $LF_8 = ES_8 = 12$

$$LF_7 = LF_8 - t_{78} = 12 - 3 = 9$$
$$LF_6 = LF_8 - t_{68} = 12 - 2 = 10$$
$$LF_5 = LF_6 - t_{56} = 10 - 3 = 7$$
$$LF_4 = \text{Min.}\ \{LF_5 - t_{54}, LF_7 - t_{47}\}$$
$$= \text{Min.}\ \{7, 8\} = 7$$
$$LF_3 = LF_4 - t_{34} = 7 - 4 = 3$$
$$LF_2 = LF_5 - t_{25} = 7 - 1 = 6$$
$$LF_1 = \text{Min.}\ \{LF_2 - t_{12}, LF_3 - t_{13}\}$$
$$= \text{Min.}\ \{4, 0\} = 0.$$

Thus the critical path is B—D—(dummy)—E—G.

5.11 PROGRAM EVALUATION AND REVIEW TECHNIQUE (PERT)

PERT was originally developed in 1958 to 1959 as part of the Polaris Fleet Ballistic Missile Program of the United States' Navy.

The primary difference between PERT and CPM is that PERT takes explicit account of the uncertainty in the activity duration estimates. CPM is activity oriented whereas PERT is event oriented. CPM gives emphasis on time and cost whereas PERT is primarily concerned with time.

In PERT, the probability distribution is specified by three estimates of the activity duration— a most likely duration (t_m), an optimistic duration (t_0) and a pessimistic duration (t_p). This type of activity duration is assumed to follow the beta distribution with

$$\text{Mean} = \frac{t_0 + 4t_m + t_p}{6}$$

and

$$\text{Variance} = \left(\frac{t_p - t_0}{6}\right)^2$$

The network construction phase of PERT is identical to that of CPM. Furthermore, once mean and variance are computed for each activity, the critical path determination is identical to CPM. The earliest and latest event times for the network are random variables. Once the critical path is determined, probability statements may be made about the total project duration and about the slack at any event.

Example 10. *A project consists of the following activities and different time estimates :*

Activity	t_0	t_m	t_p
1–2	3	5	8
1–3	2	4	8
1–4	6	8	12
2–5	5	9	12
3–5	3	5	9
4–6	3	6	10
5–6	2	4	8

(a) Draw the network.

(b) Determine the expected time and variance for each activity.

(c) Find the critical path and the project variance.

(d) What is the probability that the project will be completed by 22 days?

Solution. (*a*) Using the given information the resulting network is drawn as follows:

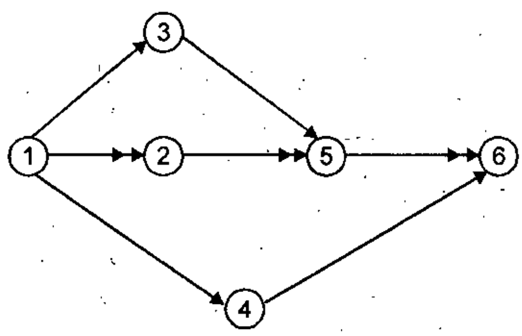


Fig. 5.10

(*b*) Expected time $= \dfrac{t_0 + 4t_m + t_p}{6} = \bar{t}_{ij}$

$$\bar{t}_{12} = 5.17 \quad \bar{t}_{25} = 8.83 \quad \bar{t}_{56} = 4.33$$
$$\bar{t}_{13} = 4.33 \quad \bar{t}_{35} = 5.33$$
$$\bar{t}_{14} = 8.33 \quad \bar{t}_{46} = 6.17$$

$$\text{Variance} = \left(\frac{t_p - t_0}{6}\right)^2.$$

$$\sigma_{12}^2 = 0.694 \quad \sigma_{25}^2 = 1.361 \quad \sigma_{56}^2 = 1$$
$$\sigma_{13}^2 = 1 \quad \sigma_{35}^2 = 1$$
$$\sigma_{14}^2 = 1 \quad \sigma_{46}^2 = 1.361$$

NOTES

(*c*) Set $ES_1 = 0$

Then
$$ES_2 = ES_1 + \bar{t}_{12} = 5.17$$
$$ES_3 = ES_1 + \bar{t}_{13} = 4.33$$
$$ES_4 = ES_1 + \bar{t}_{14} = 8.33$$
$$ES_5 = \text{Max}\{ES_3 + \bar{t}_{35}, ES_2 + \bar{t}_{25},\}$$
$$= \text{Max } \{9.66, 14\} = 14$$
$$ES_6 = \text{Max}\{ES_5 + \bar{t}_{56}, ES_4 + \bar{t}_{46},\}$$
$$= \text{Max } \{18.33, 14.5\} = 18.33$$

Set $LF_6 = ES_6 = 18.33$

Then
$$LF_5 = LF_6 - \bar{t}_{56} = 14$$
$$LF_4 = LF_6 - \bar{t}_{46} = 12.16$$
$$LF_3 = LF_5 - \bar{t}_{35} = 8.67$$
$$LF_2 = LF_5 - \bar{t}_{25} = 5.17$$
$$LF_1 = \text{Min.}\{LF_3 - \bar{t}_{13}, LF_2 - \bar{t}_{12}, LF_4 - \bar{t}_{14}\}$$
$$= \text{Min. } \{4.34, 0, 3.83\} = 0.$$

Hence the critical path is (1)—(2)—(5)—(6)

$$\text{Project variance} = \sigma_{12}^2 + \sigma_{25}^2 + \sigma_{56}^2$$
$$= 0.694 + 1.361 + 1 = 3.055.$$

(*d*) Here mean project length is 18.33.

Set $z = \dfrac{x - 18.33}{\sqrt{3.055}} \sim N(0,1)$

For $x = 22$, $z = 2.1$

Then the required probability
$$= P\ (X \le 22)$$
$$= P\ (z \le 2.1)$$
$$= 0.5 + 0.4821$$
$$= 0.9821$$

$\Rightarrow$ there is 98.21% chance that the project will be completed by 22 days.

Example 11. *A PERT network consists of 10 activities. The precedence relationships and expected time and variance of activity times, in days, are given below:*

Activity	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
Immediate predecessor (s)	–	*a*	*a*	–	*b*	*c*	*d*	*d*	*e,f,g*	*h*
Expected activity time	*4*	*2*	*6*	*2*	*3*	*9*	*5*	*7*	*1*	*10*
Variance of activity time	*1*	*1*	*2*	*1*	*1*	*5*	*1*	*8*	*1*	*16*

Construct an arrow diagram. Find the critical path based on expected times. Based on this critical path find the probability of completing the project in 25 days.

Solution. The resulting network is given in Fig. 5.11.

NOTES

Set
$$\begin{aligned}
ES_1 &= 0\\
ES_2 &= ES_1 + \bar{t}_{12} = 4\\
ES_3 &= ES_2 + \bar{t}_{23} = 6\\
ES_4 &= ES_2 + \bar{t}_{24} = 10\\
ES_5 &= ES_1 + \bar{t}_{15} = 2\\
ES_6 &= \text{Max.}\{ES_3 + \bar{t}_{36}, ES_4 + \bar{t}_{46}, ES_5 + \bar{t}_{56}\} = 19\\
ES_7 &= ES_5 + \bar{t}_{57} = 9\\
ES_8 &= \text{Max.}\{ES_6 + \bar{t}_{68}, ES_7 + \bar{t}_{78}\} = 20
\end{aligned}$$

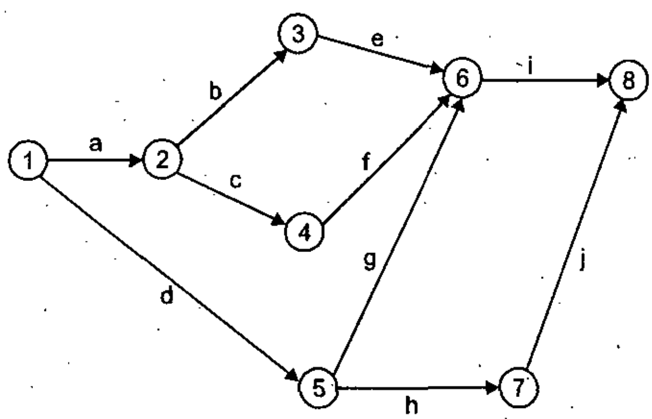


Fig. 5.11

Set
$$\begin{aligned}
LF_8 &= 20\\
LF_7 &= LF_8 - \bar{t}_{78} = 10\\
LF_6 &= LF_8 - \bar{t}_{68} = 19\\
LF_5 &= \text{Min.}\{LF_7 - \bar{t}_{57}, LF_6 - \bar{t}_{56}\} = 3\\
LF_4 &= LF_6 - \bar{t}_{46} = 10\\
LF_3 &= LF_6 - \bar{t}_{36} = 16\\
LF_2 &= \text{Min.}\{LF_3 - \bar{t}_{23}, LF_4 - \bar{t}_{24}\} = 4\\
LF_1 &= \text{Min.}\{LF_2 - \bar{t}_{12}, LF_5 - \bar{t}_{15}\} = 0
\end{aligned}$$

Hence the critical path is $a \to c \to f \to i$ on which ES = LF

Total expected time = 4 + 6 + 9 + 1 = 20

Project variance = 1 + 2 + 5 + 1 = 9

Set
$$z = \frac{x - 20}{3} \sim N(0, 1)$$

For $x = 25$, $z = 1.67$

NOTES

Then the required probability $= P(X \leq 25)$

$$= P(z \leq 1.67)$$
$$= 0.5 + \Phi\,(1.67)$$
$$= 0.5 + 0.4525$$
$$= 0.9525$$

$\Rightarrow$ There is 95.25% chance that the project will be completed by 25 days.

PROBLEMS

1. For a small project of 12 activities, the details are given below :

Activity	*Dependence*	*Duration (days)*
A	-	9
B	-	4
C	-	7
D	B,C	8
E	A	7
F	C	5
G	E	10
H	E	8
I	D,F,H	6
J	E	9
K	I,J	10
L	G	2

(*a*) Draw the network.

(*b*) Find the critical path.

2. (*a*) Draw a network for the following project:

Activities	1–2	1–3	1–4	2–5	2–6	3–6	5–7	6–7	4–7
Time (days)	8	12	4	9	3	6	5	10	5

(*b*) Determine total slack time for all activities and identify the critical path.

(*c*) Calculate total float and free floats of each activities.

3. Consider the following informations :

Job	1–2	2–3	2–4	3–4	3–5	3–6	4–5	5–6
Time (days)	10	9	7	6	9	10	6	7

(*a*) Draw the network.

(*b*) Find the critical path.

(*c*) Calculate total floats and free floats of each activities.

4. Draw the network using the given precedence conditions. Calculate the critical path and floats (total and free).

Activity	A	B	C	D	E	F	G	H	I	J
Immediate Predecessor(s)	–	A	A	A	D	D	E	F, G	C, H	B
Duration (months)	1	4	2	2	3	3	2	1	3	2

NOTES

5. A project consists of eight activities with the following time estimates:

Activity	Immediate predecessor	Time (days) t_0	t_m	t_p
A	–	1	1	7
B	–	1	4	7
C	–	2	2	8
D	A	1	1	1
E	B	2	5	14
F	C	2	5	8
G	D, E	3	6	15
H	F, G	1	2	3

(*a*) Draw PERT network.

(*b*) Find the expected time for each activity.

(*c*) Determine the critical path.

(*d*) What is the probability that the project will be completed in (*i*) 22 days, (*ii*) 18 days ?

(*e*) What project duration will have 95% chance of completion ?

6. Consider the following project :

Activity	Time estimates (in weeks) t_o	t_m	t_p	Predecessor
A	3	6	9	None
B	2	5	8	None
C	2	4	6	A
D	2	3	10	B
E	1	3	11	B
F	4	6	8	C, D
G	1	5	15	E

Find the critical path and its standard deviation. What is the probability that the project will be completed by 18 weeks ?

7. A project has the following activities and other characteristics :

Activity	Preceding activity	Time estimates (in weeks) t_0	t_m	t_p
A	–	4	7	16
B	–	1	5	15
C	A	6	12	30
D	A	2	5	8
E	C	5	11	17
F	D	3	6	15

G	B	3	9	27
H	E, F	1	4	7
I	G	4	19	28

NOTES

(a) Draw the PERT diagram.

(b) Identify the critical path.

(c) Find the probability that the project is completed in 36 weeks.

8. A small project is composed of eight activities whose time estimates are given below :

Activity	Time estimates		Pessimistic
	Optimistic	Most likely	
0 – 1	2	3	10
0 – 2	4	5	6
1 – 2	0	3	0
1 – 3	9	7	8
1 – 4	1	5	9
2 – 5	3	5	19
3 – 4	0	0	0
4 – 5	1	3	5

(a) Draw the project network.

(b) Compute the expected duration of each activity.

(c) Compute the variance of each activity.

9. Suppose the computer centre of your institute is planning to organize a national seminar. Consider the possible activities and prepare a PERT network for this seminar.

10. Draw the PERT network diagram using the given precedence conditions. Calculate the expected time, variance of each activity and the critical path.

Activity	Predecessor(s)	Duration (weeks)		
		t_0	t_m	t_p
A	–	1	2	3
B	–	1	2	8
C	A	6	7	8
D	B	1	2	3
E	A	1	4	7
F	C, D	1	5	9
G	C, D, E	1	2	3
H	F	1	2	9

NOTES

ANSWERS

1. CP : A—E—H—I—K, Project length = 40.
2. (b) CP : 1—3—6—7

   (c) TF : 6, 0, 19, 6, 7, 0, 6, 0, 19

   FF : 0, 0, 0, 0, 7, 0, 6, 0, 19
3. (b) CP : 1—2—3—4—5—6

   TF : 0, 0, 8, 0, 3, 9, 0, 0

   FF : 0, 0, 8, 0, 3, 9, 0, 0
4. A–D–E–G–H–I, Project length = 12 months.
5. (c) B—E—G—H (or, 1—3—5—6—7)

   Variance = $\frac{82}{9}$, Mean project length = 19

   (d) P $(X \leq 22) = 0.8389$ or 83.89%

   P $(X \leq 18) = 0.3707$ or 37.07%

   (e) 23.97 or 24 days.
6. CP : A—C—F, expected duration = 16 weeks

   Standard deviation = 1.374.

   P $(X \leq 18) = 0.928$.
7. (b) A—C—E—H (i.e., 1—2—4—6—7—8)

   (c) P $(X \leq 36) = 0.4207$.

10. CP : A–C–F–H, expected duration = 17 weeks

    Project variance = 1.568.

5.12 ELEMENTS OF CRASHING A NETWORK

Every activity may have two types of completion times—normal time and crash time. Accordingly costs are also two types *i.e.*, normal cost and crash cost. Obviously, the crash cost is higher than the normal cost and the normal time is higher than the crash time.

Crashing of a network implies that crashing of activities. During crashing direct cost increases and there is a trade-off between direct cost and indirect cost. So the project can be crashed till the total cost is economical. The following procedures are carried out :

(a) Calculate the critical path (CP) with normal times of the activities.

(b) Calculate the slope as given below of each activity.

$$\text{Slope} = \frac{\text{Crashing cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$$

NOTES

(c) Identify the critical activity with lowest slope.

(d) Compress that activity within crash limit. Compression time can also be calculated by taking min. (crash limit, free float limit).

If there are more than one critical path then select a common critical activity with least slope. If there is no such activity then select the critical activity with least slope from each critical path and compress them simultaneously within the crash limit.

(e) Continue crashing until it is not possible to crash any more.

(f) Calculate the total cost (TC) after each crashing as follows:

TC = Previous TC + Increase in direct cost – Decrease in indirect cost.

If the current TC is greater than the previous TC then the crashing is uneconomical and stop. Suggest the previous solution as optimal crashing solution.

Example 12. *A project consists of six activities with the following times and costs estimates :*

Activity	*Normal time (weeks)*	*Normal cost (Rs.)*	*Crash time (weeks)*	*Crash cost (Rs.)*
1–2	*9*	*400*	*7*	*900*
1–3	*5*	*500*	*3*	*800*
1–4	*10*	*450*	*6*	*1000*
2–5	*8*	*600*	*6*	*1000*
3–5	*7*	*1000*	*5*	*1300*
4–5	*9*	*900*	*6*	*1200*

If the indirect cost per week is Rs. 120, find the optimal crashed project completion time.

Solution. The slope calculations and the crash limit are given in the following table :

Activity	*Slope*	*Crash limit (weeks)*
1–2	250	2
1–3	150	2
1–4	137.5	4
2–5	200	2
3–5	150	2
4–5	100	3

Iteration 1

The CP calculations are shown in Fig. 5.12

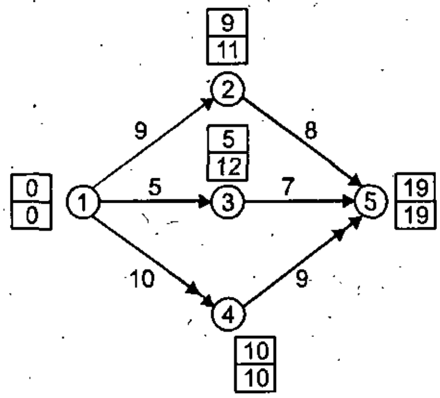


Fig. 5.12

CP : 1–4–5

Normal project duration = 19 weeks

Total direct (*i.e.*, normal) cost = Rs. 3850

Indirect cost = Rs. (19 × 120) = Rs. 2280

Total Cost (TC) = Rs. 3850 + Rs. 2280 = Rs. 6130

The slopes and crash limits of critical activities are summarised below:

Critical activity	*Slope*	*Crash limit (weeks)*
1–4	137.5	4
4–5	100*	3

Since 100 is the minimum slope, crash the activity 4–5 by 1 week *i.e.*, from 9 weeks to 8 weeks.

Iteration 2

The CP calculations are shown in Fig. 5.13

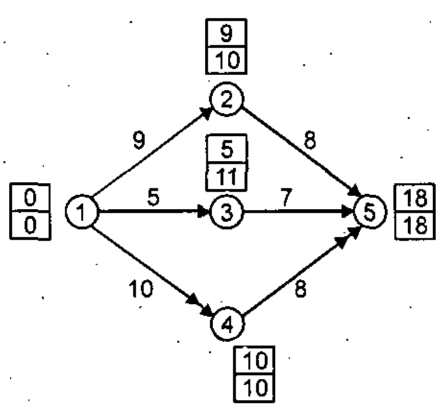


Fig. 5.13

CP : 1–4–5

Under the crashing, the project duration reduces to 18 weeks.

New TC = Rs. (6130 + 100 – 120) = Rs. 6110

NOTES

Since the new TC is less than the previous TC, the present crashing is economical and proceed for further crashing.

The slopes and crash limits of critical activities are summarised below :

Critical activity	*Slope*	*Crash limit (weeks)*
1–4	137.5	4
4–5	100*	2

Crash the activity 4–5 by 1 week *i.e.*, from 8 weeks to 7 weeks.

Iteration 3

The CP calculations are shown in Fig. 5.14

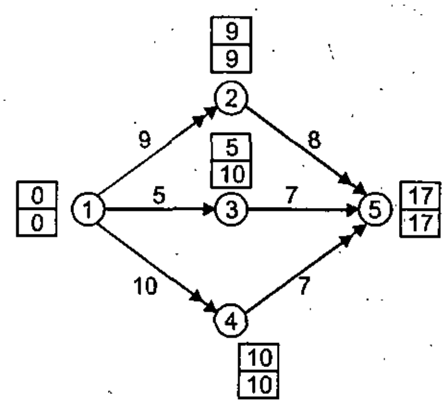


Fig. 5.14

We obtain two CPS : 1–4–5 and 1–2–5.

New TC = Rs. (6110 + 100 – 120) = Rs. 6090.

Since the new TC is less than the previous TC, the present crashing is economical and proceed for further crashing. The slopes and crash limits of critical activities are summarised below :

Critical activity	*Slope*	*Crash limit (weeks)*
1–4	137.5	4
4–5	100	1
1–2	250	2
2–5	200	2

Since there is no common critical activity, let us crash 4–5 by 1 week and 2–5 by 1 week.

Iteration 4

The CP calculations are shown in Fig. 5.15

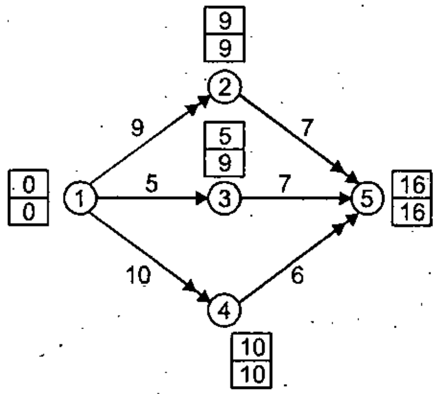


Fig. 5.15

New TC = Rs. 6090 + Rs. 100 + Rs. 200 – Rs. 120 = Rs. 6270.

Since the new TC is greater than previous TC, stop the iteration.

The previous iteration solution is the best for implementation.

Therefore, the final crashed project completion time is 17 weeks and the CPS are 1–2–5 and 1–4–5.

NOTES

5.13 DECISION TREES ANALYSIS

Decision tree is a graphical presentation of the decisions in a step by step indicating all parameters of decision like decision alternatives, states of nature, probabilities attached to states of nature, Conditional profit or loss.

Steps in decision tree analysis:

1. Locate on papers the decision points and the alternative courses of action against decision points.
2. Insert probability and associated pay off (EMY) for each course of action.
3. Start calculating EMV from extreme right end.
4. Choose the best EMV.
5. Trace backward to the next stage of decision points.
6. Repeat above steps till the first decision point is reached.
7. Identify the path to be followed from beginning to the end under different possible outcomes for the situation as a whole.

Example 13

The manager of inspection department takes the following strategies against assemble of party.

(*i*) Testing costs Rs. 15 No test incorporates Rs. 55 as guarantee accuracy with probabilities.

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(*ii*) After test, assembly actually OK, 70% and no OK 30%

(*iii*) If assy OK, it is passed as per test probability is 70%, and if fails it is 30% and cost zero and Rs. 55 respectively.

(*iv*) If any faulty, it is having probability of 30%.

(*v*) Faulty assy, it will be OK 20% and not OK 80% and costs associated are Rs. 145 and Rs. 55 respectively.

Draw the decision tree and find optimal EMV at starting node.

Solution:

EMV of node B $= 0.7 \times 0 + 0.3 \times 55 =$ Rs. 16.5

EMV of node C $= 0.2 \times 145 + 0.8 \times 55 =$ Rs 73.0

EMV of nodeA $= 0.7 \times 165 + 0.3 \times 73 =$ Rs. 33.45

EMV of node 1 via the branch testing $= 15 + 22.45 =$ Rs. 48.45

EMV of node 1 via the branch 'no test' $=$ Rs. 55 $>$ Rs. 48.45

Here we are dealing with cost, so optimal EMV of node 1 is Rs. 48.45 corresponding to the decision of testing the assembly.

SUMMARY

- The replenishments are usually instantaneous, uniform or batch. Its size refers to the quantity or size of the order to be received into inventory which may be constant or variable.
- Lead Time is the period between the time an order is placed (administrative lead time) and the time when it is received (delivery lead time). When the lead time is known, it is called deterministic. When it is not known, it can govern by a random variable.
- Economic Order Quantity is also known as 'economic lot size' or EOQ which is the optimum quantity to be purchased or produced such that the total cost of the inventory is minimized.
- The level between the maximum and minimum stock at which purchasing (or manufacturing) activities must start for replenishment is known as re-order level.
- An activity is an item of work to be done that consumes time, effort, money or other resources. It is represented by an arrow.
- An event represents a point time signifying the completion of an activity and the beginning of another new activity.
- Dummy Activity shows only preiedence relationship and they do not represent any real activity and is represented by a dashed line arrow or dotted line arrow and does not consume any time.
- The primary difference between PERT and CPM is that PERT takes explicit account of the uncertainty in the activity duration estimates. CPM is activity oriented whereas PERT is event oriented. CPM gives emphasis on time and cost whereas PERT is primarily concerned with time.

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REVIEW QUESTIONS

(Model I and III)

1. A purchase manager places order each time for a lot of 500 units of a particular item. From the available data the following results are obtained :

$$\begin{aligned} \text{Ordering cost per order} &= \text{Rs. } 600 \\ \text{Cost per unit} &= \text{Rs. } 50 \\ \text{Annual demand} &= \text{Rs. } 1000 \\ \text{Inventory carrying cost} &= 40\% \end{aligned}$$

   Find out the loss to the organization due to his ordering policy.

2. An aircraft company uses rivets at an approximately constant rate of 5000 kg per year. The rivets cost Rs. 20 per kg. and the company personnel estimate that it costs Rs. 200 to place an order, and the carrying cost of inventory is 10% per year. How frequently should orders for rivets be placed and what quantities should be ordered for?

3. The Inventory company, after an analysis of its accounting and production records, has determined that it uses Rs. 36000 per year of a component part purchased at Rs. 18 per part. The purchasing cost is Rs. 40 per order, and its annual inventory carrying charges are $16\frac{2}{3}\%$ of the average inventory. Determine

   (*a*) the most EOQ at one time.

   (*b*) the most economic number of times to order per year.

   (*c*) the average days' supply for ordering the most EOQ.

   (year = 365 days)

4. A company uses 50,000 widgets per annum which costs Rs. 10 per piece to purchase. The ordering and handling costs are Rs. 150 per order and carrying costs are 15% per annum. Find the EOQ.

   Suppose the company decides to make the widgets in its own factory and installed a machine which has capacity of 2,50,000 widgets per annum. What is the EOQ?

(Model III)

5. A contractor has to supply 10000 bearing per day to an automobile manufacturer. He finds that when he starts production run, he can produce 25000 bearings per day. The cost of holding a bearing in stock for a year is Rs. 2 and the setup cost of a production run is Rs. 1800. How frequently should production run be made? (Assume 1 year = 300 working days)

6. In a paints manufacturing unit, the changeover from one type of paint to another is estimated to cost Rs. 100 per batch. The annual sales of a particular grade of paint are 20,000 litres and the inventory carrying cost is Rs. 1 per litre. Given that the rate of production is 3 times the sales rate, determine the economic batch size and number of batches per year and total optimum yearly cost.

7. A product is sold at the rate of 30 pieces per day and is manufactured at a rate of 200 pieces per day. The set up costs of the machines are Rs. 300 and the holding cost is found to be Rs. 0.05 per piece day. Find optimum batch size, period of production and the optimum number of production run. (Assume 1 year = 365 days)

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(Model II and I)

8. An item is to be supplied at a constant rate of 100 unit per day. The ordering cost for each supply is Rs. 20, cost of holding the item in inventory is Rs. 1 per unit per day while delay in the supply of the item induces a penalty of Rs. 3 per unit per day. Find the optimal policy (Q, *t*) and optimum shortage.
9. The demand for a product is 150 units per month and the items are withdrawn uniformly. The setup cost each time a production run is Rs. 12. The holding cost is Rs. 0.25 per item per month.

   (*a*) Determine how often to make production run, if shortages are not allowed.

   (*b*) Determine how often to make production run, if shortages cost Rs. 1 per item per month.
10. The demand for a certain item is uniform at a rate of 30 units per month. The fixed cost is Rs. 10 each time a production run is made. The production cost is Rs. 2 per item and the holding cost is Rs. 0.3 per item per month. If the shortage cost is Rs. 1.5 per item per month, determine how often to make a production run and of what size should be?

(Model IV)

11. The demand for an item in a company is 24000 units per year, and the company can produce 2500 units per month. The one setup cost is Rs. 300 and the holding cost per unit per month is Rs. 0.3 and the shortage cost of one unit is Rs. 20 per month. Determine the optimum manufacturing quantity, the number of shortages and manufacturing time.

(Probabilistic Models)

12. Rework the Example 7, taking demand is uniform.
13. If the demand for a certain product has a rectangular distribution between 4000 and 500, find the optimal expected total cost if storage cost is Rs. 1 per unit and shortage cost is Rs. 7 per unit and the purchasing cost is Rs. 10 per unit, and the demand is instantaneous.
14. A certain children product is stocked by a company. The demand distribution is given below :

Demand	0	10	20	30	40	50
Probability	0.1	0.20	0.35	0.2	0.1	0.05

Inventory carrying cost is Rs. 5, the storage cost is Rs. 20, find the economic order quantity.

15. A project consists of seven activities with the following times and costs estimates:

Activity	*Normal time (weeks)*	*Normal cost (Rs.)*	*Crash time (weeks)*	*Crash cost (Rs.)*
1–2	12	500	8	900
1–3	6	600	5	700
1–4	8	700	5	850
2–5	11	500	10	820
3–5	7	1000	5	1200
4–6	6	900	4	1000
5–6	10	1200	8	1450

If the indirect cost per week is Rs. 150, find the optimal crashed project completion time.

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16. Consider the data of a project as shown in the following table :

Activity	Normal time (weeks)	Normal cost (Rs.)	Crash time (weeks)	Crash cost (Rs.)
1–2	9	500	8	600
1–3	7	800	6	1100
1–4	8	900	6	1200
2–5	6	850	5	950
3–4	10	1200	8	1400
4–5	4	700	3	870
5–6	5	1000	4	1200

If the indirect cost per week is Rs. 160, find the optimal crashed project completion time.

17. The table below provides the costs and times for a seven activity project :

Activity (i, j)	Time estimates (weeks)		Direct cost estimates (Rs. '000)	
	Normal	Crash	Normal	Crash
(1, 2)	2	1	10	15
(1, 3)	8	5	15	21
(2, 4)	4	3	20	24
(3, 4)	1	1	7	7
(3, 5)	2	1	8	15
(4, 6)	5	3	10	16
(5, 6)	6	2	12	36

(*i*) Draw the project network corresponding to normal time.

(*ii*) Determine the critical path and the normal duration and cost of the project.

(*iii*) Crash the activities so that the project completion time reduces to 11 weeks irrespective of the costs.

ANSWERS

1. Loss is about Rs. 1301.
2. $Q^* = 1000$ unit, $n^* = 5$ times in a year.
3. (*a*) $Q^* = 231$, (*b*) $n^* = 9$, (*c*) 40.6 days.
4. $Q^* = 10{,}000{,}000$ and $Q^* = 3535.53$.
5. $Q^* = 1{,}04{,}446$, $t^* = 10.44$ days.
6. $Q^* = 2449.49$, $n^* = 12.25$, total cost = Rs. 2000.
7. $Q^* = 650.79$.
8. $Q^* = 73.03$, $t^* = 0.73$ day, $Q_2^* = 18.26$.
9. (*a*) 120, (*b*) 134.16.
10. $Q^* = 49$, $t^* = 1.63$.
11. $Q^* = 4505.55$, $Q_2^* = 13.32$, Manufacturing time = 1.8 month.

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13. Rs. 49187.
14. $Q^* = 30$.
15. Total crashed cost = Rs. 10100 with crashed project completion time = 27 weeks.
16. Total crashed cost = Rs. 9990 with crashed project completion time = 24 weeks.
17. (*ii*) CP : 1–3–5–6. Normal project duration 15 weeks and normal cost = Rs. 82000.
(*iii*) All paths becomes CP.

FURTHER READINGS

- **Operations Research:** Col. D.S. Cheema, University Science Press.
- **Introductory Operations Research:** Theory & Applications 3e: Kasana, Springer
- **Operations Research:** N.P. Agarwal, Indus Valley Publication.
- **Operations Research:** Jaya Banerjee, Shree Niwas.
- **Operations Research for Library and Information Professionals:** Dariush Alimohammadi, Ess Ess Publications.